APLIKASI KAEDAH ASYMPTOTIC WAVEFORM EVALUATION (AWE) DALAM MENYELESAIKAN MASALAH TERMA DAN GETARAN

(APPLICATION OF ASYMPTOTIC WAVEFORM EVALUATION (AWE) METHOD IN SOLVING THERMAL AND VIBRATION PROBLEMS)

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FEB 2003

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Pusat Pengajian Kejuruteraan Mekanik
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This thesis is the results of my own investigations, except where otherwise stated. Other sources are acknowledged by giving explicit references. A bibliography/references are appended.

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J. S. LOH

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NOMENCLATURE

Symbol	Description	Unit
A	Area of triangular element	m^2
c	Specific heat	$J/^{o}C \cdot kg$
F	Force	N
G	Volumetric heat generation	W/m^3
h	Height	μm
I_{rms}	Root mean square current	-
k	Thermal conductivity	$W/^{o}C \cdot m$
l	Length	μm
M	Moment	-
P	Power dissipation	W
R	Resistance	Ω
T	Temperature	^{o}C
T_o	Initial temperature	^{o}C
T_w	Side wall surface temperature	-
t	Time	s
w	Width	μm
V	Volume of tetrahedral element	m^3
x,y,z	Spatial coordinate	m
Z_T	Dimensionless phase lag for temperature gradient	-
Z_q	Dimensionless phase lag for heat flux	-
α	Thermal diffusivity	m^2/s
β	Dimensionless time	-
δ	Dimensionless distance x	-
ε	Dimensionless distance y	-
θ	Dimensionless temperature	-
ρ	Density	kg/ms^{-3}
τ_T	Phase lag for temperature gradient	s
τ_q	Phase lag for heat flux	s

ABSTRAK

Asymptotic Waveform Evaluation (AWE), yang telah digunakan dalam simulasi fana litar, kini diluaskan penggunaannya dalam menyelesaikan masalah kejuruteraan mekanik. AWE adalah lebih cekap dan canggih daripada kaedah berangka tipikal kerana ia memerlukan masa pengiraan computer yang lebih singkat tetapi juga menghasilkan kejituan yang sama.

AWE berkebolehan untuk menyelesaikan persamaan pembezaan linear dengan darjah pertama, kedua, ketiga atau pun lebih tinggi. Selain itu, AWE juga boleh digunakan untuk menyelesaikan satu atau pun satu set persamaan pembezaan. Jadi, AWE sesuai digunakan bersama Kaedah Elemen Terhingga, di mana persamaan akhirnya biasanya adalah satu set persamaan pembezaan linear yang perlu diselesaikan untuk mendapatkan penyelesaian fana. Tetapi, AWE boleh menghasilkan penyelesaian yang tak stabil walaupun bagi sistem yang stabil. Anggaran dengan darjah yang lebih tinggi juga tidak dapat selalu menghasilkan penyelesaian yang lebih jitu dan stabil. Jadi, ia perlu diubahsuai untuk menghasilkan penyelesaian yang stabil.

Dalam projek ini, AWE telah membuktikan kebolehannya dalam menyelesaikan masalah pemindahan haba dan juga analisis getaran. AWE, bersama dengan Kaedah Elemen Terhingga, digunakan untuk menyelesaikan masalah sirip (fin) dalam satu dimensi dengan keadaan sempadan yang dikenakan suhu berubah atau mantap pada tapaknya. Kemudian, AWE juga digunakan untuk menyelesaikan persamaan konduksi haba hiperbolik (Non-Fourier) pada model elemen terhingga dalam dua dan tiga dimensi. Kejituan dan ketidakstabilannya juga dibincangkan dan dua skema kestabilan diperkenalkan untuk menyelesaikan masalah ketidakstabilan ini.

Selain itu, AWE juga digunakan dalam analisis getaran rasuk, di mana daya impuls, langkah atau pun daya dalam bentuk sinus dikenakan pada hujung rasuk sebagai keadaan sempadannya. Akhirnya, AWE digunakan untuk menyelesaikan persamaan pembezaan linear dengan darjah ketiga. Sepertimana yang telah dibincangkan di atas, terbukti AWE dapat menghasilkan penyelesaian fana dengan kejituan yang sama dengan kaedah Crank-Nicolson, Rungge-Kutta dan juga softwer Ansys. Akan tetapi, AWE dapat menghasilkan penyelesaian dalam masa yang lebih singkat.

ABSTRACT

Asymptotic Waveform Evaluation (AWE), which has been used in transient circuit simulation, is extended for solving mechanical engineering problems. AWE is based on the concept of approximating the original system with a reduced order system. Thus, it is efficient and powerful than conventional numerical method because it requires much less computational time but also produces the same accuracy.

AWE is capable of solving first, second, third and even higher order linear differential equation. AWE can handle one or even a set of linear differential equations. Thus, AWE is suitably used with Finite Element Method (FEM), where the final equation is usually reduced to a set of linear differential equations that is to be solved for its transient solution. However, AWE is known for producing unstable response even for stable system. Higher order approximation also will not always guarantee a more accurate and stable solution. Thus, some modification has to be made to stabilize its solution.

In this project, AWE has proved its capability in solving transient thermal and vibration problems. AWE, together with FEM, is used to solve one dimensional fin problem with varying or constant temperature boundary condition imposed at the base. Then, it is also used to solve hyperbolic (Non-Fourier) heat conduction equation on two and three dimensional finite element model. The accuracy and instability of AWE are also discussed and two stability schemes are introduced to address this problem.

Nevertheless, AWE is also used in vibration analysis of beam, where dynamic force such as impulse, step or sinusoidal force, is imposed. Lastly, AWE is used to solve third order differential equation. AWE has proved to produce transient solution as accurate as Crank-Nicolson, Rungge-Kutta and also Ansys software. However, AWE can produce the solution much faster than these three methods.

CHAPTER 1
INTRODUCTION

1.1 Background

Asymptotic Waveform Evaluation (AWE) is a numerical method that is capable of solving first, second, third and even higher order linear differential equations. AWE can handle, not only one equation but also a set of linear differential equations. Thus, AWE is suitably used with Finite Element Method (FEM) to solve various engineering problems.

