B-SPLINES BASED FINITE DIFFERENCE SCHEMES FOR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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B-SPLINES BASED FINITE DIFFERENCE SCHEMES FOR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

by

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DEDICATION

To, the reason for my being!

محمد صلى الله عليه وسلم

and

Ammi & Abbu

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In the name of Almighty Allah, the most merciful, the most gracious.

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LIST OF ABBREVIATIONS

ODEs	Ordinary differential equations
FPDEs	Fractional partial differential equations
FPDE	Fractional partial differential equation
FDE	Fractional differential equation
PDEs	Partial differential equations
PDE	Partial differential equation
FEM	Finite difference method
DEs	Differential equations
TFDE	Time fractional differential equation
CBS	Cubic B-spline
NCBS	New cubic B-spline
ExCuBS	Extended cubic B-spline
NExCuBS	New extended cubic B-spline
TFADE	Time fractional advection diffusion equation
FADE	Fractional advection diffusion equation
TFTE	Time fractional telegraph equation
FTE	Fractional telegraph equation
TFBE	Time fractional Burgers equation
FBE	Fractional Burger’s equation

TFKGE	Time fractional Klein Gordon equation
FKGE	Fractional Klein Gordon equation
KG	Klein Gordon
VRM	Variational iteration method
CAGD	Computer aided design
MCTB-DQM	Modified cubic trigonometric B-spline differential quadrature method
DQM	Differential quadrature method
1D	One dimensional
FEM	Finite element method
FVM	Finite volume method
RL	Riemann Liouville
CTB	Cubic trigonometric B-spline

LIST OF SYMBOLS

x	Space variable
t	Time variable
a	Initial point in space dimension
b	End point in space dimension
T	Total time
λ	Free parameter
θ	Weight function
h	Grid size in x-dimension
τ	Grid-size in t-dimension
O	Order of convergence
γ	Fractional order derivative
$\mathbb{N}$	Natural numbers
$\mathbb{C}$	Complex numbers
$\mathbb{Z}$	Integers
$\mathbb{R}$	Real numbers
Γ	Gamma function
${}_{0}^{C}D_{t}^{\gamma}$	Time fractional derivative of order γ in Caputo sense
N	Number of partition in space-dimension
M	Number of partition in time-dimension

ϕ, ϕ_1, ϕ_2	Initial conditions
g_1, g_2	Boundary conditions
L_∞	Maximum norm
L_2	Euclidean norm
$f(x,t)$	Forcing term
$u(x,t)$	Exact solution
$U(x,t)$	Approximated solution
δ_j^n, c_j^n, d_j^n	Unknown coefficients for $j = -1, 0 \ldots N+1$ and $n = 0, 1 \ldots M$

SKIM BEZA TERHINGGA BERASASKAN SPLIN-B UNTUK PERSAMAAN PEMBEZAAN SEPARA PECAHAN

ABSTRAK

Persamaan pembezaan separa pecahan (FPDEs) dianggap formulasi lanjutan persamaan pembezaan separa klasik (PDE). Beberapa model fizikal lebih sesuai dibangunkan dalam bentuk FPDE bagi bidang tertentu sains dan kejuruteraan. FPDE, secara umum, tidak mempunyai penyelesaian analitik yang tepat. Oleh itu, terdapat keperluan untuk membangunkan kaedah berangka baru untuk penyelesaian FPDE ruang dan masa. Penyelidikan ini memberi tumpuan kepada pembangunan kaedah berangka baru. Dua kaedah berdasarkan splin-B dibangunkan untuk menyelesaikan FPDE linear dan bukan linear. Kaedah-kaedah ini adalah kaedah penghampiran splin-B diperluaskan (ExCuBS) dan penghampiran splin-B baru yang diperluaskan (NExCuBS). Kedua-dua kaedah mempunyai fungsi asas yang sama tetapi untuk NExCuBS, pengiraan baru digunakan untuk terbitan ruang peringkat kedua. Anggaran baru ini dikira oleh gabungan linear simpul berhampiran. Terbitan pecahan masa diungkapkan dalam bentuk Caputo. Terbitan pecahan Caputo didiskretain oleh kaedah penbezaan terhingga (FDM) yang biasa dan dimensi ruang didiskretain oleh ExCuBS dan NExCuBS dengan kaedah tertimbang θ untuk kes linear dan bukan linear masing-masing. Menggunakan syarat awal dan sempadan, satu sistem segi empat tepat persamaan linear ditentukan, yang boleh diselesaikan oleh Mathematica. Kaedah ExCuBS digunakan untuk menyelesaikan persamaan pecahan masa resapan olakan (TFADE) dan persamaan pecahan masa telegraf (TFTE). NExCuBS digunakan untuk menyelesaikannya persamaan pecahan Burger tak linear (TFBE) dan persamaan pecahan masa Klein-Gordon (TFKGE). Kedua-dua kaedah ini didapati stabil tanpa syarat untuk θ dan menumpu

dalam arah x dan t. Eksperimen berangka yang dijalankan menjanakan keputusan pengiraan yang menyokong penemuan teoritis.

