SIDE SENSITIVE GROUP RUNS DOUBLE SAMPLING CHART FOR DETECTING MEAN SHIFTS

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SIDE SENSITIVE GROUP RUNS DOUBLE SAMPLING CHART FOR DETECTING MEAN SHIFTS

by

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LIST OF ABBREVIATIONS

$ANOS$	Average number of observations to signal
$ANOS_0$	In-control $ANOS$
$ANOS_1$	Out-of-control $ANOS$
ARL	Average run length
ARL_0	In-control ARL
ARL_1	Out-of-control ARL
ASS	Average sample size
ASS_0	In-control ASS
CL	Center line
CRL	Conforming run length
DS	Double sampling
$EANOS$	Expected average number of observations to signal
$EANOS_0$	In-control $EANOS$
$EANOS_1$	Out-of-control $EANOS$
EWMA	Exponentially weighted moving average
GR	Group runs
LCL	Lower control limit
SDS	Synthetic double sampling
SQC	Statistical Quality Control
SSGR	Side sensitive group runs
SSGRDS	Side sensitive group runs double sampling
tpm	Transition probability matrix

UCL	Upper control limit
$\bar{X}$	Sample mean

LIST OF NOTATIONS

$f(\delta)$	Probability density function of the standardized shift size δ
δ_{opt}	Standardized mean shift where a quick detection is important
$(\delta_{\min}, \delta_{\max})$	Standardized mean shift interval where a quick detection is important
μ_0	In-control process mean
μ_1	Out-of-control process mean
σ	Process standard deviation
$\Phi(\,.\,)$	Standard normal distribution function
n	Sample size

Notations for the DS chart

P_a	Probability that the process is in-control
P_{a1}	Probability that the process is in-control based on the first sample
P_{a2}	Probability that the process is in-control based on the combined samples

Notations for the synthetic chart

$L_{\bar{X}/S}$	LCL of the $\bar{X}$ sub-chart
$U_{\bar{X}/S}$	UCL of the $\bar{X}$ sub-chart
L_{synth}	Lower limit of the CRL sub-chart
$\mathbf{1}$	Vector of ones
$\boldsymbol{I}$	Identity matrix

$\boldsymbol{s}$	Initial probability vector
$\boldsymbol{s}^*$	Cyclical steady state probability vector
$\boldsymbol{R}$	tpm for the transient states
$\boldsymbol{P}$	tpm with the absorbing state

Notations for the SDS chart

L_3	Lower limit of the CRL sub-chart
$\boldsymbol{R}$	tpm for the transient states
P_a	Probability that the sampling stage is conforming

Notations for the GR chart

$L_{\bar{X}/\mathrm{S}}$	LCL of the $\bar{X}$ sub-chart
$U_{\bar{X}/\mathrm{S}}$	UCL of the $\bar{X}$ sub-chart
k	Control limits' constant of the $\bar{X}$ sub-chart
L	Lower limit of the extended version of the CRL sub-chart

Notations for the SSGR chart

Y_r	Value of CRL_r
k_{ssg}	Control limits' constant for the $\bar{X}$ sub-chart
L_{ssg}	Lower limit of the extended version of the CRL sub-chart
$L_{\bar{X}/\mathrm{S}}$	LCL of the $\bar{X}$ sub-chart
$U_{\bar{X}/\mathrm{S}}$	UCL of the $\bar{X}$ sub-chart

Notations for the EWMA chart

λ	Smoothing constant
k_{EWMA}	Control limits' constant
$\boldsymbol{Q}$	tpm for the transient states
p	Number of subintervals between the EWMA chart's limits
H_j	midpoint of the jth subinterval

Notations for the SSGRDS chart

L_3	Lower limit of the extended version of the CRL sub-chart
$R^*(i, j)$	Transition probability from state i to state j
R^*	tpm for the transient states
$\boldsymbol{t}$	Cyclical steady state probability vector
α^*	Size of a Type-I error

CARTA PENSAMPELAN GANDA DUA LARIAN KUMPULAN DENGAN KEPEKAAN SISI UNTUK PENGESANAN ANJAKAN MIN

ABSTRAK

Dalam tesis ini, carta pensampelan ganda dua larian kumpulan dengan kepekaan sisi (carta SSGRDS) untuk mengesan anjakan dalam min proses dicadangkan. Carta SSGRDS adalah penambahbaikan ke atas carta larian kumpulan dengan kepekaan sisi (carta SSGR) yang dicadangkan oleh Gadre dan Rattihalli pada tahun 2007 serta carta pensampelan ganda dua (carta DS) yang dicadangkan oleh Daudin pada tahun 1992. Langkah-langkah pelaksanaan carta SSGRDS dan pengukur prestasi carta tersebut iaitu kriteria bilangan cerapan purata untuk berisyarat (*ANOS*) serta jangkaan bilangan cerapan purata untuk berisyarat (*EANOS*) dijelaskan dalam tesis ini. Formula *ANOS* dan *EANOS* keadaan sifar dan keadaan mantap diterbitkan dengan menggunakan pendekatan rantaian Markov. Jadual untuk parameter carta optimum $\left(n_1,\ n_2,\ L,\ L_1,\ L_2,\ L_3\right)$ bagi carta SSGRDS untuk gabungan *ANOS* dalam kawalan $\left(ANOS_0\right)$, *EANOS* dalam kawalan $\left(EANOS_0\right)$ dan saiz sampel purata dalam kawalan $\left(ASS_0\right)$ yang berlainan telah diberikan untuk tujuan praktikal demi memudahkan pelaksanaan carta tersebut. Prosedur pengoptimuman untuk (i) meminimumkan nilai *ANOS* terluar kawal bagi saiz anjakan min yang dipiawaikan $\left(\delta_{\text{opt}}\right)$ berdasarkan gabungan nilai $ANOS_0$ dan ASS_0 yang berlainan, dan (ii) meminimumkan nilai *EANOS* terluar kawal bagi suatu julat saiz anjakan min yang dipiawaikan $\left(\delta_{\min},\delta_{\max}\right)$ berdasarkan gabungan nilai $EANOS_0$ dan ASS_0 yang berlainan adalah diberikan. Program pengoptimumam untuk semua carta ditulis

dalam perisian ScicosLab, Mathematica dan Matlab. Satu kajian kes sebenar telah digunakan untuk menunjukkan pelaksanaan carta SSGRDS. Carta SSGRDS dibandingkan dengan carta-carta sedia ada sintetik, pensampelan ganda dua (DS), pensampelan ganda dua sintetik (SDS), larian kumpulan dengan kepekaan sisi (SSGR) dan purata bergerak berpemberat eksponen (EWMA) di bawah kes *ANOS* dan *EANOS* keadaan sifar dan keadaan mantap untuk nilai ASS_0 yang berlainan. Keputusan *ANOS* dan *EANOS* keadaan sifar dan keadaan mantap menunjukkan bahawa carta SSGRDS optimum secara amnya mempunyai prestasi yang baik berbanding dengan carta optimum lain. Carta EWMA didapati berprestasi lebih baik dalam pengesanan anjakan kecil, manakala carta DS adalah lebih baik dalam pengesanan anjakan besar. Carta SSGRDS mencapai keseimbangan dengan mempunyai kelajuan pengesanan yang memuaskan untuk anjakan kecil dan besar.

