FOURTH-ORDER SPLINE METHODS FOR SOLVING NONLINEAR SCHRÖDINGER EQUATION

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2021

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by

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Thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosphy

June 2021

ACKNOWLEDGEMENT

In the name of Allah S.W.T, the creator, Most Gracious, Most Merciful.

First and foremost, I would like to thank Allah S.W.T for giving me the strength, courage and opportunity to finish this thesis in fulfilling the requirement for the degree of Doctor of Philosophy in Mathematics. Although 2020 has been hard and tough for all of human life due to the Covid-19 virus, the support and care from all the people and institutions around me makes this thesis a great success.

I would like to express my deepest appreciation and gratitude towards my supervisor, Dr. Amirah Azmi who has given me a lot of advice and assisted me until I complete my thesis successfully. I also wish to express my sincere thanks to my co-supervisors, Dr. Nur Nadiah Abd Hamid and Professor Ahmad Izani Md. Ismail for their thoughts and opinions in assisting me to do this research.

Most importantly, the person that had given me life to this world, my utmost gratitude goes to my mother, Faridah Musa and my father, Ahmad Abd. Majid who gave me motivation and support throughout my studies here in Universiti Sains Malaysia. Without their support and love, I would not be here to complete my PhD's degree. I am truly blessed to have such wonderful and caring parents with my lovely family that really support me in my studies.

My gratitude also goes to Universiti Sains Malaysia, Institute of Postgraduate Studies and School of Mathematical Sciences for allowing me to pursue my study. My sincere thanks to our Ministry of Higher Education for awarding me, the MyBrainSc scholarship for my fees and financial supports. Without the supports, this study could not have been done.

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LIST OF ABBREVIATIONS

BCuBSM	Besse Cubic B-spline Collocation Method
BCuEBSM	Besse Cubic Exponential B-spline Collocation Method
BRM	Besse Relaxation Method
BVP	boundary value problem
CAGD	Computer-Aided Geometric Design
CuBSM	Cubic B-spline Collocation Method
CuEBSM	Cubic Exponential B-spline Collocation Method
DE	differential equation
FBCuBSM	Fourth-order Besse Cubic B-spline Collocation Method
FBCuEBSM	Fourth-order Besse Cubic Exponential B-spline Collocation Method
FCuEBSM	Fourth-order Cubic Exponential B-spline Collocation Method
FCuBSM	Fourth-order Cubic B-spline Collocation Method
FDM	finite-difference method
FEM	finite-element method
NLS	Nonlinear Schrödinger
ODE	ordinary differential equation
PDE	partial differential equation

LIST OF SYMBOLS

B_j^w	j^{th} B-spline basis function of degree w
BE_j^w	j^{th} Exponential B-spline basis function of degree w
C	Continuity
D_o	differential operator
E	shift operator
h	step size for space interval, x
I_1	Mass momentum
I_2	Momentum momentum
I_3	Energy momentum
L_2	Euclidean norm
L_∞	maximum absolute norm
m	total number of x interval
n	total number of t interval
P_j	time dependent unknown at x_j
p	free parameter for Exponential B-spline function
S	approximated solution of u using CuBSM/BCuBSM
$\tilde{S}$	new approximated solution of u_{xx} using CuBSM/BCuBSM
t	time interval
u	exact solutions of NLS equation

$\tilde{u}$ approximated solution of u using FDM

U approximated solution of u using BRM

$\tilde{U}$ any approximated solution of NLS equation

V_j time dependent unknown at x_j

w order of B-spline

x space interval

Y speed of soliton

$z(x,t)$ function of two independent variables

α amplitude of soliton

δ amplification factor

η mode number

i imaginary unit

ε_{abs} absolute errors

Θ numerical order of convergence

ζ approximated solution of u using CuEBSM/BCuEBSM

$\tilde{\zeta}$ new approximated solution of u_{xx} using CuEBSM/BCuEBSM

KAEDAH-KAEDAH SPLIN PERINGKAT EMPAT UNTUK MENYELESAIKAN PERSAMAAN TAK LINEAR SCHRÖDINGER

ABSTRAK

Persamaan Tak Linear Schrödinger (TLS) adalah suatu persamaan asas dan penting dalam Fizik Matematik. Dalam tesis ini, Kaedah Kolokasi Splin-B Peringkat Empat dan Kaedah Kolokasi Splin-B Eksponen Peringkat Empat telah dihasilkan bagi menyelesaikan masalah yang melibatkan Persamaan TLS. Kaedah Kolokasi Splin-B Kubik dan Kaedah Kolokasi Splin-B Kubik Eksponen yang diketahui umum adalah ketepatan berperingkat dua. Kaedah-kaedah yang dibangunkan dalam tesis ini adalah ketepatan berperingkat empat. Dimensi masa persamaan TLS didiskretkan menggunakan Kaedah Beza Terhingga dan dimensi ruang didiskretkan berdasarkan kaedah Splin-B yang bersesuaian. Di samping itu, pendekatan Siri Taylor dan Besse digunakan bagi mengendalikan sebutan tak linear dalam persamaan TLS. Memandangkan kaedah-kaedah ini menghasilkan sistem yang tidak dapat ditentukan, maka syarat permulaan dan sempadan tambahan digunakan untuk menyelesaikan sistem. Kesemua kaedah yang dibangunkan diuji bagi menentukan kestabilan dan didapati memberi keputusan stabil tanpa syarat. Analisis ralat dan analisis penumpuan dijalankan. Kecekapan kaedah-kaedah ini dinilai pada tiga contoh ujian melibatkan soliton dan hampiran yang diperoleh didapati jitu. Di samping itu, peringkat penumpuan berangka juga dikira serta pernyataan-pernyataan teori yang berkaitan dibuktikan. Kesimpulannya, kaedah yang dicadangkan dalam tesis ini berfungsi dengan baik dan memberikan keputusan berangka yang tepat untuk persamaan TLS.

FOURTH-ORDER SPLINE METHODS FOR SOLVING NONLINEAR SCHRÖDINGER EQUATION

ABSTRACT

The Nonlinear Schrödinger (NLS) equation is an important and fundamental equation in Mathematical Physics. In this thesis, fourth-order cubic B-spline collocation method and fourth-order cubic Exponential B-spline collocation method are developed in order to solve problems involving the NLS equation. The established Cubic B-spline Collocation Method and Cubic Exponential B-spline Collocation Method are of second-order accuracy. The methods developed in this thesis are of fourth-order accuracy. The time dimension of the NLS equation is discretized using the Finite Difference Method and the space dimension is discretized based on the particular B-spline methods used. The Taylor series approach and Besse approaches are used to handle the nonlinear term of the NLS equation. Since the methods result in an underdetermined system, the supplementary initial and boundary conditions are used to solve the system. The developed methods are tested for stability and are found to be unconditionally stable. Error analysis and convergence analysis are also carried out. The efficiency of the methods are assessed on three test problems involving solitons and the approximations are found to be very accurate. Besides that, the numerical order of convergence is calculated and associated theoretical statements are proved. In conclusion, the proposed methods in this study worked well and give accurate numerical results for the NLS equation.

CHAPTER 1

INTRODUCTION

1.1 Introduction

This thesis is concerned with the numerical solutions of Nonlinear Schrödinger (NLS) equation. The focus of the thesis is on the use of finite-difference method (FDM) which utilizes B-splines. This chapter discusses the background of the research, problem statement and research question, scope of study, methodology, objectives and thesis outline.

1.2 Introduction to Solitons and NLS Equation

The theory of water waves was developed in 1834. A researcher named John Scott Russell was observing the water in Edinburgh-Glasgow Canal in Scotland. The water waves were termed the great wave of translation. The single pulse structure of the wave was later called as solitary waves (Russell, 1845).

Many phenomena around us involve travelling using waves. For example, sound, light and even the transmission of radiation from our mobile phones. The concept of water waves explains well the basic concepts of waves. For a simple example of water waves, a disturbance through a water surface which generates a movement of water waves is visualized in Figure 1.1.

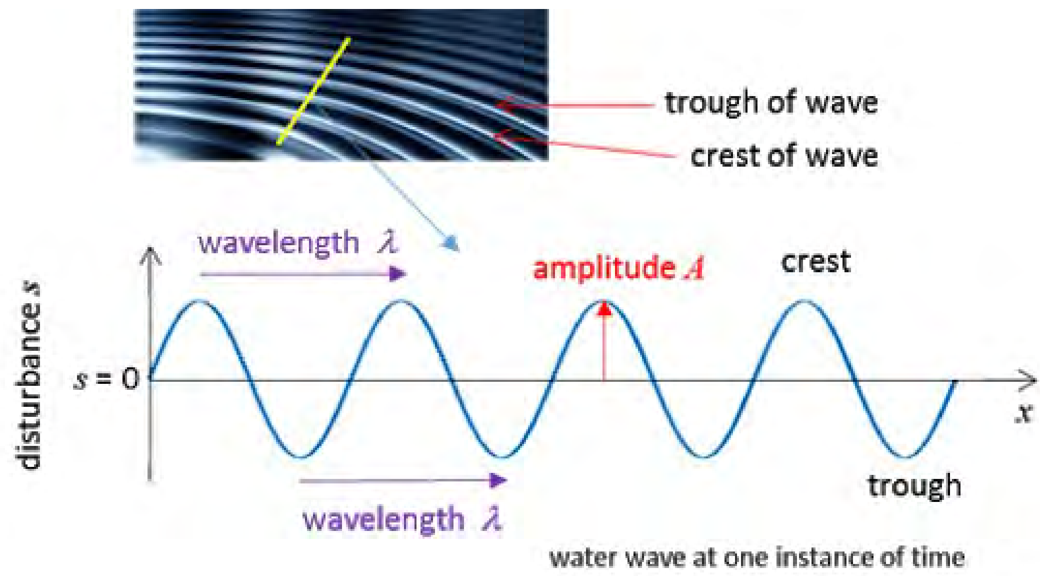

Figure 1.1: Illustration disturbance of surface water and waves propagation (Cooper, 2013)

This figure shows that the disturbance of surface water results in the propagation of waves. The crests and troughs are moving horizontally through the surface. In other words, the waves oscillate with horizontal propagation with wavelength, λ and amplitude, A.