Implementation of Finite Element Method (FEM) usually will reduce the governing differential equation into a set of linear differential equations. For instance, the governing equation for Non-Fourier heat conduction (discussed in Section 3.2.3) is a hyperbolic equation. By using FEM, this hyperbolic equation is reduced into a set of second order linear differential equation. Then, this set of equations has to be solved in time domain to obtain its transient solution.

However, in order to obtain accurate results, large number of elements has to be used, and thus this will result in a large set of linear differential equations. Solving this large set of equations is very time consuming, especially when the time step required is also very small. Usually, this set of equations is solved using conventional numerical method such as Rungge-Kutta and Crank-Nicolson, where the whole equations are solved simultaneously even though just to obtain the solution at one particular node. Using these conventional methods also, the calculation is iterated at every increment of time step to produce the transient solution.

As mentioned earlier, the size of the equations involved is usually large and thus, these conventional methods will require considerable amount of computational time. Furthermore, these conventional methods also require small time step to produce accurate results and to avoid numerical instability. With this limitation, some problems can not be solved without using high performance computer and also requiring considerable amount of time as in Section 3.2.5.

AWE, on the other hand, is independent of time step because it produces the transient solution in form of equation, rather than numerical solution at every increment of time step. AWE is also capable of producing local solution because it can obtain the solution for each node independently. Thus, it will reduce the amount of computational time significantly. It can also be shown that AWE can produce transient solution as accurate as conventional method, but much more efficient in term of time.

However, AWE suffers from instability problem, where higher order approximation will not always guarantee a more accurate and stable solution. Thus, AWE may need some modification to stabilize its solution. Two stability schemes has been introduced in Section 3.2.2 and 3.2.4 and these schemes has proved to be successful, where higher order approximation has produce accurate and also stable results.

1.2 Literature review

The inspiration of AWE concept first came from the research of Rubinstein et. al. (1983). They were doing the research in the RC-trees network simulation by using the efficient Elmore delay (first moment of the impulse response) estimate approach. But this estimate approach has not always produced an accurate result, because there are a lot of limitations in doing a transient analysis. So, the major effort of the early work was to find a solution scheme for transient analysis.

The work from McCormick (1989), gave a second spark to this concept. From his research in interconnect circuit simulation, he has shown that the circuit moments (the coefficient of expansion of circuit driving point of transfer function in a Macluarin series about s=0 in the frequency domain) could lead to lower circuit models and reasonably accurate transient response results. From the previous effort of these authors in formalization and generalization of the algorithm, eventually an n^{th} order extension of the first order Elmore delay approximation was developed and name as AWE.

With this new scheme, the response of a higher order system can be simulated by a low order approximation. This approximation consists of a few dominants poles (and zeros) in the Pade approximation, so as the order of approximation increases, the corresponding approximate transient response will asymptotically approach to the actual response, thus the nature of the approximation itself eventually inspiring the name.

Pillage (1990) has showed the capability of AWE in transient analysis, where it was used to capture the effect of interconnect on the delay with a simplified model, typically a RC tree model. Reasonable results were produced as compared with SPICE simulation, but two or three orders faster than SPICE. From his work, AWE is applicable in floating node, linear controlled sources, finite input rise time, charge sharing, bipolar circuitry, interconnect timing estimation and MOS circuit analysis.

After that, Anastakis (1992) has described a systematic approach for alleviating the inherent stability associated with AWE and moment matching method as they were applied to circuit analysis problems. In general, it is not possible to obtain a particular order of approximation due to inherent instability problems that may result in unstable models of stable circuits. In this paper, the instability problems were alleviated by considering partial Pade approximation, where a single set of dominant poles computed at an appropriately selected node can be used as a common denominator for the response approximation at any node.

Then, Lee (1992) has developed a method for calculating the sensitivities of the poles and zeros found by AWE. Using the adjoint sensitivity method, it is possible to inexpensively compute the sensitivities of the poles and zeros with respect to all the circuit parameters, as well as to suppress circuit parasitics. The sensitivities of the poles and zeros found by AWE show excellent correlation with those of the real circuit and provide useful information in both time and frequency domain. In frequency domain, measures such as the sensitivities of phase margin, gain margin, dc gain and also bandwidth are easily calculated in terms of these basic pole and zero sensitivities. In the time domain, sensitivities of delay, rise time and overshoot can be easily computed as well.

On invitation, a tutorial paper on the literature of AWE was published (Raghavan 1993). This invited paper attempted to give an overview of the algorithm of AWE, starting with the definition of moments. Then, it was shown that the time domain moments of a signal f(t) are related to the Taylor series coefficients about s=0 (Maclaurin Series) of the signal `s Laplace transform, F(s). After that, it was shown that the algorithm of AWE is divided into two main parts, namely moment computation and moment matching. The formulation given was for equation in the form of first order linear differential equation.

The instability of Pade approximation was also discussed in this paper (Raghavan 1993) since it has been historically known that Pade approximation can produce poles with positive real parts for stable system. This will cause the time domain response of the approximation to be unbounded. Then, the conventional methods of solving this instability in AWE were introduced. The concept of AWE macromodels and also distributed elements were also presented. An overview of a generalization of the AWE technique, which relates AWE to conventional numerical integration techniques, were also given.

In the same year, AWE is used to compute the time response of an arbitrary 3-D interconnect structure (Kumashiro 1993). It has been implemented in software called 3DAWE. To facilitate the application of AWE to a 3D RC mesh network model, AWE formulation was rederived based upon a nodal analysis approach. Using this software, a typical transient response of a reasonably large 3-D RC network could be obtained within a few minutes on a 15 MIPS computer.

Da (1995) presented a first paper on thermal analysis of Printed circuit board (PCB) using AWE scheme. He has used the electric thermal network analogy method to study the thermal behaviour of printed circuit board. Then, he used finite difference method to reduce the governing equation into a set of linear differential equations. This set of equations was solved with AWE and then the solution was compared with HSPICE. He has shown that the application of AWE to solve time dependent thermal analysis of printed circuit boards often resulted in two order speed-up over current iterative techniques, yet retaining comparable accuracy. However, the response formulation, poles and residues used to predict the transient temperature response for first order ordinary equation seem to be incorrect and there are no details to describe the incorporation of boundary conditions.