SIDE SENSITIVE GROUP RUNS DOUBLE SAMPLING CHART FOR DETECTING MEAN SHIFTS

ABSTRACT

In this thesis, a side sensitive group runs double sampling (SSGRDS) chart to detect shifts in the process mean is proposed. The SSGRDS chart is an improvement over the side sensitive group runs (SSGR) chart proposed by Gadre and Rattihalli in 2007 and the DS chart suggested by Daudin in 1992. The step-by-step implementation of the SSGRDS chart, as well as the performance measures of the chart, i.e. the average number of observations to signal (*ANOS*) and expected average number of observations to signal (*EANOS*) criteria are explained in the thesis. The zero state and steady state *ANOS* and *EANOS* formulae are derived using the Markov chain approach. Tables of optimal charting parameters $(n_1,\ n_2,\ L,\ L_1,\ L_2,\ L_3)$ of the SSGRDS chart for different combinations of the in-control *ANOS* $(ANOS_0)$, in-control *EANOS* $(EANOS_0)$ and in-control average sample size (ASS_0) are provided for practical reasons to facilitate the implementation of the chart. The optimization procedures in (i) minimizing the out-of-control *ANOS* value for a standardized size of the mean shift (δ_{opt}), based on different combinations of $ANOS_0$ and ASS_0 values, and (ii) minimizing the out-of-control *EANOS* value for a range of standardized sizes of the mean shifts $(\delta_{\min}, \delta_{\max})$, based on different combinations of $EANOS_0$ and ASS_0 values, are provided. Optimization programs for all the charts are written in the ScicosLab, Mathematica and Matlab software. A real case study is used to show the implementation of the SSGRDS chart. The SSGRDS chart is compared with the existing synthetic, double sampling (DS), synthetic double sampling (SDS), side sensitive group runs (SSGR) and exponentially weighted moving average (EWMA) charts, under the zero state and steady state *ANOS* and *EANOS* cases for different ASS_0 values. The zero state and steady state *ANOS* dan *EANOS* results show that, in general, the optimal SSGRDS chart has favourable performance compared to other optimal charts. The EWMA chart is found to perform better in detecting small shifts, while the DS chart is superior in detecting large shifts. The SSGRDS chart strikes a balance by having a considerable detection speed for small and large shifts.

CHAPTER 1
INTRODUCTION

1.1 Control charts

The control chart is one of the most efficient tools in Statistical Quality Control (SQC). Among the important tools in SQC are check sheet, Pareto chart, cause and effect diagram, defect concentration diagram, histogram, scatter diagram and control charts. The control chart was proposed by Dr. Walter A. Shewhart from the Bell Telephone Laboratories in the 1920s (Montgomery, 2009). During World War II, the world had noticed and realized the importance of controlling and improving its product quality. Therefore, industrial practitioners started to find the best way to apply control charts in process monitoring, in order to improve the quality of their products.

Control charts can be used to monitor either a manufacturing process or a non-manufacturing process. It helps the company to determine whether its process is in-control or otherwise. If the process is found to be out-of-control by a control chart, the company will need to find the assignable cause(s) so that corrective actions can be taken to bring the process back to the in-control state.

Till now, a lot of researches have been conducted to improve the effectiveness of control charts. Most of these researches are aimed at increasing the speed of control charts in detecting out-of-control signals as early as possible.

1.2 Basic control chart principles

A typical control chart is a graph that shows the quality characteristic of a sample versus the sample number or time. It contains a centrer line (CL), which is the average value of the said quality characteristic when the process is in-control. Two other lines that are drawn on the chart are the upper control limit (UCL) and lower control limit (LCL). These two lines are chosen based on some statistical considerations. Any process that has a quality characteristic that falls between the UCL and LCL is considered as in-control. On the other hand, a process with a quality characteristic that falls beyond the UCL/LCL interval is considered as out-of-control, where corrective actions have to be taken to eliminate the assignable causes. The points on the control chart are always connected by straight line segments, in order to show the pattern of the process change over time.

The control limits of a control chart have to be determined properly. If control limits that are far from the center line are chosen, the probability of the Type-I error (declaring the process as out-of-control when it is actually in-control) can be reduced. Unfortunately, the probability of the Type-II error (declaring the process as in-control when it is actually out-of-control) will be increased.

Additionally, a process can be considered as out-of-control if the points plotted on the chart show a non-random pattern, even though all points are within the control limits. An example of a non-random pattern is when 15 out of the last 20 points are located between the CL and LCL while another 5 points are located between the CL and UCL. In a normal case, we should always see a random pattern on a control chart. A non-random pattern may be an indication that the process is out-of-control. Investigations should be taken to find the reasons why the chart shows a non-random pattern, in order to improve the process quality.

A control chart is a test of the statistical hypotheses, $H_0 : \mu_1 = \mu_0$ versus $H_1 : \mu_1 \neq \mu_0$. Assume that we are plotting the sample means, $\overline{X}$ of a process on a control chart. If all the plotted points, $\overline{X}$, fall within the control limits, we say that the process is in-control. In other words, we say that $\mu_1 = \mu_0$, which means that we cannot reject the null hypothesis of statistical control. However, if a plotted point, $\overline{X}$

is beyond the control limits, the process is out-of-control, where we reject the null hypothesis and conclude that $\mu_1 \neq \mu_0$. In this case, we reject the hypothesis of statistical control.

Choices of sample size, control limits and sampling frequency are important criteria to consider in the design of control charts. The selections of statistical criteria and past industrial experience have brought about the guidelines in designing a control chart. More recently, practitioners prioritise designing control charts from the economic point of view. They consider the cost of sampling, losses incurred by allowing a non-conforming process to be carried on, cost of investigating an assignable cause and so on.

In addition, a chart's performance needs to be evaluated for the zero state and steady state cases. The zero state case refers to the situation where the process shift occurs at the beginning of process monitoring while the steady state case involves the situation when the process shift happens at a random time in the future after process monitoring begins.

