EXPONENTIAL PARAMETERIZED CUBIC B-SPLINE CURVES AND SURFACES

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2020

EXPONENTIAL PARAMETERIZED CUBIC B-SPLINE CURVES AND SURFACES

by

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Thesis submitted in fulfilment of the requirements for the degree of Master of Science

January 2020

ACKNOWLEDGEMENT

First and foremost, I would like to thank God Almighty for giving me the strength, knowledge, ability and opportunity to undertake this research study and to persevere and complete it satisfactorily. Without his blessings, this achievement would not have been possible.

On this moment of submission of my thesis, I would like to thank all those good human beings, whom I have come across in this path of my journey, whose lives have inspired me and from whom I've learnt to live the life.

In my journey towards this degree, I have found a teacher, an inspiration and a pillar of support in my guide, Dr. Ahmad Lutfi. He has been there providing his heartfelt support and guidance at all times and has given me invaluable guidance, inspiration and suggestions in my quest for knowledge. He has given me all the freedom to pursue my research, while silently and non-obtrusively ensuring that I stay on course and do not deviate from the core of my research. Without his able guidance, this thesis would not have been possible and I shall eternally be grateful to him for his assistance.

My acknowledgement would be incomplete without thanking the biggest source of my strength, my family. The blessings of my parents, Azra perveen, M.Yaqoob, grandmother, uncle and the love and care of my brothers and cousins, have all made a tremendous contribution in helping me reach this stage in my life. I thank them for putting up with me in difficult moments where I felt stumped and for goading me on to follow my dream of getting this degree. This would not have been possible without their unwavering and unselfish love and support given to me at all times.

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LIST OF SYMBOLS

$\mathbb{R}^m$	set of all n-tuples of real number
d	degree of curve
P_i	control points
Q_i	data points
T, U, V	knot vector
t_i	parameter values
α	represent the value for parameterization method
$\sum$	summation - sum of all values in range of series
b_1, b_2	branching contour
$\mathbf{A}$	starting position of vehicle
$\mathbf{B}$	goal position of vehicle
$\mathbf{B}_L$	left turn goal position of vehicle
$\mathbf{B}_S$	straight goal position of vehicle
$\mathbf{B}_R$	right turn goal position of vehicle
C^k	parametric continuity
G^k	geometric continuity

LIST OF ABBREVIATIONS

CAD	Computer-Aided Design
CAGD	Computer-Aided Geometric Design
NURBS	Non Uniform Rational B-spline
DTM	Digital Terrain Model
PC	Personal computer
2D	Two-Dimensional
3D	Three-Dimensional
CG	Center of Gravity
GIS	Geographic Information System

LENGKUNG DAN PERMUKAAN SPLIN-B KUBIK BERPARAMETER EKSPONEN

ABSTRAK

Penggunaan fungsi interpolasi splin-B kepada lengkung dan permukaan telah dibangunkan untuk pelbagai kegunaan. Salah satu sebab ialah kerana tahap keselanjaran dan kelicinan yang lebih tinggi. Splin-B am adalah sejenis lengkung penghampiran dan bentuknya ditentukan oleh titik kawalan. Untuk menginterpolasi titik data, pelbagai kaedah telah dilakukan oleh penyelidik terdahulu yang mengkaji pemparameteran Splin-B. Dalam tesis ini, kami membangunkan kaedah baru untuk menginterpolasi lengkung Splin-B kubik dengan menggunakan terbitan pertama dan kedua pada titik akhir dan hanya terbitan pertama di titik dalaman. Kaedah yang dicadangkan ini merupakan lanjutan teknik interpolasi Splin-B yang menggunakan terbitan sembarangan di titik akhir. Dalam menghasilkan kaedah interpolasi lengkung Splin-B, suatu algoritma untuk menginterpolasi titik data telah dibangunkan. Algoritma tersebut mengira nilai-nilai simpulan untuk kaedah pemparameteran. Nilai-nilai simpulan ini digunakan untuk membina matriks fungsi asas Splin-B dan terbitan fungsi asas tersebut. Kemudian, kami menyelesaikannya bagi mandapatkan titik kawalan dengan menggunakan kaedah penguraian LU, supaya lengkung akan melalui titik data yang diberikan. Pemilihan teknik pemparameteran yang betul adalah penting dalam proses pembinaan semula lengkung dan permukaan. Kaedah pemparameteran yang digunakan dalam kajian ini adalah kaedah pemparameteran eksponen dengan $\alpha = 0.8$. Kelebihan utama kaedah interpolasi lengkung Splin-B ialah kita boleh menjana pelbagai bentuk lengkung dengan menetapkan arah yang berbeza pada semua titik data. Kemudian, kami mengaplikasikan kaedah yang dicadangkan ini dalam pembinaan semula lengkung pada peta jalan raya daripada titik data dan arah pemanduan yang diberikan dan juga untuk perancangan laluan bagi kenderaan berautonomi yang diberi lokasi permulaan dan lokasi matlamat. Tesis ini juga melihat isu pembinaan semula permukaan yang berlaku dalam pelbagai aplikasi. Jadi kami mengembangkan kaedah interpolasi untuk membina permukaan daripada garis kontur menggunakan permukaan tergaris Splin-B. Kami mengimplimentasikan permukaan tergaris Splin-B untuk membina semula model tiga dimensi muka bumi berdasarkan titik data yang diperolehi daripada garis kontur dua dimensi. Pertama, lengkung interpolasi ruangan dijana dari garis kontur dengan menggunakan lengkung Splin-B yang telah diparameterkan. Kemudian, permukaan dibina dengan menggunakan permukaan tergaris Splin-B. Dalam proses pembinaan semula, beberapa isu seperti lubang kunci dan percabangan mungkin timbul. Oleh itu, kami mencadangkan satu kaedah yang mengendalikan objek yang bercabang dengan membina tampalan dwilinear. Kami juga menyelesaikan masalah lubang kunci dengan mengekalkan keadaan vektor simpulan yang sama pada permukaan Splin-B. Keputusan juga telah ditunjukkan bagi model dengan percabangan dan tanpa percabangan dan dibandingkan dengan kaedah sedia ada.