B-SPLINES BASED FINITE DIFFERENCE SCHEMES FOR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

ABSTRACT

Fractional partial differential equations (FPDEs) are considered to be the extended formulation of classical partial differential equations (PDEs). Several physical models in certain fields of sciences and engineering are more appropriately formulated in the form of FPDEs. FPDEs in general, do not have exact analytical solutions. Thus, the need to develop new numerical methods for the solutions of space and time FPDEs. This research focuses on the development of new numerical methods. Two methods based on B-splines are developed to solve linear and non-linear FPDEs. The methods are extended cubic B-spline approximation (ExCuBS) and new extended cubic B-spline approximation (NExCuBS). Both methods have the same basis functions but for the NExCuBS, a new approximation is used for the second order space derivative. This new approximation is calculated by a linear combination of neighbouring knots. The time fractional derivative is described in Caputo sense. The Caputo fractional derivative is discretized by the usual finite difference method (FDM) and the space dimension is discretized by ExCuBS and NExCuBS with θ weighted method for linear and non-linear cases respectively. Using the initial and boundary conditions, a square system of linear equations, which can be solved in Mathematica, is determined. The ExCuBS method is utilized for solving linear time fractional advection diffusion equation (TFADE) and time fractional telegraph equation (TFTE). The NExCuBS is utilized for solving non-linear time fractional Burgers equation (TFBE) and time fractional Klein-Gordon equation (TFKGE). Both methods are found to be unconditionally stable for θ and convergent in x and t directions. Numerical experiments

conducted indicated that the computational results support the theoretical findings.

CHAPTER 1

INTRODUCTION

1.1 Research Background

Partial differential equations (PDEs) are mathematical equations that associate some functions which contain two or more independent variables with their partial derivatives (Vvedensky, 1993). Second order PDEs have been widely and successfully used to model many problems in Science and Engineering (Strauss, 1992).

In general, fractional partial differential equations (FPDEs) are considered to be the extended formulation of classical PDEs. Several physical models, such as electron transportation (Scher and Montroll, 1975), power-law memory function (Rabotnov, 1980), visco-elastic material (Mainardi, 2010), heat conduction (Sokolov et al., 2002), high-frequency financial data (Mendes, 2009) are more appropriately developed in the form of FPDEs for certain fields of applied m athematics. FPDEs involves fractional derivatives of arbitrary order. A comprehensive history of fractional calculus were discussed in Oldham and Spanier (1974), Miller and Ross (1993) and Podlubny (1999). Many scientists (Scher and Montroll, 1975; Rabotnov, 1980; Kilbas et al., 2006) pointed out that non-integer order derivatives and integrals are more appropriate for the interpretation of certain real-life phenomena. That is FPDEs models are more suitable and accurate for the description of certain systems. For example, these models have been used successfully in diffusion processes, rheology, visco-elasticity, astro-physics, fractal networks, signal processing, turbulent flow and fluid mechanics

(Diethelm and Freed, 1999; Hilfer, 2000; Kilbas et al., 2006). The fractional derivative models are also used for models based on continuous time random walks, where the movement of a particle is dependent on past movements (Das, 2011). They also appear in the modelling of many processes in mathematical biology, chemical processes and a number of problems in engineering (Shlesinger et al., 1987; Zaslavsky et al., 1993; Diethelm and Freed, 1999; Barkai et al., 2000). In the last few decades FPDEs have generated significant interest due to their appearance in various fields. In numerical analysis various analytical and numerical methods including their applications to new problems have been proposed.

FPDEs in general, do not have exact analytical solutions (Chen et al., 2010). This is the main reason for developing new numerical methods for the solutions of FPDEs and it has become a topic of major interest. Extensive research has been carried out to obtain numerical techniques which are numerically stable for both linear and nonlinear FPDEs (Jafari et al., 2013; Siddiqi and Arshed, 2015; Esen and Tasbozan, 2015c; Sharifi and Rashidinia, 2016; Hepson, 2018; Gholamian and Nadjafi, 2018). Improvements and extensions of established methods and developing new techniques are active areas of research. Numerical techniques based on Chebyshev tau (Saadatmandi and Dehghan, 2010), finite difference (Zhang, 2009), finite element (Zheng et al., 2010), finite volume (Liu et al., 2014), quadratic B-spline method (Esen and Tasbozan, 2015a) have been employed to solve FPDEs. There are various studies on the use of splines for FPDEs such as Zahra and Elkholy (2012), Tasbozan et al. (2013), Li et al. (2014), Akram and Tariq (2016). The main advantage of this technique is that the obtained solution will be in an approximate analytical form. At any discrete point, the numerical solution can then be established from the approximate analytical solutions. This study deals with the numerical solution of second order linear and nonlinear FPDEs by the use of B-splines.