1.3 Univariate control charts

There are two types of quality characteristics that are inspected in an industrial process. One of them is when the dimensions of the quality characteristic can be measured and represented on a continuous numerical scale, such as diameter of a wire and thickness of a paper. This type of quality characteristic is categorised as variables data. There are quality characteristics that cannot be represented on a continuous scale of measurement. The process for this type of quality characteristic is classified as conforming or non-conforming, depending on whether it meets certain requirements.

For variables data, measures of central tendency and variability are good enough to describe the quality characteristic. Variables control charts monitor the central tendency and process variability. The $\bar{X}$ chart is the most common chart used in monitoring central tendency while process variability is monitored using the *R* and *S* charts. The *R* chart is based on the sample range statistic while the *S* chart is based on the sample standard deviation statistic. Most of the time, we need to use both control charts, each for the mean and variance, to monitor a process.

Univariate control charts are used when only one quality characteristic is monitored. Researches have been carried out to improve the existing univariate control charts while proposing new charts that are more efficient than existing ones. Control charts for detecting shifts in the process mean can be easily found in the literature. Recent works on univariate control charts for the process mean are the new dual CUSUM mean chart (Haq and Bibi, 2019), modified side sensitive synthetic chart (Shongwe and Graham, 2018), new adaptive EWMA control chart using auxiliary information for monitoring the process mean (Haq, 2018) and new synthetic EWMA and synthetic CUSUM control charts (Haq et al., 2014).

1.4 Research motivations

The SSGR chart for the mean proposed by Gadre and Rattihalli (2007) was found to outperform the Shewhart and synthetic charts for the mean in detecting small and moderate shifts. It was also found by Daudin (1992) that the DS chart for the mean surpasses the Shewhart chart for the mean. Owing to the effectiveness of the SSGR and DS charts in detecting process mean shifts, it is believed that an even more sensitive chart in the detection of mean shifts can be developed by combining the SSGR and DS charts. The aim in this thesis is to combine the SSGR and DS charts in developing the SSGRDS chart.

1.5 Objectives of the thesis

The objectives of the thesis are as follows:

(i) To propose the side sensitive group runs double sampling (*SSGRDS*) chart.

(ii) To compute the optimal charting parameters of the proposed *SSGRDS* chart.

1.6 Scope of the research

In this thesis, the SSGRDS chart iks developed. In order to study the performance of the SSGRDS chart, formulae are derived to compute the zero state and steady state average number of observations to signal (*ANOS*) and expected average number of observations to signal (*EANOS*) values. In addition, optimization procedures are provided to compute the optimal parameters of the SSGRDS chart in minimizing the out-of-control zero state and steady state *ANOS* and *EANOS* values.

1.7 Organization of the thesis

Chapter 1 provides some preliminaries on control charts and their basic principles. This chapter also explains the research motivations, objectives of the thesis and scope of the research. In Chapter 2, the performance measures of control charts, such as average run length (*ARL*), *ANOS* and *EANOS* criteria will be presented. Additionally, control charts related to the *SSGRDS* chart will be reviewed in Chapter 2. The implementation procedure of the *SSGRDS* chart will be presented in Chapter 3. The derivation of the *ANOS* and *EANOS* formulae for both zero state and steady state cases, as well as optimal designs of the *SSGRDS* chart are explained in Chapter 3. An illustrative example that adopts real life data is also provided in Chapter 3. In Chapter 4, the performance of the proposed *SSGRDS* chart is compared with that of existing charts, such as synthetic, double sampling (*DS*), synthetic double sampling (*SDS*), side sensitive group runs (*SSGR*) and exponentially weighted moving average (*EWMA*) charts, for both zero state and steady state cases. Finally, conclusions and suggestions for further research are discussed in Chapter 5.

CHAPTER 2
SOME PRELIMINARIES AND A REVIEW OF SELECTED CONTROL CHARTS FOR THE MEAN

2.1 Introduction

It is never an easy task to develop a new control chart which can perform better than the existing control charts. Researchers tend to combine two control charts with considerable performance to develop a new and better control chart. Modifications will be made to the existing control charting statistics in deriving new charting statistics that are applicable to the newly developed chart. New features will be added to the existing charts to develop a new chart that performs better. The performance of the new chart will be compared with existing charts using different performance measures to test the ability of the former in the detection of shifts. The objective is to develop a new chart that allows process shifts to be detected quicker.

The group runs (GR) chart was introduced by Gadre and Rattihailli (2004). Since then, numerous extensions were made on the GR charts. For example, the modified group runs (MGR) chart by Gadre and Rattihalli (2006) and the side sensitive group runs (SSGR) chart by Gadre and Rattihalli (2007) are two newer versions of GR type charts that improve the performance of the original GR chart. Additionally, Lim et al. (2015) proposed the economic and economic-statistical designs of the SSGR chart based on the *ARL* and expected average run length (*EARL*) criteria for minimizing the cost function. The SSGR chart based on estimated process parameters was proposed by You et al. (2015). The performances of GR and SSGR charts were compared by Yew et al. (2016) based on the average time to signal (*ATS*) criterion. The SSGR chart for monitoring the coefficient of variation was proposed by You et al. (2016), while the generalized GR chart that increases the speed in detecting shifts in the time-between-events was suggested by Fang et al. (2016).

Gadre and Kakade (2016) developed the multivariate GR and MGR charts for monitoring the mean vector.

Chong et al. (2017a) proposed a group runs double sampling np chart to detect increases in the fraction of non-conforming units. The performance of the GR chart for the mean was improved by Chong et al. (2017b) who proposed the GR chart based on the revised *m*-of-*k* runs rule. You (2018) extended the SSGR chart by considering the optimal design of the chart with estimated process parameters based on the EARL criterion. The performances of the SSGR chart were further examined by You et al. (2018) when the shift sizes are known and unknown. In addition, You (2018) investigated the performances of the SSGR chart with known and estimated process parameters based on the *ARL* and *EARL* criteria. Mim et al. (2019) proposed a side-sensitive modified group runs (SSMGR) chart with auxiliary information to detect process mean shifts. An optimal design of the SSMGR chart was studied by Chong et al. (2019) when process parameters are estimated. Saha et al. (2019) proposed the SSGR chart for detecting mean shifts using auxiliary information, where this chart outperforms the basic SSGR chart. Optimal designs of the MGR $\bar{X}$ chart when process paramters are estimated were suggested by Chong et al. (2020).