EXPONENTIAL PARAMETERIZED CUBIC B-SPLINE CURVES AND SURFACES

ABSTRACT

The use of B-spline interpolation function for curves and surfaces has been developed for many reasons. One reason is the higher degree of continuity and smoothness. A general B-Spline is a polynomial curve and its shape is determined by the control points. To interpolate data points, various works have been done by previous researchers who studies B-Spline parameterization. In this thesis, we develop a new way for interpolating cubic B-Spline curve by taking the first and the second derivative at endpoints and only the first derivative at inner points. The proposed method is the extension in the B-spline interpolation technique of using arbitrary derivatives at end points. In developing B-spline curve interpolation method, an algorithm is presented for interpolating data points. The algorithm computes knot values for parameterization methods. These knot values are used in constructing a matrix of B-Spline basis function and derivative of the basis function. Then, we solve it for control points by using the LU decomposition method, such that the curve will pass through the given data points. Selection of proper parametrization technique is critical for curve and surface reconstruction process. The parametrization method used in this study is an exponential parameterization method with $\alpha = 0.8$. The main advantage of developing B-spline curve interpolation method is that we can generate different shapes of curves by setting different direction at all data points. As an application, we applied the proposed method in curve reconstruction on a road map from given data points and driving directions, and also for path planning in autonomous vehicle with given starting and goal position. The thesis also look into the issue of surface reconstruction that occurs in a broad variety of applications. So we extend the interpolation method to construct surfaces from contour line using B-spline ruled surfaces. We implement B-spline ruled surfaces to reconstruct three-dimensional terrain models based on the data points obtained from two-dimensional contour lines. Firstly, spatial interpolated curves are generated from contour lines by using parameterized B-spline curve. Then surfaces are constructed by using B-spline ruled surface. In the reconstruction process, some issues such as keyholes and branching may arises. Therefore, we propose a method which handles the branching object by constructing a bilinear patch. We also solve keyholes issues by retaining the same knot vector condition on B-spline ruled surface. Results are also demonstrated for models with branching and without branching and compared to the existing method.

CHAPTER 1

INTRODUCTION

1.1 Overview

Throughout the most recent two decades the area of computer graphics has truly developed with achievement in all directions; the advent of devoted 3D hardware, computer generated liveliness and faster PCs, to name a few important occurrence. Using computer graphics enables real life modelling to be demonstrated in a computer model. A computer model makes the work a lot simpler, more efficient and progressively successful. The computer model is visualized at the start as a wire-frame model and then manipulated and rendered at the end. What's more, computer model animation can be introduced to bring computer model close to the original model. There are numerous applications being assisted by computer graphics in engineering, scientific visualization, image processing, design and entertainment (Foley et al., 1994).

One of the research development areas focuses on displaying smooth curves and surfaces that are appropriate for modelling interesting landscape, faces and other topologies. Here emerge the use of parametric curve such as Bezier, and specifically the B-spline curves. B-spline curves are comprised of an arbitrary number of control points and data points, that provide a rule of approximating and interpolating for the assessed curves. The approximation and interpolation can be modified to suit its need in various distinct ways; weighted control points, knot vector, different parameterization methods, varying the curve degree and so on.

In realistic 3D models such as virtual reality and computer-aided design (CAD) require the advanced computer graphics methods, which involves geometric modelling of curves and surfaces, differential geometry and many more. The reverse engineering process is an emerging CAD application in the manufacturing industry that uses B-spline interpolated and approximated surface patch. The interpolation and approximation is achieved by constructing parameter values using the information of given control points or data points.

One may inquire as to why parametric representation of curves and surfaces have a versatile representation. A couple of valid justification for this: automatic resolution, flexible design, easy to modify dimension in parametric programmed etc. The point of interest are various and will be thoroughly investigated.

1.2 Literature Review

B-spline method on curves and surfaces were initially proposed in 1940s but were only seriously developed in the 1970s by some researchers, especially Richard Riesenfeld (1973) as an improvement over Bezier curve to overcome some disadvantages such as continuity, global control and degree dependency on the number of control points. Riesenfeld (1973) resulted an attempt to create splines that contained local support. Since B-spline has local control property, it is possible to alter the shape of a specific curve section without influencing the whole curve. In other words, a local curve segment is built by a distinct set of variables that differ in other segments which practically means that if a P_i control point is replaced with a new P_i' control point, then new B-spline curve varies only by a few neighbouring points from the orignal curve.

This provide more control over the shape of B-spline curves (Rogers, 2000). Note that it does not means that the curve stay unchanged except its neighbourhood if a point P_i is moved to a new position $P_i^{'}$.

B-spline is a piecewise polynomial approach of curve fitting and is a linear combination of B-spline basis functions which are also piecewise polynomials with specific smoothness conditions. It has been commonly used owing to its maximum smoothness such as C^1 or C^2 that a piecewise method can have. Due to its property of smoothness and localness, B-Spline curve is a well known method in Computer Aided Geometric Design(CAGD) and other related fields. A recursive formula of B-spline basis function of any order was founded by De Boor (1972). B-spline was also extended to non-uniform rational B-spline (NURBS) at first by Versprille and James (1975) and later by Tiller (1983) and Piegl and Tiller (1987). It can interpolate and approximate infinite number of data points based on the parameterization (Hollig and Horner, 2013; Prautzsch et al., 2013; Salomon, 2007).

In many CAGD applications, parameterization on data points is one of the fundamental step and the choice of parameter values is very important for parameterization (Bartels et al., 1995; Farin, 2014). Previous researchers have used various parameterization methods such as uniform (De Boor, 2001), centripetal, exponential (Lee, 1989), chord-length (Lü, 2009), and the modified form of these methods such as universal (Lim, 1999), hybrid (Shamsuddin and Ahmed, 2004), Foley-Nielson (Foley and Nielson, 1989) and based on exponential proposed by Haron et al. (2012). Every proposed parameterization method may produce different curve and has its own characteristics when interpolating data points. Several basic concepts related to geometric modeling are summarized by Piegl and Tiller (2012), Cohen et al. (2001) and Bartels et al. (1995). Interpolation on data points by B-spline parameterization is better compared to Bezier spline especially for large number of data points, because degree of B-spline curve did not depends on the number of data point but the Bezier curve have the contrary characteristic (Andersson and Kvernes, 2003). Recently, path generating and smoothing guidance algorithm for autonomous vehicle have been studied in literature (Cheng, 2011; Katrakazas et al., 2015; Labakhua et al., 2008; Ma et al., 2012; Villagra et al., 2012; Yang and Sukkarieh, 2010). Most of the methods based on trigonometric spline interpolation, cubic spline interpolation, Bezier curve approximation and combination of circle, straight lines and clothoids (Dyllong and Visioli, 2003; Funke and Gerdes, 2016; Pérez et al., 2013).