Advection-diffusion equation can be used to describe movement and spread of a substance or conversed quantity such as energy, heat, mass etc by a fluid due to fluid's bulk motion (Dehghan, 2004). The advection-diffusion equation is a parabolic equation. Telegraph equation is a hyperbolic equation that can describe the propagation of electrical signals in an electrical transmission line and wave phenomena. Telegraph model demonstrates that the electromagnetic waves pattern can appear along the line and that waves can be reflected on the wire (Mohanty and Jain, 2001; Dehghan and Shokri, 2007). Both the advection-diffusion and telegraph equation are linear PDEs. Burgers equation has been developed as a model of turbulent fluid motion, heat conduction, gas dynamics etc (Burgers, 1948; Kutluay et al., 1999). The 1D Burgers equation is a nonlinear parabolic PDE. Klein-Gordon equation is a hyperbolic equation that can describe as a motion of rigid pendula attached a stretched wire, dislocations in crystals, nonlinear optics etc (Mittal and Bhatia, 2014; Sarboland and Amlnatael, 2015). Klein-Gordon is second order in time and space dimensions. The advection diffusion equation, telegraph equation, Burgers equation and Klein-Gordon equation are parabolic and hyperbolic PDEs which describe phenomena that are related in one way or another to transport. Hence there is a connection between them.

This thesis will focus on the fractional version of the equations discussed above.

1.2 Motivation of the Study

Motivated by the success of B-splines in the numerical solution of integer order differential equations (Goh, 2013; Siddiqi and Arshed, 2013; Abbas et al., 2014a), our aim is to investigate the use of appropriate B-splines for the numerical solution of second order linear and nonlinear FPDEs. In general, degree three B-spline can be used to solve second order differential equations. B-spline functions are powerful tools to obtain the computational outcomes due to its flexibility to approximate the solution with high accuracy at any point in the domain and also preserve the high degree smoothness at the knots. B-splines techniques give us better results as compared to the other numerical techniques in PDEs. Many authors (Caglar et al., 2006; Abbas et al., 2014b) have conducted studies that support this statement. There are many studies based on B-splines for the solutions of FPDEs (Tasbozan et al., 2013; Esen et al., 2015b; Esen and Tasbozan, 2015a; Yaseen et al., 2017a; Hepson, 2018). However, so far as we are aware, there are no studies on the use of splines for fractional advection diffusion equation (FADE), fractional telegraph equation (FTE) and nonlinear fractional Klein Gordon equation (FKGE) and limited studies which deal with nonlinear fractional Burgers equation (FBE).

However, one of the limitation according to Gang and Guo-Zhao (2008) is that the B-spline does not preserve any free parameter for the curve modification. Therefore, the shape of the curve can not be altered once the data points are determined. On the other hand, spline approximation is a global approximation, any change of the data point will require to solve all the system again. Hence, Gang and Guo-Zhao (2008) introduced an extension of cubic B-spline which developed by Han and Liu (2003) and extended it to higher degree.

In this research, extended cubic B-spline (ExCuBS) is considered for the solution of second order linear and nonlinear FPDEs due to its success in dealing with PDEs. ExCuBS is the extension of cubic B-spline (CBS) which preserves a free parameter to control the global shape of curve. A new cubic B-spline (NCBS) method has been developed by Lang and Xu (2014). This method gives good results when used to approximate the solution of nonlinear ordinary differential equations. Due to the promising results obtained by NCBS in the Lang and Xu (2014), one focus of our study is to extend this method for ExCuBS to approximate the solution of nonlinear FPDEs. The resulting spline is called new extended cubic B-spline (NExCuBS). This new method is an improvement over the NCBS method. The analytical solutions and any order derivatives can be well approximated with 4th order accuracy. It is of interest to calculate the accuracy of the methods. The stability and convergence of the method also need to be investigated. Stability means round off errors are bounded and convergence means smaller grid size ensure more accurate solution.

1.3 Objectives of the Study

The objectives of the study are

- To develop a numerical technique for solving time fractional advection diffusion equation (TFADE) by ExCuBS and to investigate the stability, convergence and accuracy of the technique.
- To develop a numerical technique for solving time fractional telegraph equation

(TFTE) by ExCuBS and to investigate the stability, convergence and accuracy of the technique.

- To formulate a NExCuBS approximation for second order FPDEs.
- To develop a numerical technique for solving nonlinear time fractional Burgers equation (TFBE) by NExCuBS and to investigate the stability, convergence and accuracy of the technique.
- To develop a numerical technique for solving nonlinear time fractional Klein-Gordon equation (TFKGE) by NExCuBS and to investigate the stability, convergence and accuracy of the technique.
- To calculate the order of convergence of ExCuBS and NExCuBS techniques.

1.4 Methodology

The methodology that will be used in this study is as follows:

1. **a)** ExCuBS which is a extension of CBS with a free parameter λ, with $-8 \leq \lambda \leq 1$ will be used in basis functions.

   b) Time fractional derivatives are evaluated by the Caputo approach. Finite difference method (FDM) will be used in the discretization of Caputo operator.

   c) A combination of Caputo fractional derivative and ExCuBS together with θ-weighted scheme is utilized.

   d) We assume that the ExCuBS is the solution of TFADE and TFTE with interpolating conditions.

e) A system of linear equations is obtained with the incorporation of initial and boundary conditions. Therefore, values of unknowns can be calculated by solving the system.

f) The stability analysis is investigated using Von Neumann method and convergence analysis is also conducted.