2.2 Measures of performance evaluation of control charts

All newly developed control charts must be evaluated to determine their ability in detecting process shifts. Different types of performance measures have been developed to evaluate control charts' performances. Several performance measures have been considered to evaluate the performances of the charts considered in this thesis. These performance measures are *ARL*, *ANOS* and *EANOS*. The following sections explain these performance measures.

2.2.1 Average run length (*ARL*)

The average run length (*ARL*) is one of the performance measures for control charts. It represents the average number of sample points that should be plotted on a control chart before a point indicates an out-of-control situation. The *ARL* of a Shewhart chart is obtained as

$$ARL = \frac{1}{p}, \tag{2.1}$$

where p is the probability that a sample point is plotted beyond the control limits of the chart (Montgomery, 2009).

2.2.2 Average number of observations to signal (*ANOS*)

The average number of observations to signal $(ANOS)$ is the expected number of observations taken from the start of process monitoring until an out-of-control signal is detected by a chart. The *ANOS* can be computed as (Khoo et al., 2011)

$$ANOS = ARL \times ASS, \tag{2.2}$$

where ASS is the average sample size.

2.2.3 Expected average number of observations to signal (*EANOS*)

The expected average number of observations to signal (*EANOS*) is the expectation of the *ANOS* over the standardized mean shift interval $(\delta_{\min}, \delta_{\max})$. When the exact shift size cannot be specified, the *EANOS* criterion should be used in place of the *ANOS* criterion. The *EANOS* value for the shift interval $(\delta_{\min}, \delta_{\max})$ is computed as follows (Khoo et al., 2015):

$$EANOS = \int_{\delta_{\min}}^{\delta_{\max}} ANOS(\delta) f(\delta) \mathrm{d}\delta. \tag{2.3}$$

Note that $ANOS(\delta)$ in Equation (2.3) is the $ANOS$ value for the shift size, δ and $f(\delta)$ is the pdf of δ. It is assumed that each shift size, δ, in the interval $\delta_{\min} \le \delta \le \delta_{\max}$ occurs with equal probability, which means that δ is uniformly distributed, i.e. $\delta \sim U(\delta_{\min}, \delta_{\max})$ (Sparks, 2000 and Wu et al., 2008). Consequently, Equation (2.3) becomes

$$EANOS = \frac{1}{\delta_{\max} - \delta_{\min}} \int_{\delta_{\min}}^{\delta_{\max}} ANOS(\delta)\,\mathrm{d}\delta \tag{2.4}$$

because $f(\delta) = \frac{1}{\delta_{\max} - \delta_{\min}}$ (Khoo et al., 2015). As the integral in Equation (2.4) is difficult to be solved, the Gauss-Legendre quadrature is used to approximate it.

2.3 A review on related charts

This section reviews the control charts for the process mean that are related to the SSGRDS chart proposed in this thesis. The charts reviewed in this section are the double sampling, synthetic, synthetic double sampling, group runs, side sensitive group runs and exponentially weighted moving average charts.

2.3.1 Double sampling chart

Daudin (1992) proposed the double sampling (DS) $\bar{X}$ chart. Compared with the Shewhart $\bar{X}$ chart, the DS $\bar{X}$ chart gives a better statistical efficiency in detecting process mean shifts. By referring to Figure 2.1, the implementation of the DS $\bar{X}$ chart is as follows:

Step 1. By using the approach mentioned in Daudin (1992), the charting parameters n_1, n_2, L, L_1 and L_2 are determined.

Step 2. Take the first sample with n_1 observations. Compute $\bar{X}_1 = \sum_{j=1}^{n_1} \frac{X_{1j}}{n_1}$ followed by $Z_1 = \frac{\bar{X}_1 - \mu_0}{\sigma / \sqrt{n_1}}$, where X_{1j} for j = 1, 2, …, n_1, are the observations in the first sample. It is assumed that the process being monitored comes from a population with a target mean value, μ_0 and an in-control standard deviation, σ.

Step 3. If Z_1 falls in the interval $I_1 \in [-L_1, L_1]$, the process is in-control. Then the control flow goes back to Step 2.

Step 4. If Z_1 falls in the interval $I_3 \in (-\infty, -L) \cup (L, \infty)$, the process is out-of-control. Then the control flow proceeds to Step 8.

Step 5. If Z_1 falls in the interval $I_2 \in [-L, -L_1) \cup (L_1, L]$, a second sample is taken from the same population where the first sample is obtained. Then, $\bar{X}_2 = \sum_{j=1}^{n_2} \frac{X_{2j}}{n_2}$ is computed, where X_{2j} for j = 1, 2, …, n_2, are the observations for the second sample.

Step 6. Compute $\bar{\bar{X}} = \frac{n_1 \bar{X}_1 + n_2 \bar{X}_2}{n_1 + n_2}$ followed by $Z = \frac{\bar{\bar{X}} - \mu_0}{\sigma / \sqrt{n_1 + n_2}}$.

Step 7. If Z is in the interval $I_4 \in (-L_2, L_2)$, the process is in-control and the control flow goes back to Step 2. Otherwise, the process is out-of-control and the control flow proceeds to Step 8.

Step 8. When the process is declared as out-of-control, corrective measures have to be taken to remove the assignable cause(s) so that the process will return to the in-control condition again, following which the control flow goes back to Step 2.