Recently, Ooi (2003) has presented a general formulation of AWE, where it is applicable in solving first and second order ordinary differential equation. The incorporation of boundary conditions has also been shown clearly by using the concept of Zero State Response (ZSR) and Zero Input Response (ZIR) that is used in control system. This AWE scheme has been used together with Finite Element Method (FEM) to solve the transient temperature rise of a simple fin, where the governing equation was reduced to a set of first order differential equations. Then, he has solved problems of second order differential equations, such as Non-Fourier heat conduction and vibration problems. With this AWE scheme also, he has solved the temperature response of micro-channel heat exchanger system.

Ooi (2003) has also shown that AWE can produce inaccurate result when higher moment approximation is used. By using the case of a simple fin problem, he has shown that AWE has produced inaccurate maximum temperature rise when 12, 14 or 16 moments are used. Thus, he has suggested the use of lower approximation to stabilize the AWE solution. However, lower moment approximation may not always yield stable solution and it may also cause the solution to be inaccurate.

From the literature review, it is clear that the application of AWE in solving mechanical engineering problems is still at the development stage, where most of the researches and application are conducted on circuit simulation. Thus, in this project, AWE scheme is further extended in solving thermal and also vibration problems. The incorporation of boundary conditions in AWE has been shown by Ooi (2003) but he has only solved problems with steady boundary condition and step input function. Therefore, problems with unsteady boundary conditions and different input function are solved in this project.

In addition, the instability of AWE when used with Finite Element Method (FEM) is also discussed. The instability and inaccuracy associated with the use of lower moment approximation is also addressed. Two stability schemes are also introduced to handle the inherent instability of AWE.

1.3 Objectives

There are six objectives in this project, which are listed as below:

- To extend the application of AWE in solving thermal and vibration problems.

- To validate the applicability of AWE in solving one dimensional, two dimensional and also three dimensional finite element problems.

- To verify the accuracy and also efficiency of AWE as compared to conventional numerical methods.

- To address the inherent instability of AWE and also the solutions to counter this problem.

- To validate the applicability of AWE in solving finite element problems with unsteady boundary conditions.

- To extend the application of AWE in solving third order linear differential equation.

CHAPTER 2

THEORY REVIEW

2.1 Overview

In this chapter, the concept and formulation of Asymptotic Waveform Evaluation (AWE) will be discussed. The details will be presented in several sections as following:

- Concept of Asymptotic Waveform Evaluation (AWE)
- Moment generation
- Moment matching
- Transient response

2.2 Concept of Asymptotic Waveform Evaluation (AWE)

The concept of AWE is to approximate the response of a system with a reduced order system. The response of a system can be represented by a polynomial equation in s-domain, where the coefficients of this polynomial are known as the moments (Pillage 1990). This polynomial equation in s-domain is substituted into the governing linear differential equation and by matching the coefficients with the term s of same power, the moments can be computed.

Then, the order of this polynomial equation is reduced by approximating it with a polynomial fraction using Pade approximation. Finally, this polynomial fraction is simplified to partial fractions, where each partial fraction contains a pole and zero. Then, each partial fraction is inversed Laplace back to time domain and they are summed up to provide the transient solution.

The algorithm of AWE is divided into two parts, which are moment generation and moment matching. In moment generation, the moments are calculated by matching the coefficients with term s of same power. The details are presented in Section 2.3. In moment matching, the poles and zeros of the partial fractions are calculated as shown in Section 2.4. Finally, the partial fractions are inversed Laplace back to time domain for the transient solution as in Section 2.5. Figure 2.1 shows the flow of AWE algorithm.

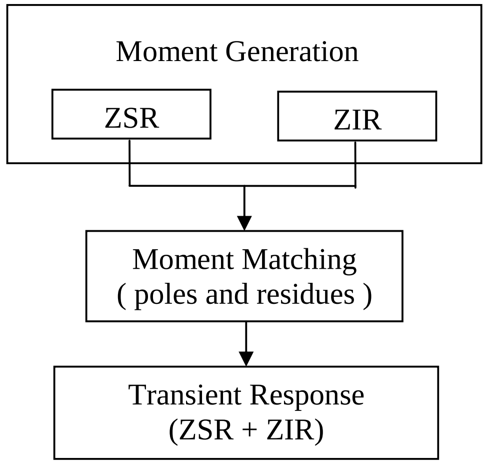

Figure 2.1: Flow of AWE algorithm

2.3 Moment generation

The first part of AWE is moment generation. However, before proceeding, the order of differentiation of the problem equation has to be determined. This is because different formulation is used according to the order of differentiation. The formulations of moment generation for first, second and third order differential equations are provided on Sections 2.3.1, 2.3.2 and 2.3.3 below.

For easier formulation, moment generation are divided into two parts, which are Zero State Response (ZSR) and Zero Input Response (ZSR). The concept of response is actually used in control system, where the response of a system is the sum of ZSR and ZIR. In ZSR, the initial conditions of the system are assumed to be zero, while in ZIR, the forcing functions are assumed to be zero. Then, the moments generated for ZSR and ZIR will be used respectively to obtain the poles and residues as in Section 2.4. Finally, the responses of ZSR and ZIR will be added to form the total system response.

2.3.1 First order linear differential equation

First order linear differential equation has the form of Equation 2.1 below.

$$C\dot{T} + KT = F(t) \tag{2.1}$$

Taking Laplace transform of Equation 2.1 to obtain Equation 2.2.

$$C\left(sT(s) - T(0)\right) + KT(s) = f \tag{2.2}$$

The solution or response, T of Equation 2.2 can be represented by a polynomial equation in s-domain as shown by Equation 2.3. The coefficients of Equation 2.3 are known as moments.

$$T(s) = \sum_{n=0}^{\infty} M_n s^n \tag{2.3}$$

Finally, moments are generated for ZSR and ZIR as following.