According to Drazin and Johnson (1989), solitary waves or solitons are permanent in form, localized within a specific region, and able to collide with each other and emerge from collision unchanged. Due to the properties, solitons are highly relevant in certain physical phenomena which are modelled using the NLS equation:

$$iu_t + u_{xx} + q|u|^2u = 0,$$

where i is the imaginary unit, u is the exact solutions of the NLS equation, q is a real parameter and subscripts x and t are the differentiation with respect to x and t. The soliton solution of the NLS equation occurs because of a balance between dispersion and nonlinear effects. It is because the equilibrium between linear dispersion (u_{xx}, which tends to break up the wave packet) and the self-focusing effect of cubic nonlinearity ($|u|^2u$, generated by the wave's self-interaction with itself) is present (Azmi, 2014).

The NLS equation models phenomena in quantum mechanics, nonlinear water waves, plasma waves, heat pulse propagation in a solid, nonlinear optics self trapping and nonlinear waves in a fluid-filled viscoelastic tube. The NLS equation explains how the structure of modulated wave groups evolves. Zakharov (1968) defined the Hamiltonian water wave structure, in which the wave amplitude roughly satisfies the NLS equation for slowly modulated wave packets.

The NLS equation is difficult to solve analytically (i.e. yielding a formula). Therefore, there is a need to use numerical methods and this thesis is concerned with a particular type of numerical methods: FDM utilizing the B-spline methods. There have been considerable amount of work using FDM in conjunction with B-spline methods by Aksoy et al. (2013), Abbas et al. (2014c), Abbas et al. (2014a), Abbas et al. (2014b), Mat Zin et al. (2014), Iqbal et al. (2018) and Iqbal et al. (2019). But there has been limited work on using FDM with Fourth-order Cubic Exponential B-spline and certain recent developments of Exponential B-spline for the NLS equation. The development of fourth-order methods result in improved accuracy. This is the main motivation of the thesis.

1.3 Problem Statements and Research Questions

There should be a considerable amount of tools that can be called upon for solving various partial differential equations (PDEs). The development of the FDM in association with Fourth-order Cubic B-spline Collocation Method (FCuBSM) and Fourth-order Cubic Exponential B-spline Collocation Method (FCuEBSM) will provide additional tools in our armoury. However, prior to using these new methods, a detailed study needs to be carried out to ensure that the new method will yield reliable and accurate results.

The research questions for our thesis are as follows:

1. Can a reliable and accurate FCuBSM and FCuEBSM be developed to solve the NLS equation?
2. How can the nonlinear term in the NLS equation be best linearized?
3. What are the theoretical properties (stability, convergence) of the FCuBSM and FCuEBSM?

1.4 Scope of Study

The focus is on B-spline for PDE because of the considerable amount of previous work carried out. Thus, we can call upon this considerable body of work. We restrict ourselves to Cubic B-spline and Cubic Exponential B-spline and focus on the NLS equation as the NLS equation arises in many areas of Mathematical Physics.

1.5 Objectives of the Study

The objectives of the thesis are:

1. To apply Cubic Exponential B-spline Collocation Method (CuEBSM) for NLS equation. It should be noted the CuEBSM has not previously been applied for the NLS equation. The results obtained will be benchmarked against results obtained using FCuBSM and FCuEBSM.

2. To develop FCuBSM and FCuEBSM for solving NLS equation.

3. To investigate linearization using Taylor series approach and Besse approach for the NLS equation.

4. To investigate the errors, convergence and stability of FCuBSM and FCuEBSM for NLS equation.

1.6 Methodology

Recall the NLS equation is of the form:

$$iu_t + u_{xx} + q|u|^2u = 0,$$

and, in practice, there will be associated initial and boundary conditions. The methodology of the research is as follows:

1. The time derivative u_t will be discretized using the standard forward-difference approximation.

2. The nonlinear term $|u|^2u$ will be linearized with the use of Taylor's expansion

and Besse approach.

3. The space derivative u_{xx} will be treated with second-order approach:

   FDM, Besse Relaxation Method (BRM), Cubic B-spline Collocation Method (CuBSM), CuEBSM, Besse Cubic B-spline Collocation Method (BCuBSM) and Besse Cubic Exponential B-spline Collocation Method (BCuEBSM).

4. In order for us to achieve a higher degree of accuracy, fourth-order approach will be developed by letting the coefficient of u_{xx} equal to one, u_{xxx} equal to zero and u_{xxxx} equal to zero to solve the NLS equation. The methods thus obtained will be called the FCuBSM, FCuEBSM, Fourth-order Besse Cubic B-spline Collocation Method (FBCuBSM) and Fourth-order Besse Cubic Exponential B-spline Collocation Method (FBCuEBSM).

5. Theoretical analysis (convergence, consistency, stability) of the schemes developed in step (4) of the methodology are carried out.

6. Numerical simulations on test problems will be conducted to validate the theoretical analysis and check the feasibility of the schemes developed in step (5) of the methodology.

1.7 Thesis Outline

There are six chapters in this thesis. The motivations, objectives, methodologies and outline of the thesis are detailed in this first chapter.

Moving on to the second chapter, a summary of literature review is given. The scope of the numerical methods used by previous researchers for the NLS equation is investigated. The basic concepts and applications of CuBSM and CuEBSM are also studied. The obtained reviews of previous research related to the topic under study are explained in this chapter.

Chapter 3 introduces the basic concepts of the derivation of FDM based on the Taylor expansion. Other than that, the basic formulas of the CuBSM and CuEBSM are discussed with focus on of how the B-spline works. An example of the interpolation technique is also given.

Proceeding to Chapter 4, the first step in discretizing the NLS equation is carried out. The Taylor linearization approach is used on the nonlinear term of the NLS equation. The FDM, CuBSM and CuEBSM are applied to the equation as the second-order B-spline method forming the approximated solutions. Besides that, the newly improved method, namely FCuBSM and FCuEBSM acts as the fourth-order B-spline method for the approximated solutions. The error analysis, stability and accuracy of the methods are also carried out for this problem.

The next step is to see alternative methods in linearizing the nonlinear term and this is elaborated in Chapter 5. Thus, the Besse approach is used with BRM, BCuBSM and BCuEBSM act as second-order methods to approximate solutions of the NLS equation. On top of that, the fourth-order methods, FBCuBSM and FBCuEBSM are used to solve the NLS equation. The results are discussed with error, stability and accuracy of the methods investigated. Lastly, the conclusions of the study are given in Chapter 6. In addition, recommendations and future works are also considered for our research.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In the era of modernization, research on mathematical equations has grown exponentially since they are able to provide physical description to almost everything around us. One of the most interesting universal equations in physical studies is the Schrödinger equation that describes the quantum state of a physical system. There are different versions of the Schrödinger equation in scientific studies for the modelling of several physical phenomena such as the propagation of optical pulses, waves in water and plasmas and self-focusing in laser pulses. Here we will be focusing on one of the specific forms of the Schrödinger equation known as the time-dependent cubic NLS equation (henceforth NLS equation) which describes the optical pulse propagation in optical fibers. Thus, this chapter will deliberate some highlights on previous research that has explored this equation starting with the introduction to the wave equation, followed by the approximate methods used to solve this equation such as the B-spline method, exponential B-spline method and Besse relaxation scheme.

2.2 Methods on solving Wave Equation

More than a century ago, Russell (1845) surmised that solitary wave is a great wave of translation after observing the water waves at the Union Canal (near Edinburgh) in 1834. Russell was doing an experiment to measure the relationship between the speed of a boat and its propelling force. In his discovery, he discovered the shape of the solitary wave, a sufficient large of the initial mass of water produces two or more independent solitary waves whereby the merging of solitary waves does not have any change and a large amplitude solitary wave travels faster than low amplitude. This great discovery is later termed as a solitary wave in recognition of its single pulse structure.

In the year 1926, the world of research on wave equations is introduced to a scientist, Erwin Schrödinger. He developed one of the famous linear PDEs, Schördinger equation. Schrödinger (1926) revolutionized the world with his equation in the world of physics. Optical solitons also used Schrödinger equation and are discussed in the next paragraph.

Shahzad and Zafrullah (2009) discovered optical solitons as natural bits for the use of telecommunications as non-linearity and anomalous group-velocity dispersion causes the shape to maintain over distance. They discovered that solitons get attracted and form a huge pulse of double amplitude and after separation, they maintain their original form.

Sanz-Serna (1984) used the FDM and finite-element method (FEM) in order to solve NLS equation. The integration of time is done using the leap-frog technique and modified Crank-Nicolson while the space dimension using both FDM and FEM schemes. He concluded the optimal rate of convergence in the L_2 norm is $O(t^2)$.

NLS equation has been an integral part of physics. Back in the year 2004, mathematician, Besse (2004) introduced a new method called, Besse relaxation scheme. This scheme is used in solving the governing equation, the NLS equation. Besse relaxation scheme has given a new idea to solve NLS equation with no nonlinear iteration step. The nonlinear part of the equation is suppressed by doing a relaxation, which means to separate the treatment of the linear part and the nonlinear part to different times. This method enabled the density and energy quantities to be conserved. Several problems are perfomed on the scheme and results were found to be more accurate. Furthermore, this method gave an advantage for solving PDE without linearizing the equation.