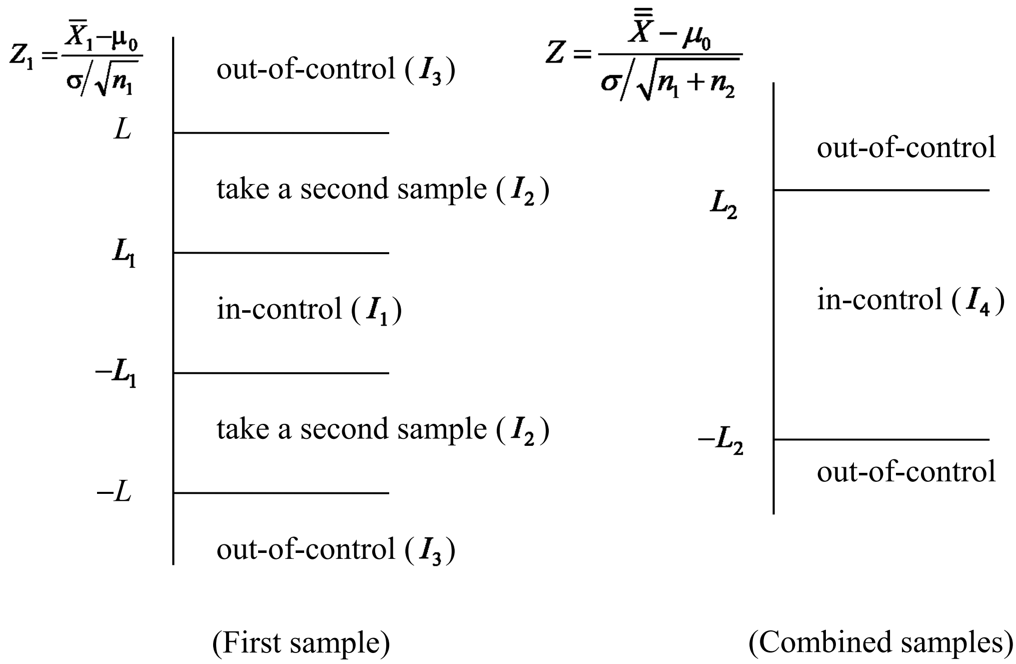

Figure 2.1. Daudin's DS $\bar{X}$ control charting procedure

According to Daudin (1992), the probability that the process is in-control, P_a, is given by $P_a = P_{a1} + P_{a2}$, where P_{a1} is the probability that the process is in-control based on the first sample and P_{a2} is the probability that the process is in-control based on the combined samples. Note that (Daudin, 1992)

$$P_{a1} = \Pr\left(Z_1 \in I_1\right) = \Phi\left(L_1 + \delta\sqrt{n_1}\right) - \Phi\left(-L_1 + \delta\sqrt{n_1}\right), \tag{2.5}$$

where $\Phi(\ .\ )$ is the standard normal distribution function and $\delta = \left(\mu_1 - \mu_0\right)/\sigma$. Here, μ_1 is the out-of-control process mean. Additionally, P_{a2} is obtained using the following equation:

$$P_{a2} = \Pr\left(Z \in I_4 \text{ and } Z_1 \in I_2\right) = \int_{z \in I_2^*} P_4 \phi(z)\, dz, \tag{2.6}$$

where $I_2^* = \left[-L + \delta\sqrt{n_1}\ ,\ -L_1 + \delta\sqrt{n_1}\right) \cup \left(L_1 + \delta\sqrt{n_1}\ , L + \delta\sqrt{n_1}\right]$ and

$$P_4 = \Pr\left(Z \in I_4 \middle| Z_1 = z\right) = \Phi\left(cL_2 + rc\delta - z\sqrt{\frac{n_1}{n_2}}\right) - \Phi\left(-cL_2 + rc\delta - z\sqrt{\frac{n_1}{n_2}}\right). \tag{2.7}$$

In Equation (2.7), $r=\sqrt{n_1+n_2}$ and $c=\sqrt{(n_1+n_2)/n_2}$ (Daudin, 1992).

Then the *ARL* of the DS $\bar{X}$ chart is computed as (Daudin, 1992)

$$ARL=\frac{1}{1-P_a} \tag{2.8}$$

and the average sample size (*ASS*) at each sampling stage is obtained as

$$ASS=n_1+n_2P_2, \tag{2.9}$$

where P_2 is the probability of taking the second sample and it is given by

$$P_2=\Pr(Z_1\in I_2)=\Phi\left(L+\delta\sqrt{n_1}\right)-\Phi\left(L_1+\delta\sqrt{n_1}\right)+\Phi\left(-L_1+\delta\sqrt{n_1}\right)-\Phi\left(-L+\delta\sqrt{n_1}\right). \tag{2.10}$$

Multiplying Equation (2.8) with Equation (2.9) gives

$$ANOS=\frac{n_1+n_2P_2}{1-P_a}. \tag{2.11}$$

The *EANOS* value of the DS $\bar{X}$ chart can be computed using Equation (2.4) by substituting $ANOS(\delta)$ in Equation (2.11), for the shift size δ into the former equation. Each sampling stage of the DS $\bar{X}$ chart may contain only the first sample or both the first and second samples. Note that the *ARL*, *ANOS* and *EANOS* formulae of the DS $\bar{X}$ chart are the same for both the zero state and steady state cases.

2.3.2 Synthetic chart

Wu and Spedding (2000) proposed the synthetic $\bar{X}$ chart which combines the $\bar{X}$ and conforming run length (CRL) sub-charts. The procedure for implementing the synthetic $\bar{X}$ chart is as follows:

Step 1. A random sample of n observations is taken and sample mean, $\bar{X}$ is calculated.

Step 2. Obtain the value of the limits $L_{\bar{X}/S}$ and $U_{\bar{X}/S}$ of the $\bar{X}$ sub-chart, where $L_{\bar{X}/S} = \mu_0 - \frac{k_{synth}\sigma}{\sqrt{n}}$ and $U_{\bar{X}/S} = \mu_0 + \frac{k_{synth}\sigma}{\sqrt{n}}$, k_{synth} is the control limit coefficient and it is usually a small decimal value. If $L_{\bar{X}/S} < \bar{X} < U_{\bar{X}/S}$, the process is conforming and the control flow goes back to Step 1. If not, the process is non-conforming and the control flow proceeds to Step 3.

Step 3. Obtain the CRL value. This value is the number of the conforming samples between the two consecutive non-conforming samples (including the current non-conforming sample).

Step 4. The process is declared as in-control if CRL > L_{synth}. Otherwise, the process is out-of-control and corrective measures have to be carried out to identify and remove the assignable cause(s). Here, L_{synth} is the lower limit of the CRL sub-chart whose value is an integer.

The Markov chain approach is used to compute the *ANOS* of the synthetic chart. Assume that $B = \Pr\left(L_{\bar{X}/S} < \bar{X} < U_{\bar{X}/S}\right)$ and $C = 1 - B$. If the value of L_{synth} is known, the $\left(L_{synth}+1\right)\times\left(L_{synth}+1\right)$ transition probability matrix (tpm) for the transient (in-control) states of the synthetic $\bar{X}$ chart, ***R*** can be constructed using the method suggested by Davis and Woodall (2002) as follows:

(i) B is placed in the first row first column while C is placed in the first row second column.

(ii) B is placed in the first column in the last row.

(iii) All the entries above the diagonal of the tpm, ***R*** are B.

(iv) The values of the other entries are zeros.

For example, when $L_{synth} = 2$, the tpm for the transient states of the synthetic $\bar{X}$ chart is as follows:

$$\boldsymbol{R} = \begin{pmatrix} B & C & 0 \\ 0 & 0 & B \\ B & 0 & 0 \end{pmatrix}. \tag{2.12}$$

For computing the *ARL*, the following equation is applied:

$$ARL = \boldsymbol{s}'\left(\boldsymbol{I} - \boldsymbol{R}\right)^{-1}\boldsymbol{1}. \tag{2.13}$$

In Equation (2.13), **1** is a $\left(L_{\text{synth}} + 1\right) \times 1$ vector of ones, ***I*** is the $\left(L_{\text{synth}} + 1\right) \times \left(L_{\text{synth}} + 1\right)$ identity matrix and ***s*** = (0, 1, 0, … , 0)′ is the $\left(L_{\text{synth}} + 1\right) \times 1$ initial probability vector. The *ANOS* of the synthetic $\bar{X}$ chart is computed using Equation (2.2) by letting $ASS = n$.