2. The TFADE and TFTE are linear but TFBE and TFKGE are nonlinear. For solving these equations first the nonlinear term will be linearized on the usual Taylor series approach. Then the above methodology is repeated for TFBE and TFKGE. Previous studies have indicated that there is only a marginal loss in accuracy when such a linearization approach is adopted.

3. Mathematica 11 and Matlab R2017a are used to achieve the stated objectives.

1.5 Organization of the Thesis

This thesis is organised into eight chapters. A description of each chapter is as follows:

- Chapter 2 provides a review of numerical methods to solve PDEs and FPDEs, especially those based on B-splines. A comprehensive history of B-spline for the solutions of fractional advection, fractional telegraph, nonlinear fractional Burgers and nonlinear fractional Klein-Gordon equations is discussed.

- Chapter 3 covers the basic concept of fractional calculus and special functions which are used in the solutions of FPDEs. This chapter also discusses the recursive formula of B-spline, some relevant properties and ExCuBS.

- Chapters 4 and 5 discuss a combined method based on Caputo's fractional derivatives and ExCuBS approach with the incorporation of θ weighted scheme for the solutions of TFADE and TFTE. The stability analysis and convergence analysis of ExCuBS will also be carried out in Chapters 4 and 5.

- Numerical technique based on NExCuBS is formulated in Chapter 6. Chapters 6 and 7 present a combination of Caputo's fractional derivatives and NExCuBS with the θ weighted scheme for the solutions of TFBE and TFKGE. The stability analysis and convergence analysis of NExCuBS approximation will also be discussed in Chapters 6 and 7.

- Finally, Chapter 8 presents the conclusions and discussions on the possibilities of future research.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In the past few years, the use of splines have generated significant interest in the field of numerical analysis. Schoenberg (1946) introduced the concept of splines and Boor (1972), Ahlberg and Ito (1975), de Boor (1978) were inspired by his work. B-spline or basis spline is formed from a linear combination of its recursive function, called B-spline basis functions. (Boor, 1972) established the recursive formula of B-splines. B-spline has been noticeably utilized to solve the solutions of ordinary differential equations (ODEs) and PDEs due to their accuracy of solutions. Recently, many researchers have been attracted towards the applications of B-spline to solve FPDEs (Tasbozan et al., 2013; Esen et al., 2015b; Sayevand et al., 2016; Yaseen and Abbas, 2019).

This literature chapter is classified into five sections. In the first section, the literature review on the applications of B-splines in solving differential equations is presented. The remaining four sections are based on the FPDEs type. First, the literature regarding fractional advection-diffusion equation (FADE) will be presented. Secondly, a brief history of methods regarding fractional telegraph equation (FTE) will be provided followed by the nonlinear fractional Burgers equation (FBE) and fractional Klein-Gordon equation (FKGE). It is hoped to present adequate information related to methods in solving the FPDEs and establish the novelty of this study.

2.2 B-spline Method

(Ahlberg and Ito, 1975) started the applications of B-splines to solve ODEs. Due to the simplicity of the B-spline method, several researchers started the use of B-splines to solve the linear and nonlinear PDEs. Kadalbajoo and Arora (2009) developed B-spline collocation method for singular perturbation problem using artificial viscosity. The method have been shown second order convergence. Zhu and Wang (2009), Zhu and Kang (2010) presented the numerical solution of Burgers and Burgers-Fisher equations using Cubic B-spline (CBS) quasi-interpolation method. The analysis of the stability of the proposed methods were also carried out. Goh et al. (2011) studied CBS method for the numerical solutions of heat and wave equations. A finite difference method (FDM) and CBS approach were used in time dimension and space dimension respectively. CBS collocation method has been used to solve nonlinear parabolic PDE with Neumann boundary conditions by Mittal and Jain (2012). CBS basis was applied to discretize space dimension. Goh et al. (2012) proposed CBS collocation method in solving 1D heat and advection-diffusion equations. In this paper, stability analysis was examined by the Von Neumann method and the efficiency of the proposed method was checked by some numerical examples. Quintic B-spline collocation method was used to solve fourth-order parabolic PDE by Siddiqi and Arshed (2013). The authors used FDM to discretize time dimension and quintic B-spline in space dimension. Abbas et al. (2014b) solved a strongly coupled reaction-diffusion system by using CBS method. They used FDM and CBS approaches for time and space dimensions respectively. A 1D hyperbolic telegraph equation with Neumann boundary conditions has been solved by Mittal and Bhatia (2014) using CBS basis functions. The stability was examined by matrix stability method. Jiwari (2015) solved hyperbolic PDEs with Dirichlet and Neumann boundary conditions using Lagrange interpolation and modified CBS differential quadrature method (DQM). Korkmaz and Dag (2016) used quartic and quintic B-spline basis functions in solving advection-diffusion equation. The stability was also examined by using matrix stability method. Sharifi and Rashidinia (2016) presented the numerical solution of hyperbolic telegraph equation using a collocation method based on redefine ExCuBS basis. The stability and convergence of the proposed method were also discussed. Alshomrani et al. (2017) proposed a numerical algorithm based on a modified CBS basis in solving hyperbolic type wave equations with Dirichlet boundary conditions.