Matuszewski et al. (2009) studied nonlinear light propagation using NLS equation in colloidal suspensions of dielectric nanoparticles within the hard-sphere interaction approximation. They analyzed the existence and properties of self-trapped beams (spatial optical solitons) using the analytical results in such media and demonstrate the existence of a bistability regime in the one-dimensional case with a lower stable branch disappears in the two-dimensional model.

Antoine et al. (2011) researched on the use of Perfectly Matched Layers and Absorbing Boundary Conditions on solving general nonlinear one-dimensional and two-dimensional Schrödinger equations. The time discretization is made using a semi-implicit relaxation scheme also known as Besse relaxation scheme which avoids any fixed point procedure while the spatial discretization required FEM. The results are compared with the analytical solutions for the one-dimensional case and shooting method for the two-dimensional case. Numerical simulations for Perfectly Matched Layers and Absorbing Boundary Conditions proved to be accurate.

Marchant (2012) mentioned in his paper the NLS equation for both one and two spatial dimensions, are derived using an averaged Lagrangian and suitable trial functions for the solitary waves in colloidal media. The semi-analytical solutions are in agreement with the numerical solutions using the averaged Lagrangian.

The study on semi-analytical solutions is continued on the temperature-dependent compressibility on a dispersive shock wave in a colloidal medium (Azmi & Marchant, 2014). It is focused on the NLS equation with implicit nonlinearity. The amplitude of the shock waves has been predicted with the one-dimensional line bore case for the amplitude of the solitary waves. Thus, the semi-analytical solutions are found to be very accurate.

Houwe et al. (2019) used three different approaches to obtain the exact traveling-wave solution and soliton solutions of the nonlinear Schrodinger equation (NLSE) for higher-order nonlinear terms of Left-handed metamaterials (LHMs). They use the csch function method, the $\exp(-\phi(\xi))-$Expansion method, and the simplest equation method. After a large number of oscillations, the new system exhibits positive behaviour of the maximal absolute error and the global norm in time.

While Kosti et al. (2020) designed an updated Runge–Kutta pair with improved periodicity and stability characteristics, motivated by the limited work done on the development of computational techniques for solving the nonlinear Schrödinger equation with time-dependent coefficients. The numerical results on the nonlinear Schrödinger equation with a periodic solution confirmed the efficiency of the new algorithm. The new approach also has a positive behaviour when it comes to the highest absolute error. Even after a large number of oscillations, the new system exhibits positive behaviour of the maximal absolute error and the global norm in time.

The approximate or numerical solutions had been very accurate with the analytical solutions of the NLS equation. But, semi-analytical solutions also play an important role in estimating the solutions of the NLS equation. This is due to the semi-analytical solutions that can give us an answer at a given spatial while approximate or numerical solutions need to be computed again at a new specific spatial. We reviewed in passing a few numerical methods for solving the NLS equation in this section. Next, we consider methods used in computer-aided geometic design or commonly known as CAGD. CAGD is the whole branch of research and development especially in designing curves and surfaces. The next section will be discussing the methods used in CAGD that are mainly will be the subject of interest of this research such as B-spline and exponential B-spline methods. These spline will be utilized to numerically solve the NLS equation.

2.3 B-spline Method

In the modern era, B-spline methods are very popular in CAGD and various fields. The usage of B-splines in the designing curve and interpolating points have been developed and studied for years by researchers. In recent years, B-splines have been used in solving a number of problems involving PDE. We will discuss recent studies involving B-splines for solving PDEs.

To start, a bit of history on the spline interpolation method will be introduced. In the late 1960s, the work of solving differential equations using the spline interpolation method was done by Bickley (1968). He worked on the second-order linear two-point boundary value problems (BVPs) where the spline function can be arbitrarily defined on a domain. Thus, the spline function is formed on the chosen domain of the problem with the boundary conditions. The results are believed to be very promising at the time.

The method by Bickley (1968) and error estimations by Curtis and Powell (1967) using cubic spline was further studied and discussed by Fyfe (1969). Fyfe produced a spline with better adjustments to the solution known as correction spline. Correction spline for the non-equal intervals had also been studied and Fyfe proved that on an interval, the method was found to be less efficient. A problem was tested with the same intervals spline producing very minimal computation. For the case of non-equal intervals, the deferred correction had given the very least improvement. Nevertheless, Fyfe claimed this method had been proven more efficient than the FDM due to the spline's ability to get approximate solutions at any point in an interval.

Albasiny and Hoskins (1969) solved a two-point BVP using cubic spline method that was introduced by Ahlberg et al. (1967). For the equation to be solved, the equation needed to be transformed into to a three-term recurrence relation that subsequently leads to a set of linear equations. The first problem was discussed without the first derivative of the two-point BVP, a basic relation between the spline solution and a representation of finite-difference is formed. For the second problem, first derivative term was available where explicit formula for the recurrence relation coefficients was derived. Both problems have shown promising results compared to the finite difference scheme.

In 2006, a two-point BVP was solved by a CuBSM presented by Caglar et al. (2006). The solution curve was modeled by the CuBSM and a matrix system was solved. Thus, forming a piecewise function that represented the approximate solutions at any point on the interval. The numerical results were found to be more accurate compared to the finite difference, finite element and finite volume methods that were carried out by Fang et al. (2002).

Goh et al. (2011) solved one-dimensional heat and wave equations by using the CuBSM. They discretized the time dimension using FDM and CuBSM on the space dimension. The approximate solutions are compared with the exact solutions and the results are very accurate. Besides that, the order of the heat and wave equations is tested with order $O(h^2+t^2)$ and $O(h^2+t)$ respectively where h is the space interval and t is the time interval.

Shortly after, Aksoy et al. (2013) solved a PDE using the Taylor B-spline collocation method also known as CuBSM. The governing equation used is the NLS equation and time-dependent NLS equation is discretized using Taylor series expansion and the resulting system of equation is solved using the B-spline collocation method. The numerical results are very encouraging with the help of L_2 and L_∞ norms to test the accuracy. The L_2 also known as the Euclidean norm is defined as the square root of the sum of the squares of the absolute values of its components and L_∞ norm or known as the maximum norm is given by the maximum of the absolute values of its components or elements.

Abbas et al. (2014b) solved numerically the Strongly Coupled Reaction-Diffusion system by using CuBSM. They discovered the numerical solutions are in good agreement. The stability is found to be unconditionally stable using von Neumann stability analysis. Furthermore, the accuracy of the scheme is tested using the L_2 and L_∞ norms.

Around 3 years later, Ahmad et al. (2017) solved NLS equation using CuBSM and FDM. In this study, they resolved the stability of the CuBSM method is unconditionally stable using von Neumann stability analysis with the accuracy of the method are found to be accurate using the L_2 and L_∞ norms.

Iqbal et al. (2018) implemented a new technique called new cubic B-spline approximation. The new scheme is fifth-order accurate with the help of Taylor series expansion. The new scheme is tested on third-order Emden–Flower type equation with the approximations are shown to be superior compared to the existing method. Their work has shown that this new technique requires less computational cost and the method is proved to be accurate.

Recently, Iqbal et al. (2019) presented his research on the numerical solutions of NLS equation using Quintic B-spline Galerkin method. His research found out that the Galerkin B-spline method is more efficient and easier to use compared to the Galerkin FEM. He also used the Crank-Nicolson scheme and FDM on the time interval and Quintic B-spline Galerkin method on space interval. The L_2 and L_∞ norms are computed and the mass and energy conservation laws are calculated as well. The computed results are compared with previous research and are found to be accurate. The convervations laws computed are well preserved.

The implementation of the B-spline method had brought a new perspective in solving the PDE numerically. Researchers are keen to find new ways to improve the B-spline method to a whole new level. The following section will discuss another method of the B-spline method called the exponential B-spline method.

2.4 Exponential B-spline method

After B-spline had been introduced, more researchers began to explore more ways to interpolate the curve in designs. Therefore, Späth (1969) proved the use of exponential spline theory with the mix of piecewise cubic and exponential splines. For the study of the exponential spline, the spline has no undesirable inflection point on each interpolated interval.

Sven Barendt (2009) studied on the interpolating of exponential splines. It is based on so-called tension parameters, which allow for tuning of its properties. The properties allow the analytic representation of the kernel which enables one to come up with space and frequency domain analysis. The frequency domain analysis, reveals that the cubic B-Spline interpolation, which suffers from ringing artefacts, yields a superior signal reconstruction efficiency. The properties exponential splines in tension as a function of the parameter enabled to close the gap between linear and cubic B-spline interpolation.

The theory of exponential splines was discussed from a perspective where exponential splines can form co-convex and co-monotone interpolants based on research by Mccartin (1991). The exponential terms are added to the governing equation which gave the exponential spline. The convergence of the spline is studied by interpolating cubic and exponential splines with identical end conditions or boundary conditions.

Nowadays, the interpolation of exponential B-spline is proposed using the infinite impulse response by Gupta and Lee (2007). The analysis of the method using Fourier approximation gave good results in high and low-frequency components.

Mohammadi (2013) solved the convection-diffusion equation using the exponential B-spline collocation method. The time dimension is discretized using the Crank-Nicolson scheme while exponential B-spline is used for the space dimension. The method is shown to be unconditionally stable by von Neumann stability analysis. Several numerical examples are tested to show the accuracy of the method. The results showed better approximations compared to some other methods in the literature.

Ersoy and Dag (2015a) applied CuEBSM in solving Korteweg-de Vries equation. A system of algebraic equations is obtained from the discretization of space and time. Thomas' algorithm is used to solve the system. The accuracy of the method is tested on some numerical experiments.