Zero State Response (ZIR)

In ZSR, the initial conditions are assumed to be zero. Thus, with $T(0) = 0$, Equation 2.2 is reduced to Equation 2.4 below.

$$\left(Cs + K\right)T(s) = f \tag{2.4}$$

Substitute Equation 2.3 into Equation 2.4 to obtain Equation 2.5 below.

$$\left(Cs + K\right)\left(M_0 + M_1 s + M_2 s^2 + ... + M_n s^n\right) = f \tag{2.5}$$

By matching the coefficients of Equation 2.5 with term s of same power, the moments are generated from Equation 2.6 below.

$$KM_0 = f$$

$$KM_n = -CM_{n-1} \qquad \text{for n = 1,2,3,......,(2q-1)} \tag{2.6}$$

Zero Input Response (ZIR)

In ZIR, the forcing function is assumed to be zero. Thus, with $f = 0$, Equation 2.2 is reduced to Equation 2.7 below.

$$\left(Cs + K\right)T(s) = CT(0) \tag{2.7}$$

Substitute Equation 2.3 into Equation 2.7 to obtain Equation 2.8 below.

$$\left(Cs + K\right)\left(M_0 + M_1 s + M_2 s^2 + ... + M_n s^n\right) = CT(0) \tag{2.8}$$

By matching the coefficients of Equation 2.8 with term s of same power, the moments are generated from Equation 2.9 below.

$$KM_0 = f$$

$$KM_n = -CM_{n-1} \qquad \text{for n = 1,2,3,……,(2q-1)} \qquad (2.9)$$

2.3.2 Second order linear differential equation

Second order linear differential equation has the form of Equation 2.10 below.

$$A\ddot{T} + C\dot{T} + KT = F(t) \qquad (2.10)$$

Taking Laplace transform of Equation 2.10 to obtain Equation 2.11.

$$A\left(s^2T(s) - sT(0) - \dot{T}(0)\right) + C\left(sT(s) - T(0)\right) + KT(s) = f \qquad (2.11)$$

The solution or response, T of Equation 2.11 can be represented by a polynomial equation in s-domain as shown by Equation 2.12. The coefficients of Equation 2.12 are known as moments.

$$T(s) = \sum_{n=0}^{\infty} M_n s^n \qquad (2.12)$$

Finally, moments are generated for ZSR and ZIR as following.

<u>Zero State Response (ZSR)</u>

In ZSR, the initial conditions are assumed to be zero. Thus, with $T(0) = 0$ and $\dot{T}(0) = 0$,

Equation 2.11 is reduced to Equation 2.13 below.

$$\left(As^2 + Cs + K\right)T(s) = f \qquad (2.13)$$

Substitute Equation 2.12 into Equation 2.13 to obtain Equation 2.14 below.

$$\left(As^2 + Cs + K\right)\left(M_0 + M_1 s + M_2 s^2 + ... + M_n s^n\right) = f \qquad (2.14)$$

By matching the coefficients of Equation 2.14 with term s of same power, the moments are generated from Equation 2.15 below.

$$KM_0 = f$$

$$KM_n = -CM_{n-1}$$

$$KM_n = -\left(AM_{n-2} + CM_{n-1}\right) \qquad \text{for n = 2,3,4,……,(2q-1)} \qquad (2.15)$$

Zero Input Response (ZIR)

In ZIR, the forcing function is assumed to be zero. Thus, with $f = 0$, Equation 2.11 is reduced to Equation 2.16 below.

$$\left(As^2 + Cs + K\right)T(s) - \left(As + C\right)T(0) - A\dot{T}(0) = 0 \qquad (2.16)$$

Substitute Equation 2.12 into Equation 2.16 to obtain Equation 2.17 below.

$$\left(As^2 + Cs + K\right)\left(M_0 + M_1 s + M_2 s^2 + ... + M_n s^n\right) - \left(As + C\right)T(0) - A\dot{T}(0) = 0 \qquad (2.17)$$

By matching the coefficients of Equation 2.17 with term s of same power, the moments are generated from Equation 2.18 below.

$$KM_0 = CT(0) + A\dot{T}(0)$$

$$KM_1 = AT(0) - CM_0$$

$$KM_n = -\left(AM_{n-2} + CM_{n-1}\right) \qquad \text{for n = 2,3,4,……,(2q-1)} \qquad (2.18)$$

2.3.3 Third order linear differential equation

Third order linear differential equation has the form of Equation 2.19 below.

$$G\dddot{T} + A\ddot{T} + C\dot{T} + KT = F(t) \qquad (2.19)$$

Taking Laplace transform of Equation 2.19 to obtain Equation 2.20.

$$\begin{matrix} G\left(s^3T(s) - s^2T(0) - s\dot{T}(0) - \ddot{T}(0)\right) + A\left(s^2T(s) - sT(0) - \dot{T}(0)\right) \\ + C\left(sT(s) - T(0)\right) + KT(s) \end{matrix} = f \qquad (2.20)$$

The solution or response, T of Equation 2.20 can be represented by a polynomial equation in s-domain as shown by Equation 2.21. The coefficients of Equation 2.21 are known as moments.

$$T(s) = \sum_{n=0}^{\infty} M_n s^n \qquad (2.21)$$

Finally, moments are generated for ZSR and ZIR as following.

Zero State Response (ZSR)

In ZSR, the initial conditions are assumed to be zero. Thus, with $T(0)=0$, $\dot{T}(0)=0$ and $\ddot{T}(0)=0$, Equation 2.20 is reduced to Equation 2.22 below.

$$\left(Gs^3+As^2+Cs+K\right)T(s)=f \tag{2.22}$$

Substitute Equation 2.21 into Equation 2.22 to obtain Equation 2.23 below.

$$\left(Gs^3+As^2+Cs+K\right)\left(M_0+M_1s+M_2s^2+....+M_ns^n\right)=f \tag{2.23}$$

By matching the coefficients of Equation 2.23 with term s of same power, the moments are generated from Equation 2.24 below.

$$KM_0=f$$
$$KM_1=-CM_0$$
$$KM_2=-CM_1-AM_0$$
$$KM_n=-\left(GM_{n-3}+AM_{n-2}+CM_{n-1}\right) \qquad \text{for n = 3,4,5,……,(2q-1)} \tag{2.24}$$

Zero Input Response (ZIR)