Jafari et al. (2013) considered the B-spline collocation method for solving linear FDE which involves Caputo fractional derivative. Numerical examples were also discussed to check the efficiency of the method. Tasbozan et al. (2013) proposed a numerical solution of fractional diffusion equation using the CBS collocation method. They developed approximation based on FEM by CBS functions with Riemann Liouville (RL) operator. Numerical examples were provided to show the accuracy of the method. Esen et al. (2015b) solved fractional diffusion and fractional diffusion wave equations using CBS collocation method. Time fractional derivatives were discretized by the Caputo formulation. The authors also examined the stability of the method. A numerical technique based on quintic B-spline was employed by Siddiqi and Arshed (2015) in solving time fractional fourth-order PDEs. The time dimension was discretized in Caputo's sense and space dimension was discretized by the quintic B-spline approach. Stability and convergence properties were also discussed. A CBS collocation method was developed by Esen and Tasbozan (2015c) and used to solve time-fractional gas dynamics equation while the Caputo's derivative was carried out for time dimension. Error norms were also calculated in this paper. Sayevand et al. (2016) solved second and fourth order time-fractional diffusion equations by using CBS approach. The time and space dimensions of diffusion equations were discretized by Caputo's derivatives and CBS method respectively. They showed that the proposed technique was unconditionally stable and convergent. Yaseen et al. (2017a) suggested a numerical technique based on cubic trigonometric B-spline (CTB) basis functions in solving time-fractional sub-diffusion equation. Time dimension was discretized by usual FDM and space derivative was discretized by CTB basis functions with the help of GrÃijnwald discretization of RL derivative. The stability was also shown to be unconditionally stable by Fourier series method. Zhu et al. (2017a) proposed a method based on exponential B-spline approach in solving fractional sub-diffusion equation. The unconditional stability was also proved. A series of numerical examples were also carried out to check the accuracy of the method. A FDM based on cubic trigonometric B-splines in solving time fractional wave equation has been presented by Yaseen et al. (2017b). They used FDM to discretize Caputo derivative in time dimension and space dimension was discretize using CTB basis. The stability, convergence and numerical examples were also discussed. Quintic B-spline approach was employed for the class of fourth-order FPDEs by Arshed (2017a). Central FDM and quintic B-spline basis were used to discretize time and space dimensions respectively. The developed technique was shown to be stable and convergent in the time dimension. A numerical method has been developed for solving time fractional hyperbolic PDEs by Arshed (2017b). The author used CBS collocation approach to discretize space dimension and Caputo's time-fractional derivative was discretized with the help of central FDM. The stability and convergence were also discussed. A numerical technique based on quadratic B-spline FEM has been suggested by Esen and Tasbozan (2017) to solve time fractional Schrödinger equation. In this research paper, the authors also conducted numerical experiments to check the accuracy of the B-spline FEM. Hepson (2018) solved generalized Burgers- Fisher equation using FEM based on ExCuBS. FEM was established by ExCuBS basis functions and discretization was based on Crank-Nicolson and FDM. The author presented two numerical examples to check the validity of the numerical method. Pitolli (2018) developed a numerical method based on fractional B-spline technique to solve predator-prey model and fractional Lotka-Volterra model. Numerical tests showed the accuracy of the proposed method. Gholamian and Nadjafi (2018) presented the CBS collocation method for a class of partial integro-differential equation. CBS was utilized for space discretization while the backward Euler formula was utilized for time discretization. Convergence and the stability of the method were also proved.

2.3 Numerical Methods for Solving Fractional Advection Diffusion Equation (FADE)

The FEM was implemented to solve space fractional advection-diffusion equation with nonhomogeneous initial and boundary conditions by Zheng et al. (2010). A priori error estimate was also derived. Wang and Wang (2011) developed a FDM for FADE. The authors claimed that the proposed method presented more accurate results than the standard implicit methods even if larger space and times step sizes used. Shen et al. (2011) considered an implicit and explicit FDM to solve space-time Riesz-Caputo FADE with initial and boundary conditions in a finite domain. Both methods were convergent but the explicit approximation was conditionally stable while an implicit was unconditionally stable. Liu et al. (2014) proposed a technique based on the finite volume method (FVM) for solving FADE. The spatial dimension was estimated by shifted GrÃijnwald formula to discretize RL fractional derivatives. Numerical results for Crank-Nicolson FVM were given to show that the consistency, stability and convergence of the technique. Bu et al. (2014) solved a class of multi-term TFADE using FDM in time direction and FEM in space direction. The stability and convergence of the method were also investigated. Parvizi et al. (2015) employed the Jacobi collocation method in solving a special kind of FADE with a nonlinear source term. The spatial derivatives were replaced by the RL derivatives, the stability and convergence of the method were also presented. Jannelli et al. (2018) analyzed TFADE involving RL derivative with nonlinear forcing term. A numerical solution was obtained by FDM for non-linear equation. The results were compared with the exact solutions and errors were shown the convergence of this method. FDM for time-space FADE with Riesz derivative has been formulated by Arshad et al. (2018). The Caputo derivative and Riesz derivative were considered in time and space directions respectively. Riesz space derivative was calculated using second-order fractional weighted and shifted Grünwald-Letnikov approximation. Further, the stability and convergence analysis were also discussed. Zhang et al. (2019) considered an implicit and explicit difference techniques for solving time-space FADE. It was shown that the implicit technique was convergent and unconditionally stable while the explicit technique was convergent and conditionally stable. The order of convergence of the both techniques was $O(\tau+h)$.