The complexity of many geometric operations greatly depends on the method of representation. Some information related to most commonly method of representing implicit surfaces (Akenine-Moller et al., 2018; De Araujo et al., 2015; Guo et al., 2010; Ilic and Fua, 2005) and parametric B-spline and NURBS (Bhattarai et al., 2017; Cashman et al., 2009; Hu et al., 2001; Lim and Haron, 2014; Zhang, 2018) also take into account. The parametric form is more natural for designing and representing shape in the computer as compared to implicit (Brigger et al., 2000). The construction of surface using parameterized B-spline and radial basis function from scattered data points is discussed by Jie et al. (2016). However, different techniques on surface reconstruction from 2D contour have been addressed in the past but some complexities are common like continuity, planarity, rapid changing, multifurcation, and keyholes (Meyers et al., 1992). Keyholes actually cause a problem with triangulation process which is due to the irregulation placement of point on contour line (Sunderland et al., 015c) and surface continuity can be achieved by applying contour interpolation method such as performing Hermite interpolation along the gradient paths (Hormann et al., 2003).

For branching object, fast reconstruction method (Shin and Jung, 2004) is implemented to generate original geometry by connecting the vertices with edges between two consecutive slices. Straight skeleton method is used to create faces on key contour by linking the key contour lines in GIS map for generating 3D terrain (Salvatore and Guitton, 2004; Sugihara and Murase, 2017; Zhu et al., 2003). Tensor product of B-spline interpolation can also be used as a modern acquisition technique for reconstruction of surfaces (Vaitkus and Várady, 2018). As a parametric function B-spline possess considerable geometric significance for constructing a ruled surface, such constructions are fundamental to many CAD systems (Jha, 2008; Lin et al., 2004). One basic problem in the study of the parametric curve is to approximate a curve with lower degree curve segment(Amirfakhrian, 2016). However, some geometric features such as singular point cannot be preserved.

1.3 Objectives

The aim of the thesis is to study and explain the concept, use and implementation of B-spline curves and surfaces thoroughly. This is mainly achieved primarily by the study, but is also practically assisted by the advanced software by showing some results. This study discusses the two main approaches.

The first objective is to develop B-spline interpolation method with direction at all points by developing knot vector generated algorithm. The developing method is also an extension of interpolation with arbitrary end derivative method by Piegl and

Tiller (2000a) and interpolation on directional constrains on few data points method by Piegl et al. (2008). The proposed method produce different shapes of curves, that is useful for curve reconstruction on a road path defined by coordinates and its direction. It can also be useful in application such as path planning as the direction is defined by data points.

For the second objective, we reconstruct smooth surface by using parametrized B-spline curve interpolation to recreate contour lines and then by using B-spline ruled surface, we construct surfaces on each contour. The issues related to branching object is also discussed, and the results are also demonstrated.

1.4 Structure of Thesis

The present study seeks to improve curve interpolation and surface reconstruction technique by developing proposed method based on exponential parameterized B-spline. The study starts with background of B-spline curve and B-spline curve interpolation and an exhaustive explanation of B-spline curve interpolation with different parameterization method in Chapter 2. In Chapter 3 the development of proposed method on the base of knot vector generated algorithm and its application on road map interpolation are discussed. The comparison of proposed results with Bezier and general B-spline curve are also discussed. In Chapter 4 detailed reconstruction process of surfaces with B-spline ruled surfaces for simple and branching models are demonstrated, and the comparison with few previous method is also given by showing some results. Finally, conclusion and future works is presented in Chapter 5.

CHAPTER 2

B-SPLINE CURVE INTERPOLATION

In this chapter, we will present background study related to B-spline curve interpolation. Let us first define the B-spline curve. B-Spline is an approximating curve in general and its shape is determined by the control points but it also has parametric nature. To interpolate data points, various works have been done by the previous researchers in parameterization of B-Spline interpolation. However, our research demonstrates that knot vector choice is no less essential than parameterization. Note that the knot vector plays a significant part in the following definitions. There are several distinct techniques for defining B-splines but recursion method is used for defining B splines in this study.

2.1 Definitions

2.1.1 B-spline Basis

The B-spline basis function of degree d (order $d+1$), defined by the knot vector $\boldsymbol{T} = \{T_0 = ... = T_d, T_{d+1}, ...T_n, T_{n+1} = ... = T_{n+d+1}\}$, with associated normalized B-splines basis $N_{i,d}$ of degree d (order $d+1$) are defined recursively as follows:

$$N_{i,0}(t) = \begin{cases} 1 & \text{if } t \in [T_i, T_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,d}(t) = \frac{t - T_i}{T_{i+d} - t_i} N_{i,d-1}(t) + \frac{T_{i+d+1} - t}{T_{i+d+1} - T_{i+1}} N_{i+1,d-1}(t) \tag{2.1}$$

for $i = 0, .., n$ and $d \geq 1$.

In Figure 2.1, some examples of B-spline basis of different orders are shown. Based on B-splines basis, one could generate a B-spline curve as shown in Figure 2.2.

2.1.2 B-spline curve

The B-spline curve of degree d (order $d+1$) with given control points $P_i \in \mathbb{R}^m, m = 2, 3,\ 0 \leqslant i \leqslant n$, parameter values $t_0, t_1, ..., t_n$ and knot vector $\boldsymbol{T}$, then a B-spline curve of degree d is defined by

$$B(t) = \sum_{i=0}^{n} P_i N_{i,d}(t) \tag{2.2}$$

where $N_{i,d}(t)$ is a normalized B-Spline Basis function with degree d.