To compute the steady state *ANOS*, $\boldsymbol{s}^*$ which is the cyclical steady state probability vector is used to replace ***s*** in Equation (2.13). $\boldsymbol{s}^*$ is obtained by solving $\boldsymbol{r} = \boldsymbol{P}'\boldsymbol{r}$, subjected to $\boldsymbol{1}'\boldsymbol{r} = 1$, where ***P*** is the tpm with the absorbing state. ***P*** is obtained from ***R*** by adding a row and a column (corresponding to the absorbing state) below the last row and after the last column of ***R***. Then, $\boldsymbol{s}^* = \left(\boldsymbol{1}'\boldsymbol{q}\right)^{-1}\boldsymbol{q}$, where ***q*** is obtained from ***r*** by deleting the entry corresponding to the absorbing state.

To compute the zero state and steady state *EANOS* of the synthetic $\bar{X}$ chart, Equation (2.4) is applied, where $ANOS\left(\delta\right)$ in the aforementioned equation is replaced with the corresponding zero state and steady state $ANOS\left(\delta\right)$ of the synthetic $\bar{X}$ chart.

2.3.3 Synthetic double sampling chart

The synthetic double sampling (SDS) $\bar{X}$ chart was proposed by Khoo et al. (2011). This chart combines the double sampling sub-chart and the CRL sub-chart. The construction of the SDS $\bar{X}$ chart is the same as that of the DS $\bar{X}$ chart in Steps 1 to 7 but with some minor modifications. In Step 1, another parameter, L_3, i.e. the lower limit of the CRL sub-chart is defined. In Steps 3, 4 and 7, the term 'process is in-control' and 'process is out-of-control' are replaced with 'sampling stage is conforming' and 'sampling stage is non-conforming', respectively. The additional steps are as follows:

Step 8. Obtain the CRL value. This value is obtained by counting the number of sampling stages between the current (included in the count) and previous non-conforming sampling stages (excluded in the count).

Step 9. If $\text{CRL} > L_3$, the process is in-control and the control flow goes back to Step 2. Otherwise, the process is out-of-control and corrective measures should be taken to remove the assignable cause(s) before returning to Step 2 again.

The SDS $\bar{X}$ chart uses the same tpm for the transient states, ***R***, as that for the synthetic $\bar{X}$ chart, explained in Section 2.3.2. The entries in ***R*** are computed as $B = P_a$ and $C = 1 - B$. In the DS $\bar{X}$ chart, P_a denotes the probability that the process is in-control. In the SDS $\bar{X}$ chart, P_a, computed using the same formula as that for the DS $\bar{X}$ chart is defined as the probability that the sampling stage is conforming. Both the zero state and steady state *ARLs* for the SDS $\bar{X}$ chart are obtained using the Markov chain approach discussed in Section 2.3.2 for the synthetic $\bar{X}$ chart. The zero state and steady state *ANOS*s can be obtained by using Equation (2.2), where *ASS* can be obtained using Equation (2.9). Additionally, the zero state and steady state *EANOS*s of the SDS $\bar{X}$ chart can be obtained using Equation (2.4) by adopting the corresponding $ANOS(\delta)$ in the formula.

2.3.4 Group runs chart

Gadre and Rattihalli (2004) proposed the group runs (GR) $\bar{X}$ chart. This chart combines the $\bar{X}$ sub-chart and an extended version of the conforming run length (CRL) sub-chart. The GR $\bar{X}$ chart is implemented as follows:

Step 1. A sample of n observations is taken and the sample mean $\bar{X}$ is computed.

Step 2. If the sample mean $\bar{X}$ falls between the limits $L_{\bar{X}/\mathrm{S}} = \mu_0 - \frac{k\sigma}{\sqrt{n}}$ and $U_{\bar{X}/\mathrm{S}} = \mu_0 + \frac{k\sigma}{\sqrt{n}}$ of the $\bar{X}$ sub-chart, the sample is conforming. Otherwise, the sample is non-conforming. Note that, μ_0, k, σ and n are the in-control process mean, control limits constant of the $\bar{X}$ sub-chart, in-control standard deviation and sample size, respectively. The control flow returns to Step 1 if the sample is conforming. If the sample is non-conforming, the control flow proceeds to Step 3.

Step 3. Count the number of conforming samples between two non-conforming ones and denote this count as *Y*. The process is declared as out-of-control if $Y_1 \leq L$ (L is the lower limit of the extended version of the *CRL* sub-chart), or for some $r > 1$, $Y_r \leq L$ and $Y_{r+1} \leq L$ for the first time. Here, Y_r represents the r^{th} *CRL* value. If the process is out-of-control, proceed to Step 4.

Step 4. Corrective actions are taken to get rid of the assignable cause(s) so that the process returns to the in-control condition.

The zero state *ARL* of the GR $\bar{X}$ chart is calculated using the following formulae (Gadre and Rattihalli, 2004):

$$ARL = \frac{1}{P\left[1-(1-P)^L\right]^2}, \tag{2.14}$$

where

$$P = 1 - \Phi\left(k - \delta\sqrt{n}\right) + \Phi\left(-k - \delta\sqrt{n}\right). \tag{2.15}$$

In Equation (2.15), δ refers to the size of the standardized process mean shift from μ_0 (target value) to $\mu_1 = \mu_0 \pm \delta\sigma$ (out-of-control value), while $\Phi(\ \cdot\ )$ denotes the standard normal cumulative distribution function. To obtain the zero state *ANOS*, Equation (2.2) is used, where $ASS = n$ for the GR $\bar{X}$ chart.

Gadre and Rattihalli (2004) proposed an optimization algorithm to obtain the optimal parameters of the GR $\bar{X}$ chart, i.e. n, k and L, in minimizing the out-of-control *ARL* (ARL_1), for the shift size δ, based on a desired in-control *ARL* (ARL_0) value.

For computing the steady state *ARL* and *ANOS* values, the Markov chain procedure given in Gadre and Rattihalli (2004) can be employed. Additionally, for computing the zero state and steady state *EANOS* values of the GR $\bar{X}$ chart, Equation (2.4) can be adopted.