Ersoy and Dag (2015b) made use of CuEBSM to solve the coupled equation which is nonlinear coupled Burgers' equation. Crank-Nicolson and CuEBSM are presented to discretize the time derivative and space dimension, respectively. Three different examples are chosen to check the accuracy of the method. As a result, the solutions obtained shown more accurate results compared to other existing methods.

Shortly after, Ersoy and Dag (2015c) devoted to studying CuEBSM applied to the reaction-diffusion system. An iterative banded algebraic matrix equation is produced and the same approach as previous research is done to get the solutions of the matrix. Problems involving linear and nonlinear terms are carried out. The results are evaluated using numerical error norms.

After that, numerical solutions of the general long-wave equation are produced using the exponential B-spline collocation method by Mohammadi (2015). In this work, the temporal direction is discretized using Crank-Nicolson FDM and CuEBSM in the spatial direction. Also, the method is found to be unconditionally stable and has a second-order convergent for both time and space.

About one year later, Ersoy and Dag (2016) implemented CuEBSM on Kuramoto-Sivashinsky (KS) equation. KS equation is transformed into linearized algebraic equations. Two different problems are discussed by comparing the approximate solutions with the exact. Therefore, CuEBSM is in good agreement with the exact solutions.

2.5 Summary

In conclusion, the research on some alternative methods in solving the wave equations is discussed at the beginning of the chapter. A few commonly used numerical methods are explained in the first section. Discussions on the CuBSM to get an overview of the B-spline method. To sum up this chapter, CuEBSM is introduced to give us different options with the B-spline method. Moreover, CuEBSM contains a global tension that we can manipulate to give us a different perspective to design the B-spline curves.

Whilst this chapter has indicated considerable research on the use of second-order B-spline to solve PDEs, there has not been much work of Cubic Exponential B-spline on PDEs due to its complexity since it involves exponential functions. Besides that, the literature review indicates no work has been done on the use of fourth-order B-spline to solve the NLS equation. So, this give us the motivation to further develop the application of Cubic Exponential and fourth-order B-splines on the NLS equation. The next chapter will elaborate more on CuBSM and CuEBSM to provide better understanding in these methods.

CHAPTER 3

BASIC CONCEPTS, THEORY AND METHODS

3.1 Introduction

This chapter will discuss the basic concepts, theory and methodologies of the FDM, CuBSM and CuEBSM. The FDM is used as the basic approach in solving the NLS equation. While CuBSM and CuEBSM acts as our primary methods to solve the NLS equation. The curve design using CuBSM and CuEBSM are discussed at the end of the section.

3.2 Finite-difference Methods

FDM is one of the famous and commonly used approximation techniques in numerical approximation techniques especially in solving ordinary differential equation (ODE) and PDE. The stability and convergence of FDM have been studied to solve ODE or PDE for the past decade. The FDM works in such a way that the independent variable in the PDE that defined an interval of finite points is changing while the dependent variable is approximated.

3.2.1 Finite-difference approximations

Originally, FDM is derived by Taylor's polynomial or known as Taylor series expansion. As FDM is deliberated, assume $z(x,t)$ as a function with two independent variables space, x and time, t respectively. A Taylor's polynomial of finite function $z(x,t)$ is introduced.

$$z(x_j+h,t_k) = z(x_j,t_k) + h\,z_x\left(x_j,t_k\right) + \frac{h^2}{2!}z_{xx}(x_j,t_k) + \frac{h^3}{3!}z_{xxx}(x_j,t_k) + ..., \tag{3.1}$$

$$z(x_j-h,t_k) = z(x_j,t_k) - h\,z_x\left(x_j,t_k\right) + \frac{h^2}{2!}z_{xx}(x_j,t_k) - \frac{h^3}{3!}z_{xxx}(x_j,t_k) + ..., \tag{3.2}$$

where h is the step size at point j.

By rearranging Equation (3.1), the first-order forward difference approximation is formulated. This operation yields

$$z_x(x_j,t_k) = \frac{z\left(x_j+h,t_k\right) - z\left(x_j,t_k\right)}{h}. \tag{3.3}$$

On the other hand, Equation (3.2) is modified to form

$$z_x(x_j,t_k) = \frac{z\left(x_j,t_k\right) - z\left(x_j-h,t_k\right)}{h}, \tag{3.4}$$

which is the backward difference approximation.

Furthermore, by subtracting Equation (3.1) from Equation (3.2) and taking the term $z_x\left(x_j,t_k\right)$ as the subject forming

$$z_x(x_j,t_k) = \frac{z_x\left(x_j+h,t_k\right) - z\left(x_j-h,t_k\right)}{2h}. \tag{3.5}$$

Equation (3.5) is also known as the first-order central difference approximation.

The second-order central difference approximation is the results of adding equations (3.1) and (3.2) to form

$$z_{xx}(x_j,t_k) = \frac{z\left(x_j+h,t_k\right) - 2z\left(x_j,t_k\right) + z\left(x_j-h,t_k\right)}{h^2}. \tag{3.6}$$

Next section will elaborate further on the B-spline methods.

3.3 B-splines

In this section, the discussion on B-spline method is started with the basic theory of B-spline. Our focus will be on the CuBSM and the CuEBSM.

3.3.1 History of B-spline

During the early years of curve design, curves are implemented and designed manually without the help of any type of machinery. During the time, drawing curves in building ships and airplanes are drawn using long, thin, flexible strips of wood. Figure 3.1 is an illustration of how the curve is drawn back in the early 90s.

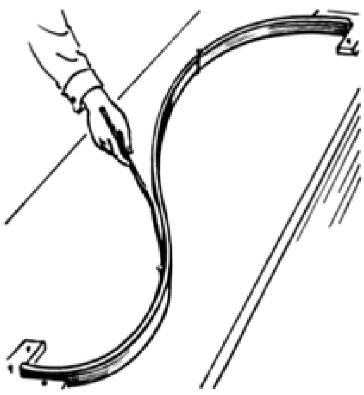

Figure 3.1: Illustration of curve being designed by carpenters (Schoenberg, 1946)

Schoenberg (1946) is the first mathematician to refer the curve as the spline curve. He defined that a spline is a simple mechanical device for drawing smooth curves.

Besides that, Barrodale and Young (1966) defined spline function as a piecewise polynomial of degree n joined smoothly with $n-1$ continuous variable.

Definition 3.1. Suppose a spline is a piecewise-polynomial of a real function, (Van Loan, 2000)

$$S : [a,b] \rightarrow \mathbb{R},$$

where S is on the interval $[a,b]$ composed of order w disjoint subintervals $[x_j, x_{j+1}]$ for $j = 0,1,...,n-1$ such that

$$a = x_0 < x_1 < ... < x_{n-1} < x_n = b.$$

The knots of S are $x_0, x_1, ..., x_n$ and the constraint of S to an interval j is a polynomial of

$$S_j : \left[x_j, x_{j+1}\right] \rightarrow \mathbb{R}.$$

The chosen polynomials assured the smoothness of S and spline of order w gave the meaning that S is continuously differentiable up to order $w-1$.

3.3.2 B-spline basis function

B-spline basis function is constructed using a piecewise function and referred as Cox-De Boor recursion formula, defined as (De Boor, 2001)

$$B_j^1(x) = \begin{cases} 1, & x \in \left[x_j, x_{j+1}\right), \\ 0, & \text{otherwise}, \end{cases} \tag{3.7}$$

$$B_j^w(x) = \left(\frac{x - x_j}{x_{j+w-1} - x_j}\right) B_j^{w-1}(x) + \left(\frac{x_{j+w} - x}{x_{j+w} - x_{j+1}}\right) B_{j+1}^{w-1}(x). \tag{3.8}$$

Equations (3.7) and (3.8) formed the B-spline basis of order $w \geq 1$. Figure 3.2 shows the triangular computation scheme of $B_j^6(x)$.

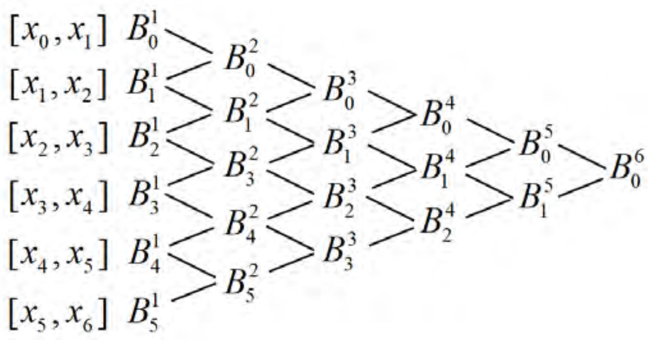

Figure 3.2: Computation scheme for the B-spline basis function

The first-order B-spline basis function is defined in Equation (3.7). The following knots, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 4$ are considered for this example with $h = 1$, defined as $h = x_{j+1} - x_j$. The knots represented the points where the piecewise polynomial ends of the B-spline basis function. For every basis, w order followed by $(w+1)$ number of knots. Equation (3.9) and Figure 3.3 demonstrates degree zero or order 1 for basis $B_0^1(x)$

$$B_0^1(x) = \begin{cases} 1, & x \in [0,1), \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

Equations (3.10) and (3.11) define the B-spline of order 2 or also known as degree 1,

$$B_j^2(x) = \left(\frac{x - x_j}{x_{j+1} - x_j}\right) B_j^1(x) + \left(\frac{x_{j+2} - x}{x_{j+2} - x_{j+1}}\right) B_{j+1}^1(x), \tag{3.10}$$