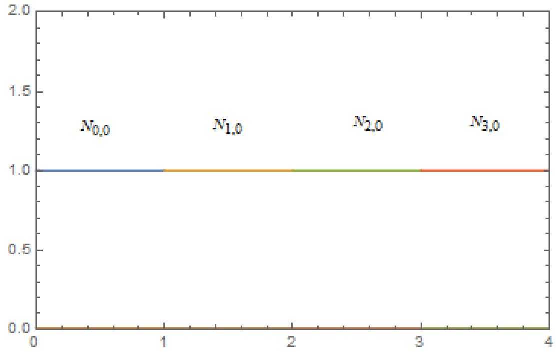
2.1.3 B-spline curve interpolation

If the list of data points $Q_i \in \mathbb{R}^m, m = 2, 3.\ i \in [0, n]$ are given, then the B-Spline curve interpolation of degree d is to find

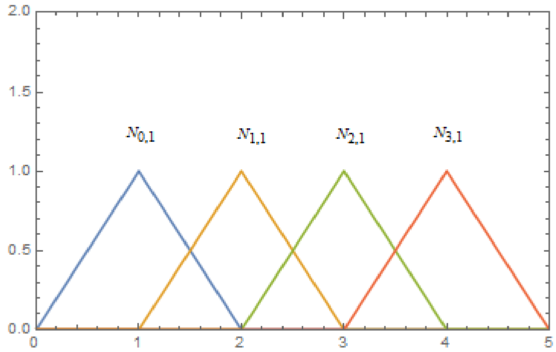
- The knot vector $\boldsymbol{T} = (T_0 = ... = T_d, ..., T_n, T_{n+1} = ... = T_{n+d+1})$
- The parametric values t_i, for each Q_i, $i \in [0, n]$ and
- The control points P_i such that the resulting B-Spline curve

$$B(t) = \sum_{i=0}^{n} N_{i,p}(t) P_i \tag{2.3}$$

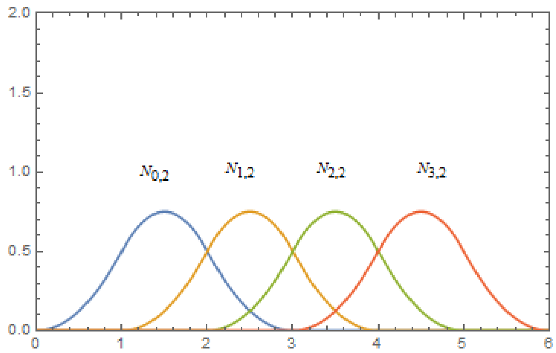
has the property $B(t_i) = Q_i$, $i \in [0, n]$.



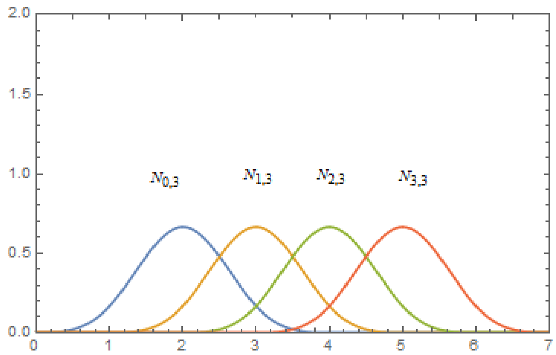
(a) Order 1



(b) Order 2



(c) Order 3



(d) Order 4

Figure 2.1: Normalized B-spline Basis

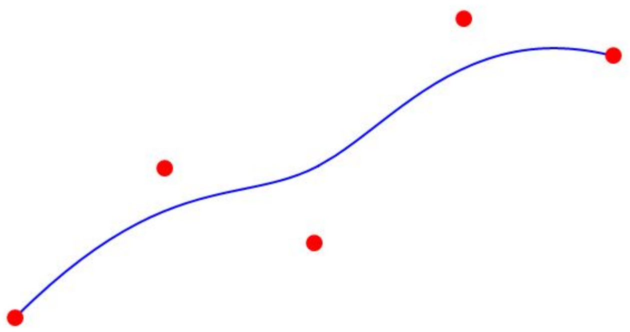


Figure 2.2: B-spline curve and its control points

2.2 Computation of B-spline curve interpolation

The computation of B-Spline curve interpolation is indicated as follows. A B-Spline curve of degree d defined in Equation (2.2) with control points P_i and normalized B-Spline Basis function $N_{i,d}(t)$ define over the knots $\boldsymbol{T}$ and a set of data points

$$Q_i, \qquad i = 0, ..., n, \tag{2.4}$$

is given. Then the curve $B(t)$ is needed to pass through these points i.e. the curve assumes these points at certain parameter values t_i, such that

$$B(t_i) = \sum_{i=0}^{n} N_{i,d}(t_i) P_i \qquad i = 0, ..., n. \tag{2.5}$$

For curve interpolation, the parameter values $t_0, t_1, ..., t_n$, knot vector $\boldsymbol{T}$ and the control points $P_0, P_1, ..., P_n$ must be computed.

The best curve fitting depends on the parameter values and the parameter values are obtained by different parameterization methods. First, we write the general formula for finding the parameter values by (Lee, 1989)

$$
\begin{aligned}
t_0 &= 0 \\
t_n &= 1 \\
t_k &= \frac{1}{L}\sum_{i=1}^{k}(|Q_i - Q_{i-1}|)^{\alpha}
\end{aligned}
\tag{2.6}
$$

where

$$L = \sum_{i=1}^{n}(|Q_i - Q_{i-1}|)^{\alpha}$$

For different values of α, this general formula represents the different parameterization method such that

- if $\alpha = 0$, it is called uniform parameterization,
- if $\alpha = 0.5$, it is called centripetal parameterization,
- if $\alpha = 1$, it is called chord length parameterization.

After the parameters are computed, the next step is to find the knot vector. Knot values are found by using knot averaging method suggested by De Boor (2001)

$$T_{j+d} = \frac{1}{d}\sum_{i=j}^{j+d-1} t_i \qquad j = 1, \cdots, n-d \tag{2.7}$$

and the end knots are set to 0 and 1 respectively,

$$T_0 = \cdots = T_d = 0 \quad , \quad T_{n+1} = \cdots = T_{n+d+1} = 1$$

Once we have made the parameterization, the following equation is solved to find the control points

$$P_0 N_{0,d}(t_i) + P_1 N_{1,d}(t_i) + \cdots\cdots + P_n N_{n,d}(t_i) = Q_i \qquad i \in [0,n]. \tag{2.8}$$

In matrix form, the equation becomes

$$\mathbf{AP{=}Q}, \tag{2.9}$$

where

$$\mathbf{A} = \begin{pmatrix} N_{0,d}(t_0) & N_{1,d}(t_0) & \cdots & N_{n,d}(t_0) \\ N_{0,d}(t_1) & N_{1,d}(t_1) & \cdots & N_{n,d}(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ N_{0,d}(t_n) & N_{1,d}(t_n) & \cdots & N_{n,d}(t_n) \end{pmatrix}$$

$$\mathbf{P} = (P_0, P_1, ..., P_n)^T \qquad and \qquad \mathbf{Q} = (Q_0, Q_1, ..., Q_n)^T$$

By solving the system of equations, we obtain the control points. Then by substituting these control points we can produce a curve that exactly passes through each of the data point. The quality of interpolating the curve depends on the selection of parametric values t_i for the given points Q_i.