In ZIR, the forcing function is assumed to be zero. Thus, with $f=0$, Equation 2.20 is reduced to Equation 2.25 below.

$$\left(Gs^3+As^2+Cs+K\right)T(s)-\left(Gs^2+As+C\right)T(0)-\left(Gs+A\right)\dot{T}(0)-G\ddot{T}(0)=0 \tag{2.25}$$

Substitute Equation 2.21 into Equation 2.25 to obtain Equation 2.26 below.

$$\begin{matrix}\left(Gs^3+As^2+Cs+K\right)\left(M_0+M_1s+....+M_ns^n\right)\\ -\left(Gs^2+As+C\right)T(0)-\left(Gs+A\right)\dot{T}(0)-G\ddot{T}(0)\end{matrix}=0 \tag{2.26}$$

By matching the coefficients of Equation 2.26 with term s of same power, the moments are generated from Equation 2.27 below.

$$KM_0=CT(0)+A\dot{T}(0)+G\ddot{T}(0)$$
$$KM_1=-CM_0+AT(0)+G\dot{T}(0)$$
$$KM_2=-CM_1-AM_0+GT(0)$$

$$KM_n = -\left(CM_{n-1} + AM_{n-2} + GM_{n-3}\right) \qquad \text{for n = 3,4,5,……,(2q-1)} \qquad (2.27)$$

2.4 Moment matching

By using conventional methods such as Rungge-Kutta and Crank-Nicolson, all the nodes of a system have to be solved simultaneously, even though only the solutions of certain nodes are of interest. In contrast, AWE can provide local solution because it is capable of obtaining the solution for each node independently and thus, this will reduce the amount of calculation significantly.

Each moment generated previously in Section 2.3 contains a set of values, where each value corresponds to each node. Thus, in order to obtain the solution for one chosen node i, only the moment values that corresponded to that chosen node i are used in moment matching as shown in Equation 2.28.

$$[m_n]_i = [M_n]_i \qquad \text{for n = 0,1,2,….(2q-1)} \qquad (2.28)$$

As shown in Equations 2.3, 2.12 and 2.21, the response of a system can be represented by a polynomial equation in s-domain, T(s). The process of moment matching is to reduce the order of the response, $T_i(s)$, at chosen node i. By using Pade approximation, $T_i(s)$ can be approximated by a lower order polynomial fraction, as shown in Equation 2.29.

$$T_i(s) = m_0 + m_1 s + m_2 s^2 + ... + m_n s^n = \frac{b_0 + b_1 s + ... + b_n s^{q-1}}{1 + a_1 s + ... + a_n s^q} \qquad (2.29)$$

By using Pade approximation, the coefficients of the denominator polynomial in Equation 2.29 can be obtained by Equation 2.30 (Pillage 1990).

$$\begin{bmatrix} m_0 & m_1 & m_2 & \cdot & m_{q-1} \\ m_1 & m_2 & m_3 & \cdot & m_q \\ m_2 & m_3 & m_4 & \cdot & m_{q+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{q-1} & \cdot & \cdot & \cdot & m_{2q-2} \end{bmatrix} \begin{Bmatrix} a_q \\ a_{q-1} \\ a_{q-2} \\ \cdot \\ a_1 \end{Bmatrix} = \begin{Bmatrix} -m_q \\ -m_{q+1} \\ -m_{q+2} \\ \cdot \\ -m_{2q-1} \end{Bmatrix} \qquad (2.30)$$

In order to inverse Laplace the response $T_i(s)$ in s domain back to response $T_i(t)$ in time domain, the reduced order polynomial fraction can be simplify to partial fractions as in Equation 2.31.

$$\frac{b_0 + b_1 s + ... + b_n s^{q-1}}{1 + a_1 s + ... + a_n s^q} = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + ... + \frac{k_q}{s - p_q} \tag{2.31}$$

From Equation 2.31, it can be seen that the poles (coefficients p) of the partial fractions are the roots of the reduced order denominator polynomial as shown in Equation 2.32.

$$1 + a_1 s + ... + a_n s^q = (s - p_1)(s - p_2)....(s - p_q) \tag{2.32}$$

Thus, by solving Equation 2.33, the poles, p , can be obtained.

$$1 + p_1 s + p_2 + ... + p_n s^q = 0 \tag{2.33}$$

The residues (coefficients k) of the partial fractions in Equation 2.31 can be obtained from Equation 2.34 (Pillage 1990).

$$\begin{bmatrix} p_1^{-1} & p_2^{-1} & p_3^{-1} & \cdot & p_q^{-1} \\ p_1^{-2} & p_2^{-2} & p_3^{-2} & \cdot & p_q^{-2} \\ p_1^{-3} & p_2^{-3} & p_3^{-3} & \cdot & p_q^{-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_1^{-q} & p_2^{-q} & p_3^{-q} & \cdot & p_q^{-q} \end{bmatrix} \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \\ \cdot \\ k_q \end{Bmatrix} = \begin{Bmatrix} -M_0 \\ -M_1 \\ -M_2 \\ \cdot \\ -M_{q-1} \end{Bmatrix} \tag{2.34}$$

2.5 Transient response

The transient response at node i is given by the sum of ZSR and ZIR in time domain as shown in Equation 2.35.

$$T_i(s) = ZSR(s) + ZIR(s)$$

$$T_i(t) = ZSR(t) + ZIR(t) \tag{2.35}$$

The formulation for ZSR and ZIR depends on the input function. Thus, when the input function is impulsive, step, ramp or sinusoidal, the formulation of ZSR and ZIR will vary accordingly as shown in the following sections.