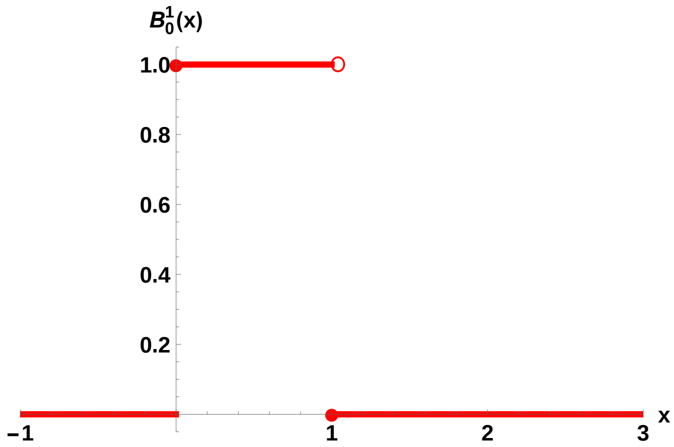

Figure 3.3: B-spline basis for $B_0^1(x)$, for $h = 1$

$$B_j^2(x) = \begin{cases} \frac{x - x_j}{h}, & x \in [x_j, x_{j+1}), \\ \frac{x_{j+2} - x}{h}, & x \in [x_{j+1}, x_{j+2}). \end{cases} \tag{3.11}$$

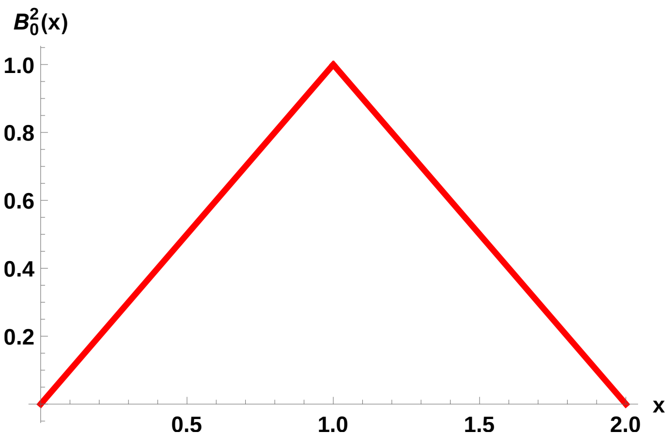

Figure 3.4: B-spline basis for $B_0^2(x)$, for $h = 1$

Figure 3.4 displays the plot of B-spline basis of order 2 based on Equations (3.10) and (3.11). Equations (3.12) and (3.13) are the B-spline basis function of second degree or known as third order.

$$B_j^3(x) = \frac{x - x_j}{x_{j+2} - x_j} B_j^2(x) + \frac{x_{j+3} - x}{x_{j+3} - x_{j+1}} B_{j+1}^2(x), \tag{3.12}$$

$$B_j^3(x) = \begin{cases} \frac{(x-x_j)^2}{2h^2}, & x \in [x_j, x_{j+1}), \\ \frac{(x-x_j)(x_{j+2}-x)+(x_{j+3}-x)(x-x_{j+1})}{2h^2}, & x \in [x_{j+1}, x_{j+2}), \\ \frac{(x_{j+3}-x)^2}{2h^2}, & x \in [x_{j+2}, x_{j+3}). \end{cases} \tag{3.13}$$

Based on the Equations (3.12)-(3.13) above, Figure 3.5 represents the B-spline basis for degree 2.

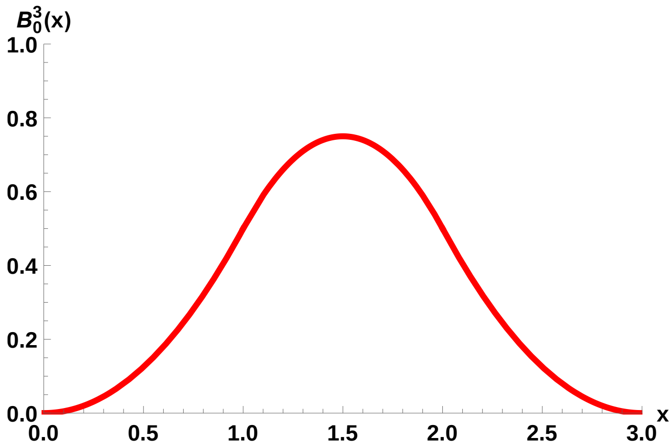

Figure 3.5: B-spline basis for $B_0^3(x)$, for $h = 1$

The next section will provide more information on the B-spline basis function with degree 3.

3.3.2(a) Cubic B-spline basis function

In the previous section we discussed the basis functions of order 1,2 and 3. This section is related with the basis function of order 4 or degree 3. The basis function of degree 3 is also known as Cubic B-spline basis function, where the basis of cubic

B-spline, is defined by

$$B_j^4(x) = \left(\frac{x - x_j}{x_{j+3} - x_j}\right) B_j^3(x) + \left(\frac{x_{j+4} - x}{x_{j+4} - x_{j+1}}\right) B_{j+1}^3(x), \tag{3.14}$$

$$B_j^4(x) = \begin{cases} \frac{(x-x_j)^3}{6h^3}, & x \in \left[x_j, x_{j+1}\right), \\ \frac{(x-x_j)^2(x_{j+2}-x)+(x-x_j)(x_{j+3}-x)(x-x_{j+1})+(x_{j+4}-x)(x-x_{j+1})^2}{6h^3}, & x \in \left[x_{j+1}, x_{j+2}\right), \\ \frac{(x-x_j)(x_{j+3}-x)^2+(x-x_{j+1})(x_{j+4}-x)(x_{j+3}-x)+(x_{j+4}-x)^2(x-x_{j+2})}{6h^3}, & x \in \left[x_{j+2}, x_{j+3}\right), \\ \frac{(x_{j+4}-x)^3}{6h^3}, & x \in \left[x_{j+3}, x_{j+4}\right), \end{cases} \tag{3.15}$$

where $h = x_{j+1} - x_j$. Equations (3.14) and (3.15) mentioned are the basis for Cubic B-spline function. Figure 3.6 shows the composite curve of degree 3 polynomial with joining knots $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 4$ with $h = 1$.

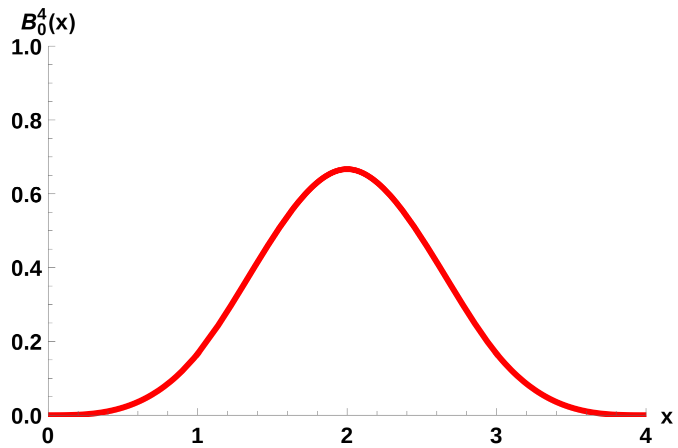

Figure 3.6: B-spline basis for $B_0^4(x)$, for $h = 1$

We can generate many copies of the B-spline, $B_j^4(x)$ by translating each basis function according to the knots. The blue curve in Figure 3.7 shows the curve of $B_0^4(x)$ on the interval $[0,4]$ and the plotting of $B_0^4(x)$, $B_1^4(x)$, $B_2^4(x)$ and $B_3^4(x)$ combined.

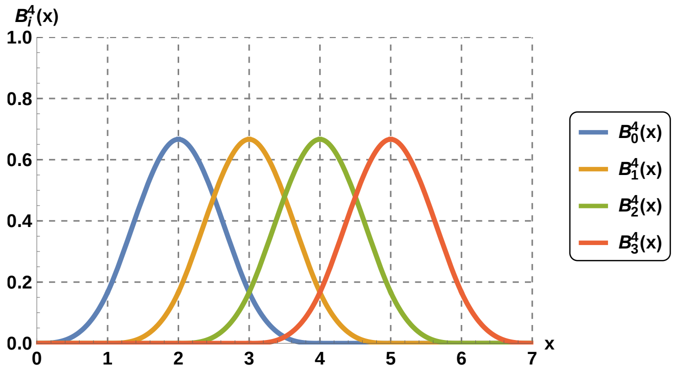

Figure 3.7: Collection of B-spline basis for $B_j^4(x)$ for $j = 0, 1, 2, 3$, for $h = 1$

3.3.2(b) Cubic B-spline Interpolation

In this section, the properties of cubic B-spline are discussed. Caglar et al. (2006) simplified and derived the cubic B-spline functions into a simpler form. The cubic B-spline function $S(x)$ is derived as a linear combination of cubic B-spline basis of B_j^4 and $S(x)$ is a piecewise polynomial of order 4 (degree 3) with a continuity up to C^2 as:

$$S(x) = \sum_{j=-3}^{n-1} P_j B_j^4(x), \qquad x \in [x_0, x_n], \qquad n \geq 1, \tag{3.16}$$

where P_j is known as the coefficient of the function yet to be determined.

Based on Figure 3.7, the B-spline basis B_j^4, is non-zero on the interval $[x_j, x_{j+4}]$. Afterwards, we will be using the interval $[x_j, x_{j+1}]$ in order to solve our equations. Figure 3.8 illustrates the basis we require to solve the NLS equation.