2.3.5 Side sensitive group runs chart

Gadre and Rattihalli (2007) proposed the side sensitive group runs (SSGR) $\bar{X}$ chart. This chart is a combination of the $\bar{X}$ sub-chart and an extended version of the CRL sub-chart. The SSGR $\bar{X}$ chart improves the GR $\bar{X}$ chart's performance by having an extra feature, i.e. the side sensitivity feature. The SSGR $\bar{X}$ chart outperforms the Shewhart $\bar{X}$, synthetic $\bar{X}$ and GR $\bar{X}$ charts in detecting process mean shifts. Note that similar to the synthetic $\bar{X}$ and GR $\bar{X}$ charts, CRL for the SSGR $\bar{X}$ chart is the number of conforming samples between two consecutive non-conforming ones (including the ending non-conforming sample). Let Y_r represent the value of CRL_r, for the *r*th non-conforming sample.

The implementation of the SSGR $\bar{X}$ chart is as follows (Gadre and Rattihalli, 2007):

Step 1. n successive items constituting a sample are taken. The sample mean $\bar{X}$ is computed.

Step 2. If the sample mean $\bar{X}$ falls between the limits $L_{\bar{X}/S} = \mu_0 - \frac{k_{ssg}\sigma}{\sqrt{n}}$ and $U_{\bar{X}/S} = \mu_0 + \frac{k_{ssg}\sigma}{\sqrt{n}}$ of the $\bar{X}$ sub-chart, the sample is conforming. Otherwise, the sample is declared as non-conforming. Hence, k_{ssg} is the control limit coefficient of the $\bar{X}$ sub-chart. The control flow returns to Step 1 if the sample is conforming. If the sample is non-conforming, the control flow proceeds to Step 3.

Step 3 The process is declared as out-of-control if $Y_1 \leq L_{\text{ssg}}$ (L_{ssg} is the lower limit of the extended version of the CRL sub-chart), or for some $r > 1$, $Y_r \leq L_{ssg}$ and $Y_{r+1} \leq L_{ssg}$ for the first time and the corresponding sample mean values lie on the same side of the target value, μ_0 on the $\bar{X}$ sub-chart. When the process is out-of-control, the control flow proceeds to Step 4.

Step 4. Corrective actions are taken to get rid of the assignable cause(s) so that the process returns to the in-control condition.

The zero state ARL of the SSGR $\bar{X}$ chart can be calculated using the following formula (Gadre and Rattihalli, 2007):

$$ARL = \frac{\left[1-\alpha\left(1-\alpha\right)A^2\right]}{PA^2\left[1+\alpha\left(1-\alpha\right)\left(A-2\right)\right]}, \tag{2.16}$$

where

$$P = 1-\Phi\left(k_{ssg}-\delta\sqrt{n}\right)+\Phi\left(-k_{ssg}-\delta\sqrt{n}\right), \tag{2.17}$$

$$\alpha = \Pr\left[\bar{X} > U_{\bar{X}/\mathrm{S}} \middle| \left(\bar{X} > U_{\bar{X}/\mathrm{S}} \text{ or } \bar{X} < L_{\bar{X}/\mathrm{S}}\right)\right] = \frac{1-\Phi\left(k_{ssg}-\delta\sqrt{n}\right)}{P} \tag{2.18}$$

and

$$A = \Pr\left(Y_r \le L_{ssg}\right) = 1-\left(1-P\right)^{L_{ssg}}. \tag{2.19}$$

By using Equation (2.2), the zero state *ANOS* of the SSGR $\bar{X}$ chart can be obtained, where $ASS = n$ in this case.

Gadre and Rattihalli (2007) proposed an optimization algorithm to compute the optimal parameters of the SSGR $\bar{X}$ chart, i.e. n, k_{ssg} and L_{ssg}, to minimize ARL_1 for the shift size, δ, based on a specified ARL_0 value.

For computing the steady state *ARL* and *ANOS* values, Gadre and Rattihalli (2007) suggested the Markov chain procedure. Additionally, to compute the zero state and steady state *EANOS* values, Equation (2.4) can be used.

2.3.6 Exponentially weighted moving average chart

The exponentially weighted moving average (EWMA) $\bar{X}$ chart statistic at sample *i* is defined as follows:

$$Z_i = \lambda \bar{X}_i + \left(1-\lambda\right)Z_{i-1} \text{ for } i = 1, 2, \ldots, \tag{2.20}$$

where λ $\left(0 < \lambda \le 1\right)$ is a smoothing constant and $Z_0 = \mu_0$. The control limits of the EWMA $\bar{X}$ chart are computed as

$$\text{LCL} = \mu_0 - k_{\text{EWMA}}\sigma\sqrt{\frac{\lambda}{(2-\lambda)n}} \tag{2.21}$$

and

$$\text{UCL} = \mu_0 + k_{\text{EWMA}}\sigma\sqrt{\frac{\lambda}{(2-\lambda)n}}, \tag{2.22}$$

where k_{EWMA} is predetermined to give the desired in-control and out-of-control performances.

Zhang et al. (2009) presented the following tpm for the transient states, $\boldsymbol{Q}$ of the EWMA $\bar{X}$ chart:

$$\boldsymbol{Q} = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1p} \\ Q_{21} & Q_{22} & \cdots & Q_{2p} \\ \vdots & \vdots & & \vdots \\ Q_{p1} & Q_{p2} & \cdots & Q_{pp} \end{pmatrix}, \tag{2.23}$$

where

$$Q_{ij} = \Phi\left[\left(\frac{H_j + a - (1-\lambda)H_i}{\lambda} - \delta\right)\sqrt{n}\right] - \Phi\left[\left(\frac{H_j - a - (1-\lambda)H_i}{\lambda} - \delta\right)\sqrt{n}\right]. \tag{2.24}$$

In Equation (2.21), $p = 2m+1$ is the number of subintervals (each of width $2a$) between the LCL and UCL and $a = \frac{\text{UCL} - \text{LCL}}{2p}$ (see Figure 2.2). Additionally, H_j, for $j = -m, -m+1, ..., -1, 0, 1, ..., m-1, m,$ represents the midpoint of the jth subinterval. The same Markov chain formulae adopted for the synthetic $\bar{X}$ chart is used to compute the zero state and steady state *ARL*s and *ANOS*s of the EWMA $\bar{X}$ chart but by replacing $\boldsymbol{R}$ in Equation (2.13) with $\boldsymbol{Q}$ in Equation (2.21). Both the zero state and steady state *EANOS*s of the EWMA $\bar{X}$ chart can be computed using Equation (2.4). Note that for the EWMA $\bar{X}$ chart, the initial probability vector is $\boldsymbol{s}$ =

(0, …, 0, 1, 0, …, 0)′ while the steady state probability vector, $\boldsymbol{s}^*$ is obtained using the method discussed in Section 2.3.2 for the synthetic $\bar{X}$ chart.