2.3 Selection of parameterization method

The quality of an interpolating curve $B(t)$ is strongly dependent on selecting parametric values for specified information points. There are different parameterization methods, but for comparison purposes, we are taking into account the three main methods such as uniform, centripetal and chord length. The further detail related to advantages and disadvantages of these method is discussed in Haron et al. (2012). In our study all work is done by exponential parameterization method. Because this method is the generalization of all other three methods with characterization of freely selecting the curve and give good result with better curvature profile as discussed in Section 2.3.1.

2.3.1 Results and discussion

We select three sets of data points for experimental work as shown in Table 2.1. Data Set 1 and 2 are taken from the previous study (Piegl and Tiller, 2000b; Yuksel et al., 2011), while Data Set 3 is a generated point data set and these three sets of data points has different features. For example, In Data Set 1 there is no collinear points and no two consecutive points so this is one type of simple data set. The Data Set 2 has two successive and wide distance between two data points while the Data Set 3 is a combination of the characteristics of two prior data sets.

The results of 4 parameterization methods and their curvature profile over Data Set 1, Data Set 2 and Data Set 3 are shown in Figure 2.3, 2.4 and 2.5 respectively. The interpolated curves are getting by using Equation (2.8), while curvature profile's are getting by using curvature formula given in Equation 2.10. If a curve is defined in a parameteric form over the parameter t by the equations $x = x(t), y = y(t)$, then the curvature of a parametrically defined curve is expressed by the formula

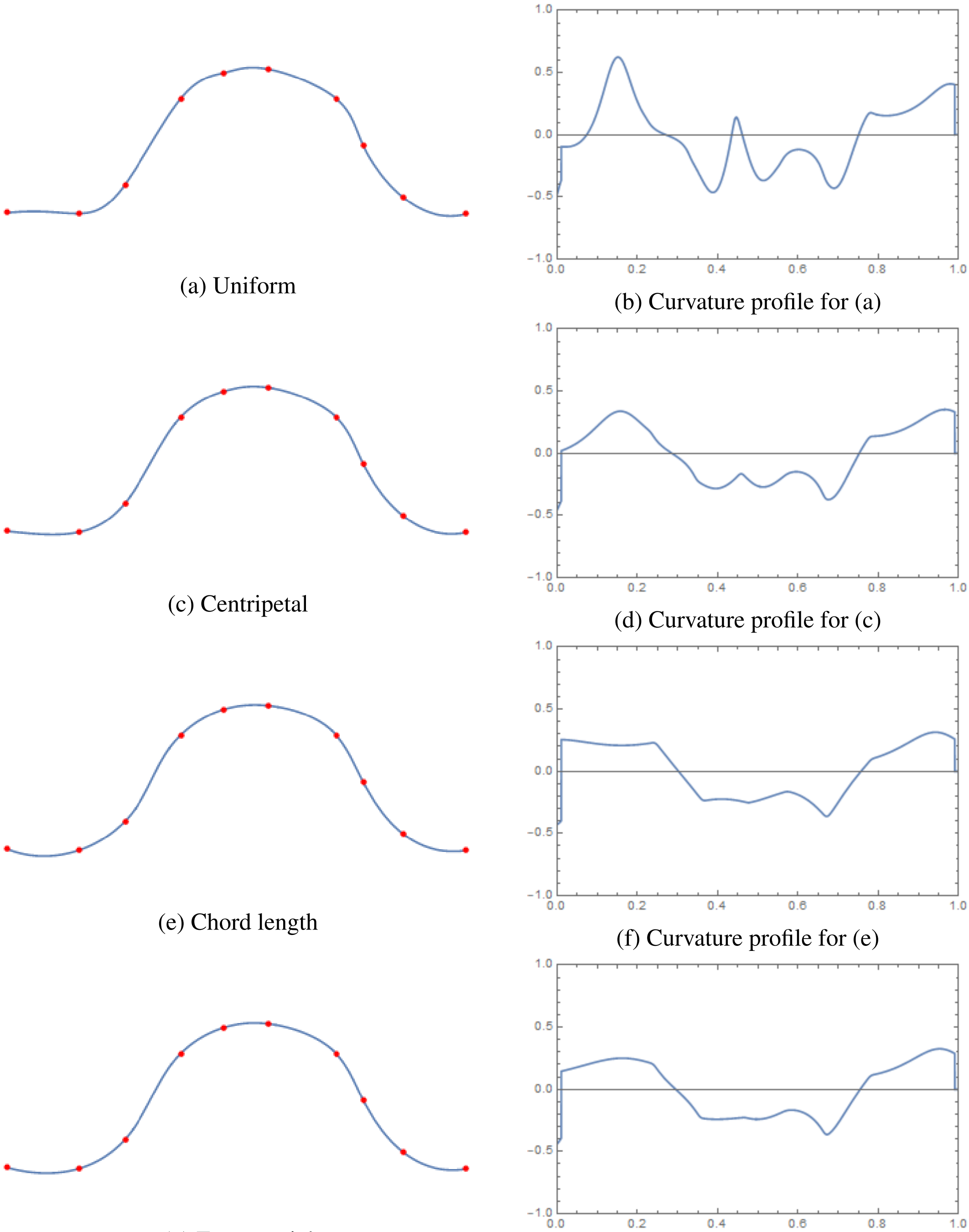
$$\kappa(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{[(x'(t))^2 + (y'(t))^2]^{3/2}} \tag{2.10}$$

where primes refer to derivatives with respect to t.