Zhu et al. (2017b) proposed a numerical technique based on an efficient differential quadrature method (DQM) for FADE. CTB was used for DQM in solving 1D TFADE. This method was evaluated by some numerical experiments. Finally, the results were compared with other numerical methods discussed in the literature. Badr et al. (2018)

suggested a numerical method for solving 1D TFADE using B-spline finite volume element method. The time-fractional derivative was calculated with the help of Caputo discretization. The analysis of the stability of this method and some numerical examples were also presented.

2.4 Numerical Methods for Solving Fractional Telegraph Equation (FTE)

Li and Cao (2012) presented a technique based on FDM for a kind of linear FTE. They derived FTE from the classical telegraph equation by replacing the second-order time derivative with the fractional derivative of order $1 < \gamma < 2$. The analysis of stability and convergence have been proven by the energy method. Wang et al. (2014) discussed and analyzed Galerkin mixed FEM for the numerical solution of TFTE. The time dimension was discretized by the Caputo fractional derivative and space dimension is discretized by FDM. They presented optimal order of convergence in space-time dimensions and also derived the stability of Galerkin mixed finite element technique. Saadatmandi and Mohabbati (2015) developed a computational technique for solving TFTE using Tau method and Legendre polynomials. The Caputo operator has been used to describe the fractional-order derivatives. Few examples were tested to check the validity and applicability of this method. Asgari et al. (2016) obtained the solution of TFTE via Bernstein polynomials' operational matrices. The operational matrices of fractional differential and collocation methods are used to reduce TFTE to a linear system of equations. Convergence of the proposed technique has also been proven. They used Mathematica 9 to obtain the computational results. Hashemi and Baleanu (2016) proposed a numerical method for the solution of TFTE using the Caputo fractional derivative in temporal direction, a combination of group preserving scheme and method of line for spatial direction. Some numerical experiments have been conducted to check the accuracy of this method. The reproducing kernels has been presented for the solution of TFTE with initial boundary conditions using piecewise technique by Wang et al. (2017). They claimed that the method is more accurate than the traditional reproducing kernel technique. Three numerical examples have been presented to check the effectiveness of this technique. Wang and Mei (2017) proposed a method for solving TFTE via Legendre spectral Galerkin method and generalized FDM. The generalized FDM was used to discretize the time dimension and Legendre spectral Galerkin method was used in space dimension. The stability and convergence analysis have also been discussed. Liu (2018) discussed the Caputo fractional difference method and Grünwald formula for the solution of TFTE with Dirichlet boundary conditions. The stability and convergence of difference methods have also been discussed. Kamran and Ali (2018) constructed the solution of TFTE using localized kernel-based method. The Laplace transform and local radial basis functions method have been used to solve TFTE. The resulting integral form representation was solved by quadrature rule. Kumar et al. (2019) suggested FDM for a TFTE with generalized derivative terms. The scale and weighted functions have been used for generalized fractional derivatives. These derivatives have been reduced in terms of the Caputo and RL derivatives. The stability, convergence and numerical examples have also been presented.

The quadratic B-spline Galerkin method has been employed in solving TFTE by Tasbozan and Esen (2017). The fractional time derivatives were discretized in Caputo's sense. Three numerical examples were presented to check the accuracy of this method. Yaseen and Abbas (2018) proposed CTB method for solving TFTE. FDM was used to discretized the Caputo sense time-fractional derivatives while the CTB was used to in space dimension. The stability analysis and computational examples were also presented.