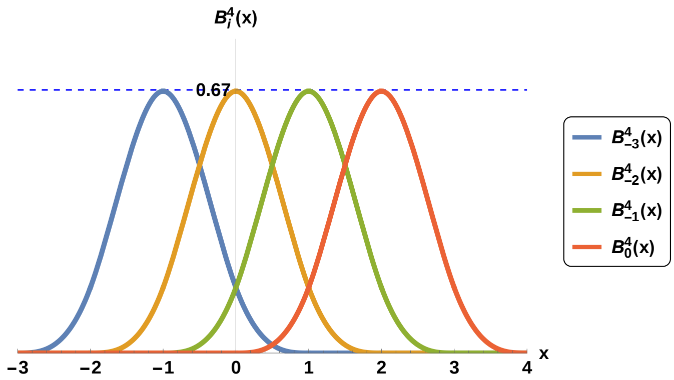

Figure 3.8: B-spline basis for $B_j^4(x)$ for $x \in [-3,4]$, for $h = 1$

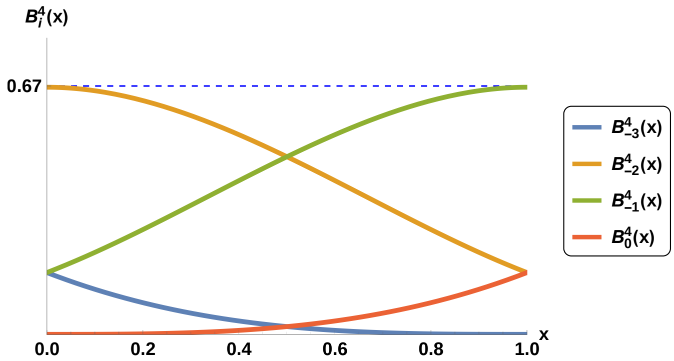

Figure 3.9: B-spline basis for $B_{-3}^4(x), B_{-2}^4(x), B_{-1}^4(x)$ and $B_0^4(x)$ on interval $x \in [0,1]$

Figure 3.9 shows non-zero basis functions $B_{-3}^4(x), B_{-2}^4(x), B_{-1}^4(x)$ and $B_0^4(x)$ on the interval $[0,1]$. Then $S(x)$ on the interval $[x_j, x_{j+1}]$ is written as

$$\begin{aligned} S(x) &= \sum_{j=-3}^{n-1} P_j B_j^4(x) \\ &= P_{-3} B_{-3}^4(x) + P_{-2} B_{-2}^4(x) + \ldots + P_{n-1} B_{n-1}^4(x) \\ &= P_{j-3} B_{j-3}^4(x) + P_{j-2} B_{j-2}^4(x) + P_{j-1} B_{j-1}^4(x) + P_j B_j^4(x), \quad x \in [x_j, x_{j+1}]. \end{aligned} \tag{3.17}$$

The basis functions $B_{j-3}^4(x), B_{j-2}^4(x), B_{j-1}^4(x)$ and $B_j^4(x)$ are redefined as translated basis function where the basis functions are defined as follows:

$$
\begin{aligned}
&B_j^4(x) = \tfrac{1}{6h^3}(x-x_j)^3, && x \in [x_j, x_{j+1}],\\
&B_{j-1}^4(x) = \tfrac{1}{6h^3}\left(h^3+3h^2(x-x_j)+3h(x-x_j)^2-3(x-x_j)^3\right), && x \in [x_j, x_{j+1}],\\
&B_{j-2}^4(x) = \tfrac{1}{6h^3}\left(4h^3-6h(x-x_j)^2+3(x-x_j)^3\right), && x \in [x_j, x_{j+1}],\\
&B_{j-3}^4(x) = \tfrac{1}{6h^3}(x_{j+1}-x)^3, && x \in [x_j, x_{j+1}].
\end{aligned}
\tag{3.18}
$$

Equation (3.18) is substituted into Equation (3.17), then equation $S(x)$ becomes

$$
S(x) = \frac{1}{6h^3}\begin{pmatrix} P_{j-3}(x_{j+1}-x)^3 + P_{j-2}\left[4h^3-6h(x-x_j)^2+3(x-x_j)^3\right] + \\ P_{j-1}\left[h^3-3h^2(x-x_j)+3h(x-x_j)^2-3(x-x_j)^3\right] + P_j\left(x-x_j\right)^3 \end{pmatrix}.
\tag{3.19}
$$

Next, the derivatives of Equation (3.19) are determined by differentiating Equation (3.17) with respect to x. The first derivative is

$$
S'(x) = \frac{d}{dx}\left[\sum_{j=-3}^{n-1} P_j B_j^4(x)\right] = \sum_{j=-3}^{n-1} P_j\left[\frac{d}{dx}B_j^4(x)\right],
\tag{3.20}
$$

where

$$
\frac{d}{dx}\left[B_j^4(x)\right] = \frac{1}{2h^3}\begin{cases} (x-x_j)^2, & x \in [x_j, x_{j+1}],\\ h^2+2h(x-x_{j+1})-3(x-x_{j+1})^2, & x \in [x_{j+1}, x_{j+2}],\\ -4h(x-x_{j+2})+3(x-x_{j+2})^2, & x \in [x_{j+2}, x_{j+3}],\\ -(x_{j+4}-x)^2, & x \in [x_{j+3}, x_{j+4}]. \end{cases}
\tag{3.21}
$$

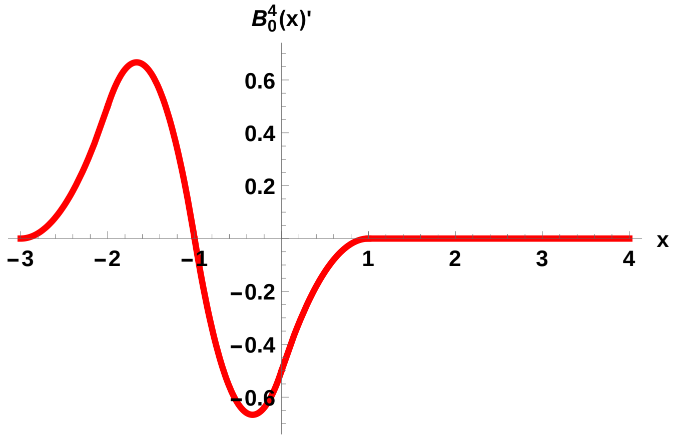

Figure 3.10: First derivative B-spline basis for $\frac{d}{dx}\left[B^4_{-3}(x)\right]$

The second derivative of Cubic B-spline basis function is as follows

$$S''(x) = \frac{d^2}{dx^2}\left[\sum_{j=-3}^{n-1} P_j B_j^4(x)\right] = \sum_{j=-3}^{n-1} P_j \left[\frac{d^2}{dx^2} B_j^4(x)\right], \tag{3.22}$$

where

$$\frac{d^2}{dx^2}\left[B_j^4(x)\right] = \frac{1}{h^3}\begin{cases} (x-x_j), & x \in \left[x_j, x_{j+1}\right], \\ h+3(x_{j+1}-x), & x \in \left[x_{j+1}, x_{j+2}\right], \\ -2h+3(x-x_{j+2}), & x \in \left[x_{j+2}, x_{j+3}\right], \\ (x_{j+4}-x), & x \in \left[x_{j+3}, x_{j+4}\right]. \end{cases} \tag{3.23}$$

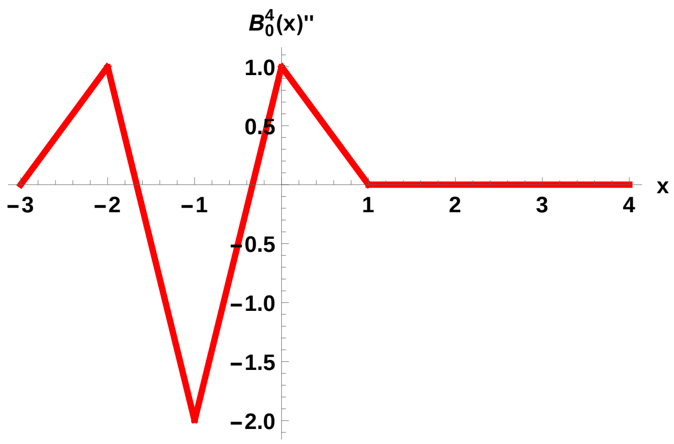

Figure 3.11: Second derivative B-spline basis for $\frac{d^2}{dx^2}\left[B^4_{-3}(x)\right]$

Figures 3.10 and 3.11, represented the first derivative of $B_j^4(x)$ and second derivative of $B_j^4(x)$ respectively. Thus, the curve of B-spline from Equation (3.19) is C^2 continuity. Equation (3.21) is substituted into Equation (3.20) and the same for Equation (3.23) is substituted into Equation (3.22) respectively, and evaluated on the interval $x \in \left[x_j, x_{j+1}\right]$, the following equations are obtained:

$$
\begin{aligned}
S'(x) &= P_{j-3}\frac{d}{dx}\left[B_{j-3}^4(x)\right] + P_{j-2}\frac{d}{dx}\left[B_{j-2}^4(x)\right] + P_{j-1}\frac{d}{dx}\left[B_{j-1}^4(x)\right] + P_j\frac{d}{dx}\left[B_j^4(x)\right] \\
&= \frac{1}{2h^3}\begin{pmatrix} P_{j-3}[-h^2 - 2h(x_j - x) - (x_j - x)^2] + P_{j-2}[4h(x - x_j) + 3(x - x_j)^2] + \\ P_{j-1}[h^2 + 2h(x - x_j) - 3(x - x_j)^2] + P_j(x - x_j)^2. \end{pmatrix},
\end{aligned} \tag{3.24}
$$

$$
\begin{aligned}
S''(x) &= P_{j-3}\frac{d^2}{dx^2}\left[B_{j-3}^4(x)\right] + P_{j-2}\frac{d^2}{dx^2}\left[B_{j-2}^4(x)\right] + P_{j-1}\frac{d^2}{dx^2}\left[B_{j-1}^4(x)\right] + P_j\frac{d^2}{dx^2}\left[B_j^4(x)\right] \\
&= \frac{1}{h^3}\begin{pmatrix} P_{j-3}[h + x_j - x] + P_{j-2}[-2h + 3(x - x_j)] + \\ P_{j-1}[h - 3(x - x_j)] + P_j(x - x_j), \end{pmatrix},
\end{aligned} \tag{3.25}
$$

where $x \in \left[x_j, x_{j+1}\right]$ and $j = 0, 1, ..., n-1$. Equations (3.19), (3.24) and (3.25) are arranged in order to form

$$
\begin{aligned}
S(x_j) &= P_{j-3}\left(\frac{1}{6}\right) + P_{j-2}\left(\frac{2}{3}\right) + P_{j-1}\left(\frac{1}{6}\right), \\
S'(x_j) &= P_{j-3}\left(\frac{1}{2h}\right) + P_{j-2}(0) + P_{j-1}\left(-\frac{1}{2h}\right), \\
S''(x_j) &= P_{j-3}\left(\frac{1}{h^2}\right) + P_{j-2}\left(-\frac{2}{h^2}\right) + P_{j-1}\left(\frac{1}{h^2}\right).
\end{aligned} \tag{3.26}
$$

Based on Equation (3.26), the function value at the knots are shown in Table 3.1.