In Figure 2.3, uniform parameterization method has an inflection point at near the top of the curve and all other methods has the same results. But in curvature profile, our observation based on Table 2.2 is that the exponential method has better curvature profile as compared to uniform, centripetal and chord length parameterization methods.

Table 2.1: Data points used for comparison of different parameterization methods

Data Set 1		
i	x	y
1	0.58072	2.08688
2	3.50755	2.05734
3	5.36585	3.24051
4	7.64228	6.74273
5	9.40767	7.79716
6	11.1963	7.94949
7	13.9837	6.73456
8	15.0859	4.83752
9	16.7247	2.71041
10	19.2799	2.03701
Data Set 2		
1	1.83761	1.33271
2	0.72649	2.65040
3	4.82906	7.40482
4	5.12821	7.15185
5	8.41880	1.62069
6	8.84615	2.08095
7	9.44444	6.46124
Data Set 3		
1	32	14
2	34	10
3	36	9
4	38	12
5	38.2	11.7
6	42	14
7	44	12
8	46	13
9	48	7
10	50	6
11	52	10
12	54	10
13	56	10
14	58	10



(a) Uniform

(b) Curvature profile for (a)

(c) Centripetal

(d) Curvature profile for (c)

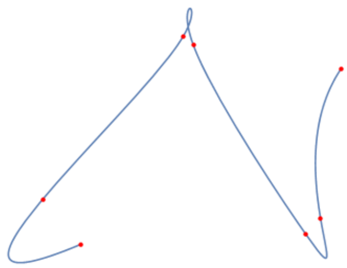
(e) Chord length

(f) Curvature profile for (e)

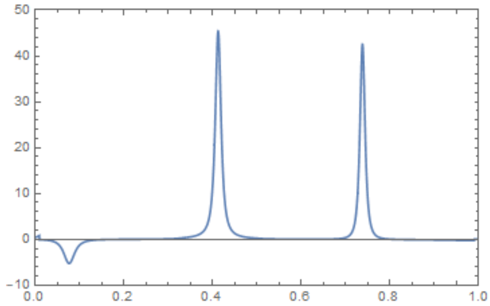
(g) Exponential

(h) Curvature profile for (g)

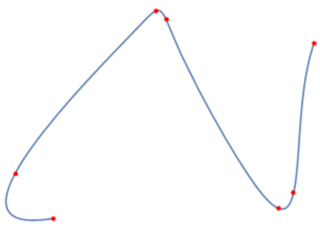
Figure 2.3: Curve interpolation of Data Set 1 with different parameterization methods and their curvature profiles



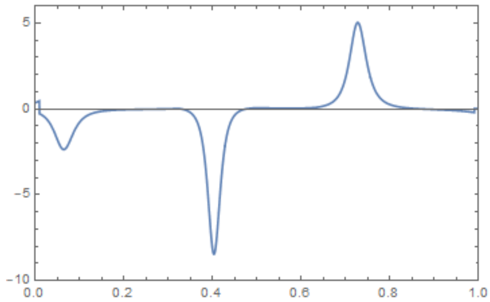
(a) Uniform



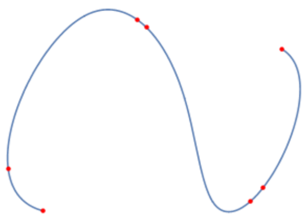
(b) Curvature profile for (a)



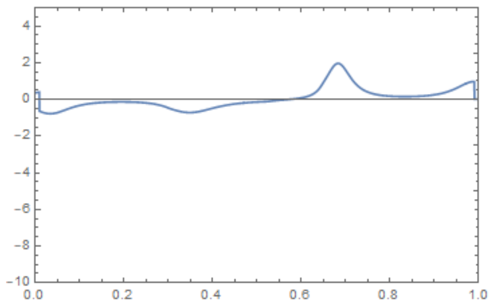
(c) Centripetal



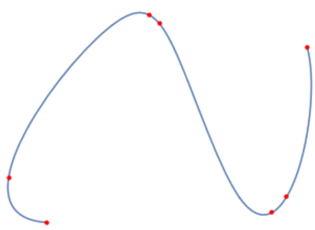
(d) Curvature profile for (c)



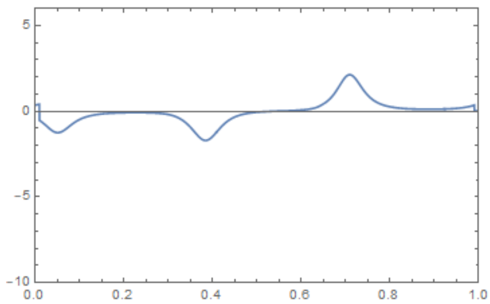
(e) Chord length



(f) Curvature profile for (e)

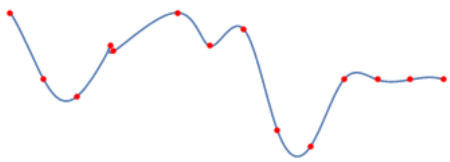


(g) Exponential

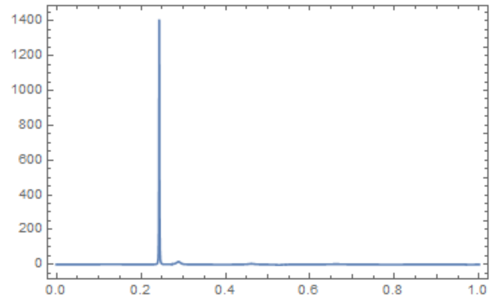


(h) Curvature profile for (g)

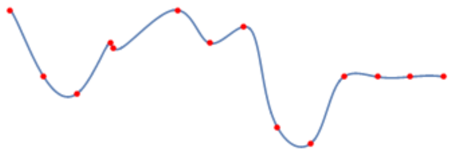
Figure 2.4: Curve interpolation of Data Set 2 with different parameterization methods and their curvature profiles



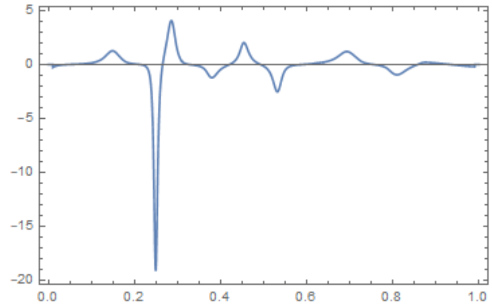
(a) Uniform



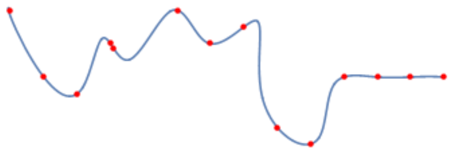
(b) Curvature profile for (a)