2.5.1 Impulse input

In this case, the input is an impulse function, $F(s) = 1$.

$$ZSR(s) = \sum_{r=1}^{q} \left(\frac{k_r}{s - p_r} \right)$$

$$ZSR(t) = \sum_{r=1}^{q} k_r \left(e^{p_r t} - 1 \right) \tag{2.36}$$

$$ZIR(s) = \sum_{r=1}^{q} \left(\frac{k_r}{s - p_r} \right)$$

$$ZIR(t) = \sum_{r=1}^{q} k_r \left(e^{p_r t} - 1 \right) \tag{2.37}$$

Substituting Equation 2.36 and 2.37 into Equation 2.35, transient response at node i for impulse input is given by Equation 2.38.

$$T(t) = \sum_{r=1}^{q} k_r \left(e^{p_r t} - 1 \right) + \sum_{r=1}^{q} k_r \left(e^{p_r t} - 1 \right) \tag{2.38}$$

2.5.2 Step input

In this case, the input is a step function, $F(s) = \frac{1}{s}$.

$$ZSR(s) = \sum_{r=1}^{q} \left(\frac{k_r}{s - p_r} \times \frac{1}{s} \right)$$

$$ZSR(t) = \sum_{r=1}^{q} \frac{k_r}{p_r} \left(e^{p_r t} - 1 \right) \tag{2.39}$$

$$ZIR(s) = \sum_{r=1}^{q} \left(\frac{k_r}{s - p_r} \right)$$

$$ZIR(t) = \sum_{r=1}^{q} k_r \left(e^{p_r t} - 1 \right) \tag{2.40}$$

Substituting Equation 2.39 and 2.40 into Equation 2.35, transient response at node i for step input is given by Equation 2.41.

$$T(t) = \sum_{r=1}^{q} \frac{k_r}{p_r} \left(e^{p_r t} - 1 \right) + \sum_{r=1}^{q} k_r \left(e^{p_r t} - 1 \right) \tag{2.41}$$

2.5.3 Ramp input

In this case, the input is a ramp function, $F(s) = \frac{1}{s^2}$.

$$ZSR(s) = \sum_{r=1}^{q} \left(\frac{k_r}{s - p_r} \times \frac{1}{s^2} \right)$$

$$ZSR(t) = \sum_{r=1}^{q} \frac{k_r}{p_r^2} \left(e^{p_r t} - 1 - pt \right) \tag{2.42}$$

$$ZIR(s) = \sum_{r=1}^{q} \left(\frac{k_r}{s - p_r} \right)$$

$$ZIR(t) = \sum_{r=1}^{q} k_r \left(e^{p_r t} - 1 \right) \tag{2.43}$$

Substituting Equation 2.42 and 2.43 into Equation 2.35, transient response at node i for ramp input is given by Equation 2.44.

$$T(t) = \sum_{r=1}^{q} \frac{k_r}{p_r^2} \left(e^{p_r t} - 1 - pt \right) + \sum_{r=1}^{q} k_r \left(e^{p_r t} - 1 \right) \tag{2.44}$$

2.5.4 Sinusoidal input

In this case, the input is a sinusoidal function, $F(s) = \frac{w}{s^2 + w^2}$.

$$ZSR(s) = \sum_{r=1}^{q} \left(\frac{k_r}{s - p_r} \times \frac{w}{s^2 + w^2} \right)$$

$$ZSR(t) = \sum_{r=1}^{q} \frac{k_r \cdot w}{p_r^2 + w^2} \left(e^{p_r t} - \cos(wt) - \frac{p_r}{w} \sin(wt) \right) \tag{2.45}$$

$$ZIR(s) = \sum_{r=1}^{q} \left(\frac{k_r}{s - p_r} \right)$$

$$ZIR(t) = \sum_{r=1}^{q} k_r \left(e^{p_r t} - 1 \right) \tag{2.46}$$

Substituting Equation 2.45 and 2.46 into Equation 2.35, transient response at node i for sinusoidal input is given by Equation 2.47.

$$T(t) = \sum_{r=1}^{q} \frac{k_r \cdot w}{p_r^2 + w^2} \left(e^{p_r t} - \cos(wt) - \frac{p_r}{w} \sin(wt) \right) + \sum_{r=1}^{q} k_r \left(e^{p_r t} - 1 \right) \tag{2.47}$$

CHAPTER 3
RESULTS AND DISCUSSION

3.1 Overview

Asymptotic Waveform Evaluation (AWE) is used to solve various engineering problems to show its applicability, accuracy and also efficiency. This chapter will cover the application of AWE in the following fields:

a) Heat transfer
b) Vibration
c) Third order differential equation

Besides that, this chapter will also discuss the numerical instability in AWE and its solutions.

3.2 Solving thermal problems using finite element method and AWE

A simple one dimensional fin problem is solved using finite element method and also AWE. Step by step calculation is provided to show the basic algorithm of using AWE in solving a system of first order linear differential equations. A stability scheme is also introduced to stabilize the AWE solution by neglecting the unstable positive poles. Then, AWE is used to solve Non-Fourier heat conduction equation (hyperbolic equation) on a two dimensional and non-dimensionlized strip. Another stability scheme, called partial Pade approximation, is also introduced to stabilize the AWE solution. After that, the same Non-Fourier heat conduction equation is used to model the self-heating characteristic of a three dimensional VLSI interconnection. Finally, a simple one dimensional fin subjected to unsteady boundary condition is solved with AWE.

3.2.1 Simple one dimensional fin problem

In this case, a simple one dimensional fin is subjected to higher temperature at the base, as shown in Figure 3.1. This problem is taken from Logan (2000).

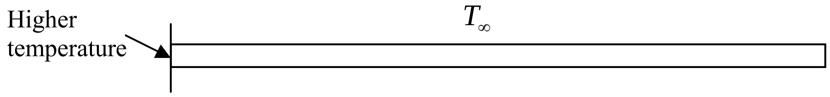

Figure 3.1: Simple one dimensional fin

Finite element method is used and thus, the fin is meshed with line elements. For easier understanding, only three line elements are used. After assembly, the global matrix equation is shown in Equation 3.1 and Equation 3.2.

$$C\dot{T} + KT = f \tag{3.1}$$

$$\begin{bmatrix} 0.1398 & 0.0699 & 0 \\ 0.0699 & 0.2796 & 0.0699 \\ 0 & 0.0699 & 0.1398 \end{bmatrix} \begin{Bmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \end{Bmatrix} + \begin{bmatrix} 0.5089 & -0.4995 & 0 \\ -0.4995 & 1.0179 & -0.4995 \\ 0 & -0.4995 & 0.5089 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ \dot{T}_3 \end{Bmatrix} = \begin{Bmatrix} 0.2356 \\ 0.4712 \\ 0.2356 \end{Bmatrix} \tag{3.2}$$