2.5 Numerical Methods for Solving Fractional Burgers Equation (FBE)

Song and Zhang (2007) derived the homotopy analysis method, an explicit and numerical solutions of FBE for the first time. They used the Caputo operator for fractional derivative. Numerical solution of space and time FBE have been proposed by Inc (2008) using the VRM. The exact and numerical solutions have been compared with results obtained by the Adomian decomposition method. The author claimed that VRM method is more efficient and stable than the Adomian decomposition method. Liu and Hou (2011) presented numerical solutions of time and space FBE using generalized differential transform method. This numerical method was based on generalized Taylor series formula and the Caputo formula. Numerical experiments have been demonstrated that the effectiveness of this method. Khan and Ara (2012) proposed a generalized differential transform method and homotopy perturbation method for solving TFBE. Fractional derivatives have been described in the Caputo sense. El-Danaf and Hadhoud (2012) developed a numerical method for the solution of TFBE using cubic parametric splines. Time fractional derivative has been discretized by the Caputo formula and the Von Neumann method was applied to check the stability of this method. The truncation error of the proposed method has also been analyzed. Gupta and Ray (2014) used numerical technique based on two dimensional Legendre wavelet method in solving fractional Burgers type equation. They described time Fractional derivative in the Caputo sense. Yokus and Kaya (2017) solved TFBE for numerical solutions using Cole-Hopf transformation and expansion method. FDM has been used to discretize the Caputo sense derivative. TFBE has been linearized by using Cole-Hopf method and the stability has been investigated by the Von Neumann Fourier method. Asgari and Hosseini (2018) established two semi-implicit pseudospectral methods in solving generalized TFBE. The Caputo operator has been utilized for the fractional-order derivative. They presented the stability, consistency and convergence of the methods. The results of the proposed methods have also been compared with the exact analytical solution. Oruç et al. (2019) described a unified finite difference Chebyshev wavelet method for solving TFBE numerically. Chebyshev wavelet method and Caputo formula have used for the discretization of space and time directions respectively. Senol et al. (2019) introduced the residual power series method in solving TFBE using conformable fractional derivative.

Esen and Tasbozan (2015a) developed a numerical method for the solution of TFBE using quadratic B-spline Galerkin method. FDM has been used to discretized the Caputo operator. Three numerical examples have also been presented to check the efficiency of this method. Esen and Tasbozan (2016) solved TFBE numerically using CBS finite elements. FEM based CBS basis functions have been used for space direction and the Caputo derivative has been utilized for time direction. This study demonstrated that the method preserves the ability to obtain good numerical results. Yaseen and Abbas (2019) presented a numerical method based on CTB for the solution of TFBE. The Caputo fractional derivative has been used in time direction while the CTB has been utilized in space direction. They also presented the stability and convergence of this method.

2.6 Numerical Methods for Solving Fractional Klein-Gordon Equation (FKGE)

Golmankhaneh et al. (2011) has been applied homotopy perturbation method for the analytical solution of FKGE. The authors claimed that this algorithm provides good numerical solutions without discretization. Three examples have been given to show that the applicability of this method. A high order compact difference scheme for nonlinear FKGE with Neumann boundary conditions has been proposed by Vong and Wang (2014). The stability and convergence of FDM have been analyzed by matrix form. Alqahtani (2015) applied the spectral collocation method using Legendre polynomial for the solution of non-linear TFKGE. Fractional derivative has been described in the Caputo sense. The results of this method have been shown a good agreement with the exact solution. Merdan and Oral (2016) provided an application of the local fractional decomposition method for the approximate solution of nonlinear FKGE. This study demonstrated that the computational results of FKGE with RL derivative obtained by the local fractional decomposition method are very efficient and useful. Abuteen et al. (2016) presented a numerical algorithm based on fractional reduced differential transform technique for finding the solution of non-linear FKGE. These findings have been provided a comparative study between implicit Runge-Kutta technique and fractional reduced differential technique. Chen et al. (2017) established a numerical technique for the solution of TFKGE in a bounded domain. Time fractional derivative has been described in the Caputo operator form. This technique has been presented on the basis of FDM discretization in time dimension and Legendre spectral method in space dimension. Nagy (2017) solved non-linear TFKGE numerically using Sinc-Chebyshev method. The Caputo operator has been utilized for fractional derivative. The Sinc functions and the shifted Chebyshev polynomials have been utilized in space and time directions respectively. Lyu and Vong (2018) proposed a new grid function FDM for solving TFKGE. Some numerical examples have been provided to justify the theoretical results. Kanwal et al. (2018) implemented Genocchi polynomial and Ritz-Galerkin method on FKGE and fractional diffusion wave equation. A linear system of equations was obtained using Genocchi polynomial and Ritz-Galerkin methods. Singh et al. (2019) discussed a numerical technique for the solution of FKGE. This numerical technique is based on the Legendre scaling functions and Caputo fractional operator.

A numerical technique based on B-spline collocation FEM has been developed for the numerical solution of FKGE by Karaagac et al. (2019). The CBS basis functions and the Caputo fractional derivative incorporation with FEM and FDM have been employed to discretized space and time dimensions. They also showed the numerical error norms L_∞ and L_2 to check the efficiency of this technique.

B-spline is very useful technique for the solutions of differential equations but there are some limitations that associate with the use of B-spline method for solving numerical problems. For the curve modification, it does not possess any free parameter. The shape of curve can not be changed once the data points are determined. Moreover, any change of data point will require solving the system again which is computationally very expensive. Identifying the weaknesses of the B-spline approximation Han and Liu (2003) introduced the blending function of degree four which is the extension of cubic B-spline function. Later, Gang and Guo-Zhao (2008) modified the extension cubic B-spline blending function to degree five and six. The extended cubic B-spline provides a free parameter for controlling the shape of curve.