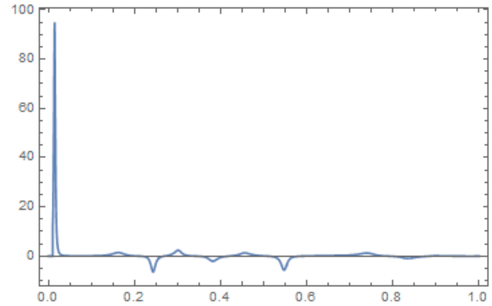
(c) Centripetal



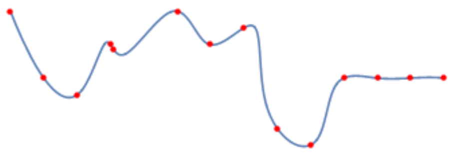
(d) Curvature profile for (c)



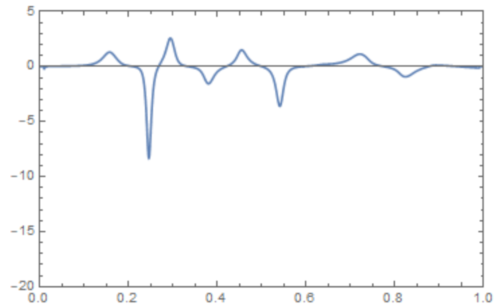
(e) Chord length



(f) Curvature profile for (e)



(g) Exponential



(h) Curvature profile for (g)

Figure 2.5: Curve interpolation of Data Set 3 with different parameterization methods and their curvature profiles

Table 2.2: Maximum and minimum curvature values for 3 sets of data points

Methods	Data set 1		Data set 2		Data set 3	
	Maximum curvature	Minimum curvature	Maximum curvature	Minimum curvature	Maximum curvature	Minimum curvature
Uniform	0.3011	-0.4768	2.9879	-0.0660	3.89513	-0.9941
Centripetal	0.2741	-0.4531	2.67163	-4.033	1.9724	-11.6952
Chord length	0.2560	-0.4262	0.8829	-0.4652	2.26896	-1.91046
Exponential	0.2466	-0.4371	1.5657	-1.5045	1.4948	-1.5353

In Figure 2.4, uniform parameterization method produce small loop with high curvature profile and the chord length method has bulky result as compared to centripetal and exponential parameterization methods. In comparison to the curvature profile based on Table 2.2 over the Data Set 2, exponential method show better Maximum and Minimum curvature values as compared to centripetal and uniform, but chord length method show much better curvature values as compared to exponential.

Similarly in Figure 2.5, the chord length method showed a twisty result at the beginning of the curve and the centripetal method has few bulky result on the curve. But in comparison to the curvature profile, exponential method have better Maximum and Minimum curvature values as compared to other which is clearly seen through the Table 2.2.

Our main focus in comparison the curve interpolation of all parameterization methods is at curvature profile along with the good curve that do not have any loop, twist and bulk. So we can easily seen that with three different sets of data points, the exponential curves shown good result along with better curvature profile as compared to other parameterization methods as shown in Table 2.2. So, base on these experimental results, we will use exponential parameterization method with $\alpha = 0.8$ for this study.

2.4 Summary

B-spline is one of the most frequently used curve and surface method in CAGD. This is due to its property of locality and smoothness. It also have simple manipulation for parametric representation which is preferred in literature. parameterization is quite important in B-spline curve interpolation because it greatly affects the quality of curves. In this chapter, we have discussed four parameterization methods for curve interpolation and constructed various interpolated cubic B-spline curves along with their curvature profile. After discussing all curves with geometric aspect and comparison through their curvature profiles, we select exponential parameterization method with ($\alpha = 0.8$) for further implementation of B-spline in proposed method. In B-spline interpolation, parameter values are much important along with the knot vector for generating curves with efficient computation of interpolation. The next chapter will extend the existing study and focus on the interpolation of data points with directional constrains at all points and some application on road map interpolation, path planning are also discussed in detail.

CHAPTER 3

B-SPLINE CURVE INTERPOLATION BY PROPOSED METHOD

There has been a lot of discussion on parameterization for interpolation (Epstein, 1976; Foley and Nielson, 1989; Lee, 1989) and every author claims that his method is the one to be used in the design practice. Practical experience shown that parameterization depends on the data and that none of the method is quite acceptable when used in isolation from the unknown, e.g knots. Since parameter are also required to reflect the flow of points, and the interpolation technique must be generalized with no derivatives. The parameters are selected according to the parameterization methods described in Section 2.3. This means that the existence of directional limitation should not affect the flow of points considerably.

3.1 B-Spline curve interpolation with arbitrary end and inner points derivative

For a smoother B-Spline curve (Piegl and Tiller, 2000a) used derivative up to $d-1$ at end points. If end derivative is not given, one can estimate the derivative proposed by Piegl and Tiller (2000a). In some applications such as path planning, we may have the information of direction with respect to the data points Q_i. Therefore, we propose a method of taking the derivative at all data points and cubic B-Spline curve (degree $d=3$) is used for this study. Recall Equation (2.2), the j^{th} derivative of Equation (2.2) is given as

$$B^{(j)}(t) = \sum_{i=0}^{n} N_{i,d}^{(j)}(t)\, P_i \tag{3.1}$$

where

$$N_{i,d}^{(j)}(t) = d(\frac{N_{i,d-1}^{(j-1)}(t)}{t_{i+d} - t_i} - \frac{N_{i+1,d-1}^{(j-1)}(t)}{t_{i+d+1} - t_{i+1}}).$$

As discussed in Section 2.2, we must compute the parameters, knots and control points for the curve interpolation. The parameter values are computed in the same way as in the case of general B-spline curve interpolation using Equation (2.6). So, our next step is to find the values of knot vector. Algorithm 1 is constructed for finding the knot values in the knot vector

$$\boldsymbol{T} = \{T_0 = ... = T_d, T_{d+1}, T_{d+2}, ..., T_{n-1}, T_n, T_{n+1} =, ..., = T_{n+d+1}\} \tag{3.2}$$