Incorporating the boundary conditions: $T_1 = 85$ and $\dot{T}_1 = 0$, Equation 3.2 becomes the following equations:

$$\begin{bmatrix} 0.2796 & 0.0699 \\ 0.0699 & 0.1398 \end{bmatrix} \begin{Bmatrix} \dot{T}_2 \\ \dot{T}_3 \end{Bmatrix} + \begin{bmatrix} -1.0179 & -0.4995 \\ -0.4995 & 0.5089 \end{bmatrix} \begin{Bmatrix} T_2 \\ \dot{T}_3 \end{Bmatrix} = \begin{Bmatrix} 0.4712 \\ 0.2356 \end{Bmatrix} - 85 \begin{Bmatrix} 0.0699 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 0.2796 & 0.0699 \\ 0.0699 & 0.1398 \end{bmatrix} \begin{Bmatrix} \dot{T}_2 \\ \dot{T}_3 \end{Bmatrix} + \begin{bmatrix} -1.0179 & -0.4995 \\ -0.4995 & 0.5089 \end{bmatrix} \begin{Bmatrix} T_2 \\ \dot{T}_3 \end{Bmatrix} = \begin{Bmatrix} 42.9287 \\ 0.2356 \end{Bmatrix}$$

Thus,

$$C = \begin{bmatrix} 0.2796 & 0.0699 \\ 0.0699 & 0.1398 \end{bmatrix}$$

$$K = \begin{bmatrix} -1.0179 & -0.4995 \\ -0.4995 & 0.5089 \end{bmatrix}$$

$$f = \begin{Bmatrix} 42.9287 \\ 0.2356 \end{Bmatrix}$$

For calculating Zero State Response (ZSR), Equation 2.6 is used to calculate the moment, M.

$$KM_0 = f$$

$$M_0 = K^{-1} f$$

$$M_0 = \begin{bmatrix} 1.0179 & -0.4995 \\ -0.4995 & 0.5089 \end{bmatrix}^{-1} \begin{Bmatrix} 42.9287 \\ 0.2356 \end{Bmatrix} = \begin{Bmatrix} 81.8003 \\ 80.7523 \end{Bmatrix}$$

$$KM_1 = -CM_0$$

$$M_1 = -K^{-1}CM_0$$

$$M_1 = -\begin{bmatrix} 1.0179 & -0.4995 \\ -0.4995 & 0.5089 \end{bmatrix} \begin{bmatrix} 0.2796 & 0.0699 \\ 0.0699 & 0.1398 \end{bmatrix} \begin{Bmatrix} 81.8003 \\ 80.7523 \end{Bmatrix} = \begin{Bmatrix} -85.6834 \\ -117.5199 \end{Bmatrix}$$

Same as the calculation for M_1, the values for other moments are as follows.

$$M_2 = \begin{Bmatrix} 102.6790 \\ 144.8354 \end{Bmatrix} \qquad M_4 = \begin{Bmatrix} 151.4116 \\ 214.1377 \end{Bmatrix}$$

$$M_3 = \begin{Bmatrix} -124.6179 \\ -176.2073 \end{Bmatrix} \qquad M_5 = \begin{Bmatrix} -183.9835 \\ -260.2081 \end{Bmatrix}$$

The node at the tip of fin, that is node 2, is chosen to be calculated. Thus, only the second value for each moment, M, is used. These values are substituted into Equation 2.30 to obtain Equation 3.3.

$$a = \begin{bmatrix} 80.7523 & -117.5199 & 144.8354 \\ -117.5199 & 144.8354 & -176.2073 \\ 144.8354 & -176.2073 & 214.1377 \end{bmatrix}^{-1} \begin{Bmatrix} 176.2073 \\ -214.1377 \\ 260.2081 \end{Bmatrix} \tag{3.3}$$

$$a = \begin{Bmatrix} 0.0084 \\ 0.2145 \\ 1.3860 \end{Bmatrix}$$

The roots of equation 3.4 are the poles.

$$0.0084p^3 + 0.2145p^2 + 1.3860p + 1 = 0 \tag{3.4}$$

$$p = \begin{Bmatrix} -15.1502 \\ -9.5396 \\ -0.8230 \end{Bmatrix} \tag{3.5}$$

Then, Equation 3.5 is substituted into Equation 2.34 to obtain Equation 3.6.

$$k = \begin{bmatrix} -0.0660 & -0.1048 & -1.2151 \\ 0.0044 & 0.0110 & 1.4765 \\ -0.0003 & -0.0012 & -1.7942 \end{bmatrix}^{-1} \begin{Bmatrix} -80.7523 \\ 117.5199 \\ -144.8354 \end{Bmatrix} \tag{3.6}$$

$$k = \begin{Bmatrix} 0.0000 \\ -166.6400 \\ 80.8311 \end{Bmatrix}$$

By substituting the poles, p and zeros, k into Equation 3.7, the Zero State Response (ZSR) can be obtained.

$$ZSR = \sum_{1}^{4} \frac{k_i}{p_i}\left(e^{p_i t} - 1\right) \tag{3.7}$$

For calculating Zero Input Response (ZIR), Equation 2.9 is used to calculate the moment, $\overline{M}$.

$$K\overline{M}_0 = CT(0)$$

$$\overline{M}_0 = K^{-1}CT(0)$$

$$\overline{M}_0 = \begin{bmatrix} 1.0179 & -0.4995 \\ -0.4995 & 0.5089 \end{bmatrix}^{-1} \begin{bmatrix} 0.2796 & 0.0699 \\ 0.0699 & 0.1398 \end{bmatrix} \begin{Bmatrix} 25 \\ 25 \end{Bmatrix} = \begin{Bmatrix} 26.3125 \\ 36.1281 \end{Bmatrix}$$

$$K\overline{M}_1 = -C\overline{M}_0$$

$$\overline{M}_1 = -K^{-1}C\overline{M}_0$$

$$\overline{M}_1 = -\begin{bmatrix} 1.0179 & -0.4995 \\ -0.4995 & 0.5089 \end{bmatrix}^{-1} \begin{bmatrix} 0.2796 & 0.0699 \\ 0.0699 & 0.1398 \end{bmatrix} \begin{Bmatrix} 26.3125 \\ 36.1281 \end{Bmatrix} = \begin{Bmatrix} -31.5470 \\ -44.5032 \end{Bmatrix}$$