2.7 Conclusion

A brief history of B-spline methods to solve the PDEs and FPDEs have been discussed in this chapter. In particular, numerical methods to solve FADE, FTE, FBE and FKGE have been reviewed. There are a large number of numerical techniques already established for solving FPDEs. However, as far as we are aware there are no such studies on the use of B-splines in solving linear TFADE, TFTE and non-linear TFBE and TFKGE. Thus, this is a gap we seek to fill. Therefore it would be of great interest to solve 1D FPDEs by using ExCuBS and NExCuBS. The addition of more methods to the pool of numerical methods for TFADE, TFTE, TFBE and TFKGE will benefit researchers investigating phenomena modeled by TFADE, TFTE, TFBE, TFKGE.

CHAPTER 3

BASIC CONCEPT, TECHNIQUE AND THEORY

3.1 Introduction

Fractional calculus have generated significant interest in past decades. An extensive study on this topic is discussed in Oldham and Spanier (1974), Ross (1977) and Podlubny (1999). This chapter covers the basic knowledge of the special functions, fractional calculus, B-spline, particularly extended cubic B-spline which is required for the next chapters. A recursive formula of B-spline and its properties will also be reviewed. These extended cubic B-spline functions will be utilized for the solution of linear and nonlinear fractional partial differential equations.

3.2 Special Functions

Some useful special functions are defined in this section. These functions are important in fractional calculus. The Euler Gamma function and Mittag-Leffler function are the two basic functions of fractional calculus. These functions play a major role in the theory of FDEs.

3.2.1 Euler Gamma Function

This function is a generalization of the factorial $\alpha!$ for $\alpha \in \mathbb{N}$ and is denoted by $\Gamma(.)$. For complex number m with $Re(m) > 0$, Euler Gamma function $\Gamma(m)$ is defined as (Podlubny, 1999; Gradshteyn and Ryzhik, 1980):

$$\Gamma(m) = \int_0^\infty t^{m-1} e^{-t} dt.$$

Some useful properties of Euler Gamma function are defined below:

i: $\Gamma(1+m) = m\Gamma(m), \ m \in \mathbb{C}, Re(m) > 0$

ii: $\Gamma(\alpha) = (\alpha - 1)!, \quad \alpha \in \mathbb{N}$

iii: $\Gamma(1-m) = -m\Gamma(-m), \quad m \in \mathbb{C}, Re(m) > 0.$

3.2.2 Mittag-Leffler Function

This function was first introduced by Mittag-Leffler (1903). For $m \in \mathbb{C}$ Mittag-Leffler function E_γ in one parameter form is defined as (Podlubny, 1999):

$$E_\gamma(m) = \sum_{k=0}^{\infty} \frac{m^k}{\Gamma(\gamma k + 1)}, \quad \gamma > 0.$$

The generalized function in two parameters form is defined as:

$$E_{\gamma,\delta}(m) = \sum_{k=0}^{\infty} \frac{m^k}{\Gamma(\gamma k + \delta)}, \quad \gamma > 0, \delta > 0.$$

Mittag-Leffler function converges for $m \in \mathbb{C}$.

3.3 Fractional Calculus

Fractional calculus appeared with some simple questions which were associated to the derivative concept: i.e., first order derivative shows the slope of a function but what is a $\frac{1}{2}$ order derivative represent? As a result, many mathematicians managed to open a new window of opportunity to mathematical and real world, which has arisen many new and fascinating results. Simply, we can say fractional calculus is a theory of differentiation and integration of arbitrary order (Podlubny, 1999).

3.3.1 Fractional Integrals

Fractional integral is the name of the integral of an arbitrary order (Podlubny, 1999). For $\gamma > 0$, fractional integral operator of γ order is denoted by

$$ {}_cI_t^{\gamma} g(t) \quad or \quad {}_cD_t^{-\gamma} g(t), $$

where $g(t)$ is any dependent function, c and t are the limits of fractional integral operators and usually these limits called as terminals (Ross, 1977; Podlubny, 1999). The RL integral can be defined as follows:

$$ {}_cI_t^{\gamma} g(t) = {}_cD_t^{-\gamma} g(t) = \frac{1}{\Gamma(\gamma)} \int_c^t \frac{g(\tau)}{(t-\tau)^{1-\gamma}} d\tau, \quad Re(\gamma) > 0. \tag{3.1} $$

Another approach is to describe the notion of (3.1) by considering the n-fold integral of a function $g(t)$ in the following way

$$ {}_cD_t^{-n} g(t) = \int_c^t d\tau_1 \int_c^{\tau_1} d\tau_2 ... \int_c^{\tau_{n-1}} g(\tau_n) d\tau_n. \quad \tau_i, i = 1, 2, ...n. $$

Repeated integrals can be written as a single integral in the form of Cauchy's formula as

$$ {}_cI_\tau^n = \frac{1}{(n-1)!} \int_c^t \frac{g(\tau)}{(t-\tau)^{1-n}} d\tau. \tag{3.2} $$