Table 3.1: Values of $B_j^4(x)$, $B_j^{4'}(x)$,$B_j^{4''}(x)$

x	x_j	x_{j+1}	x_{j+2}	x_{j+3}	x_{j+4}
$B_j^4(x)$	0	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0
$B_j^{4'}(x)$	0	$\frac{1}{2h}$	0	$-\frac{1}{2h}$	0
$B_j^{4''}(x)$	0	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$	0

Cubic B-spline and its derivatives at x_j are simplified to a combination of P_{j-3}, P_{j-2} and P_{j-1} as listed in Equation (3.26). The simplified equations are very helpful for us to solve numerical problems using the CuBSM. Equations (3.26) are used to form a matrix abbreviated as

$$\mathbf{B_1 P_j} = \mathbf{F}, \tag{3.27}$$

where

$$\mathbf{B_1} = \begin{pmatrix} \frac{1}{2h} & 0 & -\frac{1}{2h} & & \dots & & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & \dots & & 0 \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ 0 & & \dots & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & & \dots & & \frac{1}{2h} & 0 & -\frac{1}{2h} \end{pmatrix}_{(n+3)\times(n+3)},$$

$$\mathbf{P_j} = \begin{pmatrix} P_{-3} \\ P_{-2} \\ \vdots \\ \vdots \\ P_{n-2} \\ P_{n-1} \end{pmatrix}_{(n+3)\times 1}, \qquad \mathbf{F} = \begin{pmatrix} f_{-3} \\ f_{-2} \\ \vdots \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{pmatrix}_{(n+3)\times 1},$$

where **F** is a function. From matrix in Equation (3.27), the coefficients are determined by finding inverse of matrix $\mathbf{B_1}$ that leads to $\mathbf{P_j} = \mathbf{B_1}^{-1}\mathbf{F}$.

Example 3.1 below shows the interpolation function $S(x)$ with polynomial, $P_i \in \mathbb{R}$ and Example 3.2 polynomial, $P_i \in \mathbb{R}^2$.

Example 3.1. Taking an example of interpolation of a few data points for $S(x) \in \mathbb{R}$ where I_0, I_1,I_2, ... I_{20} be the interpolation points. Let $f(x) = e^{i(2x)}\,\mathrm{sech}\,(x)$ be the function that we interpolate where i is the imaginary number. The given data points are $(x_j, f(x_j))$, $j = 0,1,2,...,20$. Table 3.2 below is the given data points in tabulated form.

The following boundary conditions are as follows:

$$\frac{\partial S}{\partial x}(x_0) = 0, \ \ \frac{\partial S}{\partial x}(x_{20}) = 0.$$

Table 3.2: Data points, $(x_j, | f(x_j) |)$ for $j = 0, 1, 2, ..., 20$

j	x_j	$\| f(x_j) \|$
0	-10	9.1×10^{-5}
1	-9	2.5×10^{-4}
2	-8	6.7×10^{-3}
3	-7	1.8×10^{-3}
4	-6	5.0×10^{-3}
5	-5	1.3×10^{-2}
6	-4	3.7×10^{-2}
7	-3	1.0×10^{-1}
8	-2	2.7×10^{-1}
9	-1	6.5×10^{-1}
10	0	1.0×10^{0}
11	1	6.5×10^{-1}
12	2	2.7×10^{-1}
13	3	1.0×10^{-1}
14	4	3.7×10^{-2}
15	5	1.3×10^{-2}
16	6	5.0×10^{-3}
17	7	1.8×10^{-3}
18	8	6.7×10^{-3}
19	9	2.5×10^{-4}
20	10	9.1×10^{-5}

By using the scheme from Equation (3.27), we will obtain the following coefficients

$$P_j = \begin{bmatrix} 0.000469578 \\ 0.000237564 \\ 0.000469578 \\ \vdots \\ \vdots \\ 0.000237564 \\ 0.000469578 \end{bmatrix}.$$

The coefficients obtained are substituted into Equation (3.17) to form an interpolating polynomial

$$
S(x) = P_{j-3}B^4_{j-3}(x) + P_{j-2}B^4_{j-2}(x) + P_{j-1}B^4_{j-1}(x) + P_jB^4_j(x), \quad x \in [x_j, x_{j+1}],
$$
$$
= \begin{cases}
-0.0230824 - 0.008943x - 0.0010934x^2 - 0.00004308x^3, & x \in [-10,-9], \\
0.0705068 + 0.0222534x + 0.00237287x^2 + 0.00008530x^3, & x \in [-9,-8], \\
0.110057 + 0.0370845x + 0.00422676x^2 + 0.000162542x^3, & x \in [-8,-7], \\
\vdots \\
\vdots \\
0.0705068 - 0.0222534x + 0.00237287x^2 - 0.00008530x^3, & x \in [8,9], \\
-0.0230824 + 0.00894298x - 0.0010934x^2 + 0.0000431x^3, & x \in [9,10].
\end{cases} \tag{3.28}
$$

The obtained piecewise polynomial in Equation (3.28) is used to plot the interpolation curve through the chosen interpolation points as in Figure 3.12.

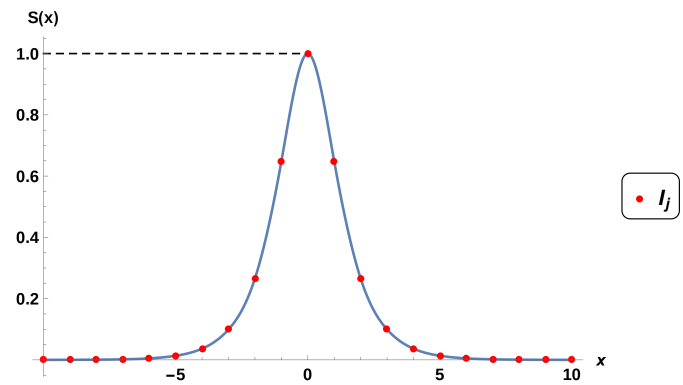

Figure 3.12: The interpolation curve through interpolation points, I_j in Example 3.1

It can be seen that the interpolation curve, $S(x)$ mimic the shape of $f(x)$ nicely in Figure 3.13. The next example referred to the polynomial, $P_j \in \mathbb{R}^2$.

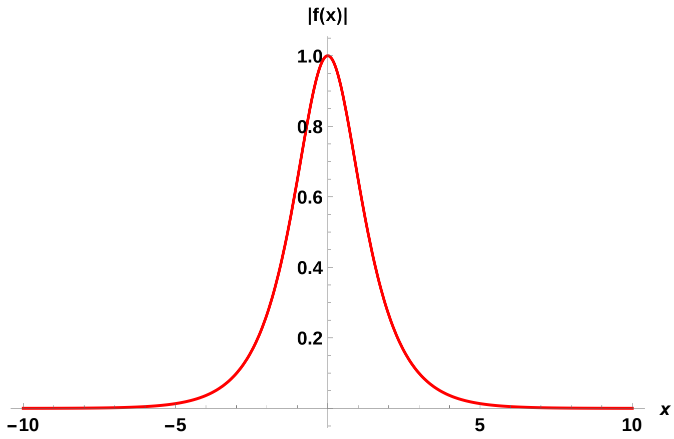

Figure 3.13: The plot of $|\ f(x)\ |$ for $x \in [-10, 10]$

Example 3.2. To understand further the interpolation of cubic B-spline in $\mathbb{R}^2$. Given

$$I_0 = \left(\frac{5}{3}, \frac{50}{3}\right), \quad I_1 = \left(\frac{25}{3}, \frac{40}{3}\right), \quad I_2 = \left(\frac{25}{3}, -\frac{40}{3}\right),$$

with the following boundary conditions:

$$\frac{\partial}{\partial x} S(x_0) = (5, 10), \quad \frac{\partial}{\partial x} S(x_2) = (-5, -20).$$

The interpolating points are interpolated by polynomial $S(x)$ of degree 3 based on the scheme from Section 3.3.2(b) and Equation (3.27). We obtain the following coefficients:

$$P_0 = (0, 0), \quad P_1 = (0, 20), \quad P_2 = (10, 20), \quad P_3 = (10, -20), \quad P_4 = (0, -20).$$

The interpolation curve is given by

$$S(x) = P_0 B_0^4(x) + P_1 B_1^4(x) + P_2 B_2^4(x) + P_3 B_3^4(x) + P_4 B_4^4(x), \quad x \in [x_0, x_2]. \tag{3.29}$$

The obtained coefficients are used to plot the interpolation curve, $S(x)$ through the chosen interpolation points given where the red curve is the first curve and blue curve is the second curve as shown in Figure 3.14.