Same as the calculation for M_1, the value for other moments are as follows.

$$\overline{M}_2 = \begin{Bmatrix} 38.2891 \\ 54.1404 \end{Bmatrix} \qquad \overline{M}_4 = \begin{Bmatrix} 56.5295 \\ 79.9497 \end{Bmatrix}$$

$$\overline{M}_3 = \begin{Bmatrix} -46.5216 \\ -65.7945 \end{Bmatrix} \qquad \overline{M}_5 = \begin{Bmatrix} -68.6908 \\ -97.1496 \end{Bmatrix}$$

The node at the tip of fin, that is node 2, is chosen to be calculated. Again, only the second value for each moment, $\overline{M}$, is used. These values are substituted into Equation 2.30 to obtain Equation 3.8.

$$\overline{a} = \begin{bmatrix} 36.1281 & -44.5032 & 54.1404 \\ -44.5032 & 54.1404 & -65.7945 \\ 54.1404 & -65.7945 & 79.9497 \end{bmatrix}^{-1} \begin{Bmatrix} 65.7945 \\ -79.9497 \\ 97.1496 \end{Bmatrix} \tag{3.8}$$

$$\bar{a} = \begin{Bmatrix} -0.0947 \\ -0.8543 \\ 0.5763 \end{Bmatrix}$$

The roots of the Equation 3.9 are the poles.

$$-0.0947\bar{p}^3 - 0.8543\bar{p}^2 + 0.5763\bar{p} + 1 = 0 \tag{3.9}$$

$$\bar{p} = \begin{Bmatrix} -9.5396 \\ 1.3447 \\ -0.8230 \end{Bmatrix} \tag{3.10}$$

Then, Equation 3.10 is substituted into Equation 2.34 to obtain Equation 3.11.

$$\bar{k} = \begin{bmatrix} -0.1048 & 0.7437 & -1.2151 \\ 0.0110 & 0.5531 & 1.4765 \\ -0.0012 & 0.4113 & -1.7942 \end{bmatrix}^{-1} \begin{Bmatrix} -36.1281 \\ 44.5032 \\ -54.1404 \end{Bmatrix} \tag{3.11}$$

$$\bar{k} = \begin{Bmatrix} -5.1785 \\ 0.0000 \\ 30.1785 \end{Bmatrix}$$

By substituting the poles, $\bar{p}$ and zeros, $\bar{k}$ into Equation 3.12, the Zero Input Response (ZIR) can be calculated as below.

$$ZIR = \sum_{1}^{4} k_i \left(e^{p_i t}\right) \tag{3.12}$$

Finally, the temperature response, T is given by Equation 3.13.

$$T(t) = ZSR + ZIR \tag{3.13}$$

From the calculation above, the temperature response, T, for the node at the tip of the fin can be plotted against time, as shown in Figure 3.2. The temperature response for other nodes can be obtained similarly according to the calculation above except that the values of the moments used are according to the node chosen.

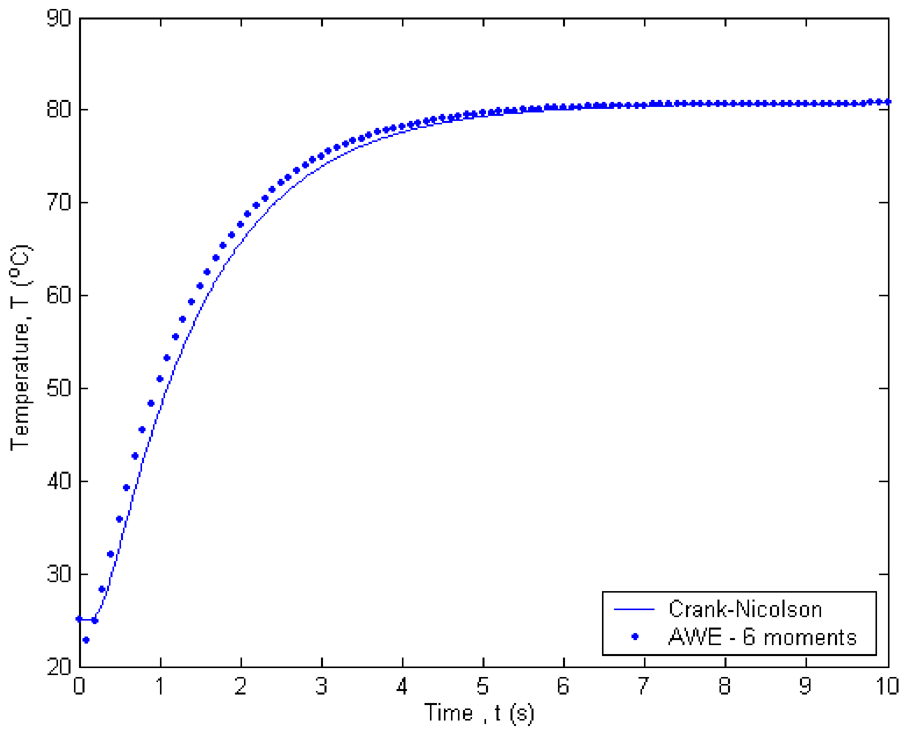

Figure 3.2: Temperature response for node at the tip of the fin

3.2.2 Stabilizing the unstable solution produced by AWE

From Figure 3.2, the solution from AWE agrees well with the solution produced using Crank-Nicolson. However, when meshing elements used are increased and thus there are more nodes, it happens that there are a few nodes where AWE will produce unstable solution due to unstable poles. This is because Pade approximation is known for producing unstable result for stable model even though higher order of approximation is attempted (Anastakis 1992).

Generally, AWE suffers from two main problems (Anastakis 1992). The first one is that it is difficult to select the order of approximation since the total number of dominant poles is unknown. Secondly, it is not always possible to obtain a particular order of approximation