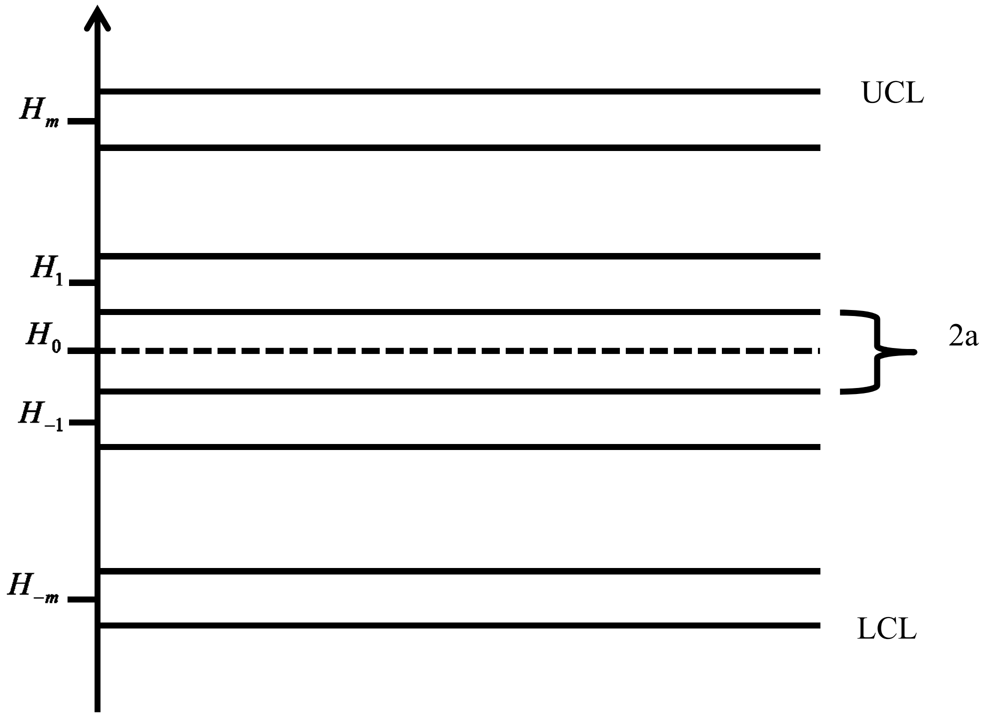

Figure 2.2 Interval between LCL and UCL divided into $2m$+1 subintervals, each of width $2a$.

2.4 Summary

Recent researches have shown that the performance of the GR chart can be enhanced by incorporating an additional feature, such as side sensitivity, so that the basic GR chart becomes the SSGR chart. It is known that the DS $\bar{X}$ chart has a better efficiency than the Shewhart $\bar{X}$ chart. Therefore, an integration of the DS scheme into the SSGR chart will produce the SSGRDS chart which will be more efficient than the basic GR and DS charts. The proposed SSGRDS chart will be discussed in the next chapter.

CHAPTER 3
A PROPOSED SIDE SENSITIVE GROUP RUNS DOUBLE SAMPLING (SSGRDS) CHART

3.1 Introduction

This section provides a brief summary of the topics covered in this chapter. In this chapter, the implementation of the SSGRDS chart is discussed in Section 3.2, where a step-by-step procedure is provided to implement the SSGRDS chart. The idea and concept of the SSGRDS chart is explained in Section 3.2. The zero state and steady state *ANOS* and *EANOS* formulae, as well as optimal designs of the SSGRDS chart are presented in Section 3.3. Detailed explanations are provided in the derivations of formulae and computations of optimal parameters of the SSGRDS chart. In Section 3.4, a real case study showing the implementation of the SSGRDS chart is given. A table containing the charting statistics and figures showing the implementation of the SSGRDS chart are given to facilitate the discussion.

3.2 Implementation of the SSGRDS Chart

The SSGRDS chart is implemented based on the idea of the SSGR and DS control charting concepts. The SSGRDS chart consists of the DS sub-chart and an extended version of the CRL sub-chart. In the SSGRDS chart, the CRL represents the number of sampling stages between two consecutive non-conforming sampling stages (including the ending or current non-conforming sampling stage). Figure 2.1 can be used to explain the implementation of the SSGRDS chart by replacing the in-control and out-of-control regions with conforming and non-conforming regions, respectively. The same notations discussed in Chapter 2 will be adopted here. The SSGRDS chart is implemented as follows:

Step 1. The values of the charting parameters n_1, n_2, L, L_1, L_2 and L_3 are specified. In the SSGRDS chart, L_3 is the lower limit of the extended version of the CRL sub-chart.

Step 2. The first sample with n_1 observations is taken from the population with an in-control mean, μ_0 and standard deviation, σ. Then $\bar{X}_{1,i} = \sum_{j=1}^{n_1} \frac{X_{1j}}{n_1}$ and $Z_{1,i} = \frac{\bar{X}_{1,i} - \mu_0}{\sigma / \sqrt{n_1}}$ are computed where X_{1j}, for j = 1, 2, …, n_1 are the observations of the first sample in sampling stage *i*.

Step 3. If $Z_{1,i}$ falls in I_1 (see Figure 2.1), sampling stage *i* is declared as conforming. Then the control flow goes back to Step 2.

Step 4. If $Z_{1,i}$ falls in I_3 (see Figure 2.1), sampling stage *i* is declared as non-conforming. Then the control flow proceeds to Step 8.

Step 5. If $Z_{1,i}$ falls in I_2 (see Figure 2.1), a second sample with n_2 observations is drawn from the same population as that of the first sample. Then $\bar{X}_2 = \sum_{j=1}^{n_2} \frac{X_{2j}}{n_2}$ is computed. Note that X_{2j} for j = 1, 2, …, n_2 are the observations from the second sample in sampling stage *i*.

Step 6. Compute the overall mean, $\bar{\bar{X}}_i = \frac{n_1 \bar{X}_1 + n_2 \bar{X}_2}{n_1 + n_2}$ followed by $Z_i = \frac{\bar{\bar{X}}_i - \mu_0}{\sigma / \sqrt{n_1 + n_2}}$.

Step 7. If Z_i falls in I_4 (see Figure 2.1), sampling stage *i* is declared as conforming and the control flow goes back to Step 2. Otherwise, sampling stage *i* is declared as non-conforming and the control flow proceeds to Step 8.