Algorithm 1: Algorithm for finding knot vector

Input: Q_i: Data points
t_i: Parameters , $i = 0, ..., n$
n: number of data points
$d = 3$: Degree of curve
Output: $T_i, i = 0, ..., 2n + d + 4$
Initialisation
$k = 2n + 3$
for $i = 0$ to d
$\quad\quad T_i = t_i$
$\quad T_{K+i+1} = t_n$
end
Do
$\quad$ **for** $i = 1$ to n
$\quad\quad T_{2i+2} = (t_{i-1} + t_i)/2$
$\quad$ **for** $i = 1$ to $n - 1$
$\quad\quad T_{2i+3} = (t_{i-1} + t_i + t_{i+1})/3$
end

However, we need one more knot value which we can find by taking average of any two consecutive knot values from T_{d+1} to T_{k-1} and the placement of this T_{new} knot value is selected arbitrarily. For example, if we want T_{new} in between T_{k-2} and T_{k-1}. then, we will take the average of T_{k-2},T_{k-1} as $T_{new} = (T_{k-2} + T_{k-1})/2$. After finding this new value, we merge it in Equation (3.2) to become

$$\boldsymbol{T} = \{T_0 = ... = T_d, T_{d+1}, ..., T_{k-2}, T_{new}, T_{k-1}, T_k, T_{k+1} =, ..., = T_{k+d+1}\} \tag{3.3}$$

So Equation (3.3) is the final knot vector. The last step is to find the control points. The system of equation can be setup in many ways, but for numerical stability, we compute the system of equation in (3.4) where D_0^1 and D_0^2 represent the first and second derivative values at parameter t_0, D_1^1 represent the first derivative value at parameter t_1 and so on. Similarly, D_n^1 and D_n^2 represent the first and second derivative values at parameter t_n. A relation between B-splines of consecutive degrees in a stable and efficient fashion is discussed in Cox (1972).

By solving the system of equations in (3.4), we find the values of control points. For solving this system, we also need all derivative values. As the shape of curve depends on derivative values along with parameter values, changing the values of derivative yield different shapes of curves. So, the user can choose different choices of derivative values according to his needs (Piegl and Tiller, 2012).

$$
\begin{bmatrix}
N_{0,d}(t_0) & N_{1,d}(t_0) & \cdots & N_{m,d}(t_0) \\
N_{0,d}^{(1)}(t_0) & N_{1,d}^{(1)}(t_0) & \cdots & N_{m,d}^{(1)}(t_0) \\
N_{0,d}^{(2)}(t_0) & N_{1,d}^{(2)}(t_0) & \cdots & N_{m,d}^{(2)}(t_0) \\
N_{0,d}(t_1) & N_{1,d}(t_1) & \cdots & N_{m,d}(t_1) \\
N_{0,d}^{(1)}(t_1) & N_{1,d}^{(1)}(t_1) & \cdots & N_{m,d}^{(1)}(t_1) \\
\vdots & \vdots & \ddots & \vdots \\
N_{0,d}(t_{n-2}) & N_{1,d}(t_{n-2}) & \cdots & N_{m,d}(t_{n-2}) \\
N_{0,d}^{(1)}(t_{n-2}) & N_{1,d}^{(1)}(t_{n-2}) & \cdots & N_{m,d}^{(1)}(t_{n-2}) \\
N_{0,d}^{(1)}(t_{n-1}) & N_{1,d}^{(1)}(t_{n-1}) & \cdots & N_{m,d}^{(1)}(t_{n-1}) \\
N_{0,d}(t_{n-1}) & N_{1,d}(t_{n-1}) & \cdots & N_{m,d}(t_{n-1}) \\
N_{0,d}^{(2)}(t_n) & N_{1,d}^{(2)}(t_n) & \cdots & N_{m,d}^{(2)}(t_n) \\
N_{0,d}^{(1)}(t_n) & N_{1,d}^{(1)}(t_n) & \cdots & N_{m,d}^{(1)}(t_n) \\
N_{0,d}(t_n) & N_{1,d}(t_n) & \cdots & N_{m,d}(t_n)
\end{bmatrix}
\begin{bmatrix}
P_0 \\ P_1 \\ \vdots \\ P_{m-1} \\ P_m
\end{bmatrix}
=
\begin{bmatrix}
Q_0 \\ D_0^1 \\ D_0^2 \\ Q_1 \\ D_1^1 \\ \vdots \\ Q_{n-2} \\ D_{n-2}^1 \\ D_{n-1}^1 \\ Q_{n-1} \\ D_n^2 \\ D_n^1 \\ Q_n
\end{bmatrix}
\tag{3.4}
$$

3.2 Experimental results and discussion

In this section, we present data points, results and analysis with previous research related to B-Spline curve interpolation. We are using three sets of data points as shown in Table 3.1. Data Set 1 and Data Set 3 are generated data sets. These two data sets are simple. Data set 2 is taken from (Haron et al., 2012) which is a type of linear data set. There are 5 data points in Data Set 1 and 3, 8 data point for Data Set 2.

Parameters and knot distribution of three set of data points are presented in Figure 3.1, 3.4 and 3.7. The red circle represent the knots and the blue circle represent the parameters. The parameter values are obtained with exponential parameterization method by using Equation (2.6) and the knots in Figure 3.1(a), 3.4(a) and 3.7(a) are obtained by knot averaging method suggested by De Boor (2001) while in Figure 3.1(b), 3.4(b) and 3.7(b) are obtained by using Algorithm 1.

Table 3.1: Data points used for generating results by the proposed method

Data set 1		
i	x	y
1	1.4	2.8
2	3.8	5.4
3	6.2	6.1
4	7.2	4.2
5	11.0	3.3
Data set 2		
1	0	9
2	0	5
3	0	3
4	0	1
5	1	0
6	3	0
7	5	0
8	9	0
Data set 3		
1	1	2
2	2	1
3	3	3
4	4	1
5	5	2

In many figures we are using the word general method, which means that the result obtained by Exponential B-spline parameterization method at $\alpha = 0.8$. In previous study, the researcher used the method of knot insertion but in this case in Figure