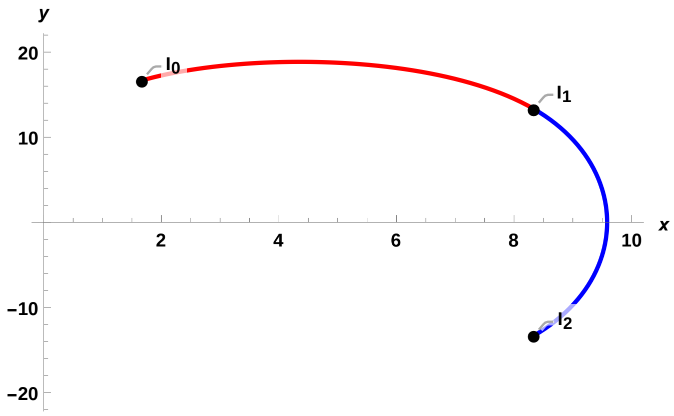

Figure 3.14: The interpolation curve through interpolation points, I_0, I_1, I_2 using the coefficients P_0, P_1, P_2, P_3, P_4 in Example 3.2

Concluding this section, cubic B-spline is one of the approximate methods in solving numerical problems. Apart from this, there are other types of approximate methods that are also suitable to solve numerical problems such as Taylor's approximation. The next section will discuss the fundamental concepts of Exponential B-splines.

3.4 Exponential B-splines

The next B-spline method, this section discusses the description and properties of the Cubic Exponential B-spline. The interval $[x_j, x_{j+1}] \subset [a,b]$ with $j = 0, 1, 2, ..., n-1$ as mentioned in Definition 3.1 is considered. Together on this interval points, x_j for $j = -3, -2, -1, n+1, n+2, n+3$ outside of the domain are added to define exponential B-splines, $BE_j^4(x)$, mentioned by Mohammadi (2013) are as follows:

$$\zeta(x) = \sum_{j=-3}^{n-1} V_j BE_j^4(x), \tag{3.30}$$

where

$$BE_j^4(x) = \begin{cases} b_2((x_j - x - 2h) - \frac{1}{p}(\sinh(p(x_j - x - 2h)))), & x \in [x_j, x_{j+1}), \\ a_1 + b_1(x_{j+2} - x - 2h) + c_1 \exp(p(x_{j+2} - x - 2h)) & \\ +d_1 \exp(p(x_{j+2} - x - 2h)), & x \in [x_{j+1}, x_{j+2}), \\ a_1 + b_1(x - x_{j+2} - 2h) + c_1 \exp(p(x - x_{j+2} - 2h)) & \\ +d_1 \exp(p(x - x_{j+2} - 2h)), & x \in [x_{j+2}, x_{j+3}), \\ b_2((x - x_{j+4} + 2h) - \frac{1}{p}(\sinh(p(x - x_{j+4} + 2h)))), & x \in [x_{j+3}, x_{j+4}), \\ 0, & otherwise, \end{cases} \tag{3.31}$$

$$a_1 = \frac{phc}{phc - s},\ b_1 = \frac{p}{2}\left[\frac{c(c-1)+s^2}{(phc-s)(1-c)}\right],\ b_2 = \frac{p}{2(phc-s)},$$

$$c_1 = \frac{1}{4}\left[\frac{\exp -ph(1-c) + s(\exp -ph - 1)}{(phc-s)(1-c)}\right],\ d_1 = \frac{1}{4}\left[\frac{\exp ph(c-1) + s(\exp ph - 1)}{(phc-s)(1-c)}\right],$$

$$s = \sinh ph,\ c = \cosh ph,\ h = \frac{b-a}{n},$$

BE is a basis, V are unknown coefficients and $p > 0$ is a free parameter. (Mohammadi, 2013)

Exponential B-spline basis function are denoted as $BE_j^4(x)$, with Figure 3.15 illustrating the basis for Cubic Exponential B-spline function.

The parameter p is used to get special forms of Cubic Exponential B-spline basis function, for our case $p = 1.0$. The first and second derivatives of the exponential B-spline basis functions from Equation (3.30) are in Equations (3.32) and (3.33).

$$\zeta'(x) = \frac{d}{dx}\left[\sum_{j=-3}^{n-1} V_j BE_j^4(x)\right] = \sum_{j=-3}^{n-1} V_j \left[\frac{d}{dx} BE_j^4(x)\right], \tag{3.32}$$

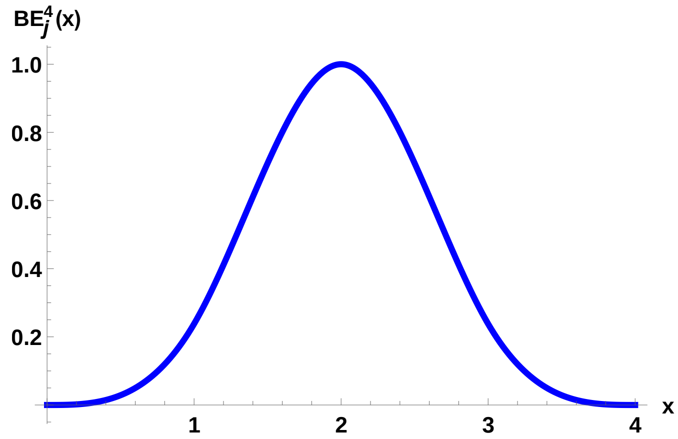

Figure 3.15: Basis function of cubic Exponential B-spline with $p = 1.0$

$$\zeta''(x) = \frac{d^2}{dx^2}\left[\sum_{j=-3}^{n-1} V_j BE_j^4(x)\right] = \sum_{j=-3}^{n-1} V_j \left[\frac{d^2}{dx^2} BE_j^4(x)\right]. \tag{3.33}$$

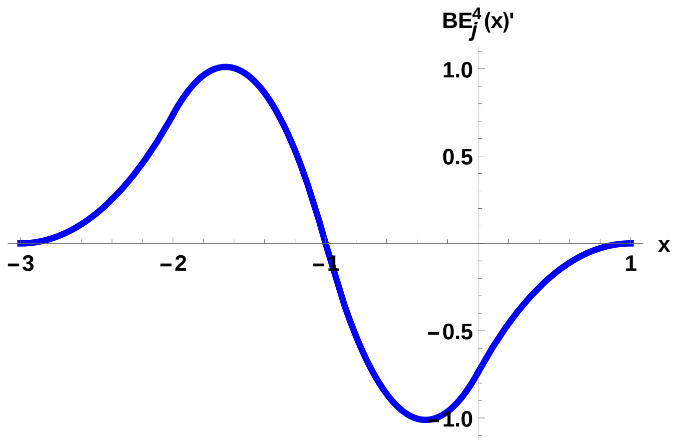

Figure 3.16: First derivative Exponential B-spline basis for $\frac{d}{dx}\left[BE_j^4(x)\right]$ with $p = 1.0$

The derivatives are illustrated in Figures 3.16 and 3.17. Thus, the curve of cubic Exponential B-spline basis, $BE_j^4(x)$ from Equation (3.30) is C^2 continuity.

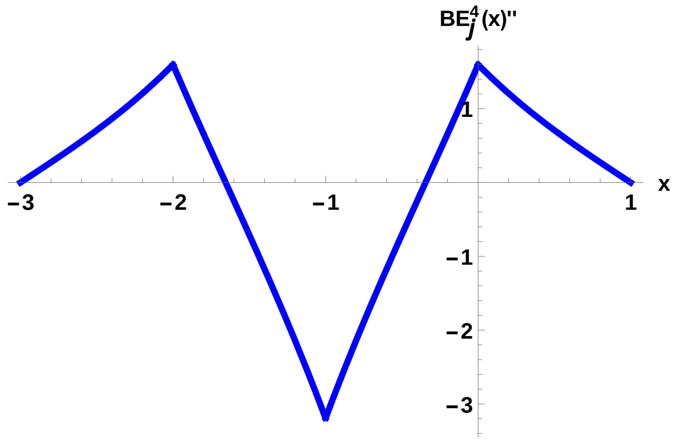

Figure 3.17: Second derivative Exponential B-spline basis for $\frac{d^2}{dx^2}\left[BE_j^4(x)\right]$ with $p = 1.0$

Table 3.3 summarizes the values of $BE_j^4(x)$, $BE_j^{4'}(x)$ and $BE_j^{4''}(x)$ at points x_j which are taken from Equation (3.31).

Table 3.3: Values of $BE_j^4(x)$, $BE_j^{4'}(x)$,$BE_j^{4''}(x)$

x	x_j	x_{j+1}	x_{j+2}	x_{j+3}	x_{j+4}
$BE_j^4(x)$	0	Ω_1	Ω_2	Ω_1	0
$BE_j^{4'}(x)$	0	Ω_3	0	Ω_4	0
$BE_j^{4''}(x)$	0	Ω_5	Ω_6	Ω_5	0

Equations (3.34) are obtained from Equations (3.30), (3.32) and (3.33) as follows:

$$\begin{aligned} \zeta(x_j) &= V_{j-3}(\Omega_1) + V_{j-2}(\Omega_2) + V_{j-1}(\Omega_1), \\ \zeta'(x_j) &= V_{j-3}(\Omega_3) + V_{j-2}(0) + V_{j-1}(\Omega_4), \\ \zeta''(x_j) &= V_{j-3}(\Omega_5) + V_{j-2}(\Omega_6) + V_{j-1}(\Omega_5). \end{aligned} \tag{3.34}$$

where

$$\Omega_1 = \frac{s - ph}{2(phc - s)}, \quad \Omega_2 = 1, \quad \Omega_3 = \frac{p(1-c)}{2(phc - s)},$$

$$\Omega_4 = -\frac{p(1-c)}{2(phc-s)}, \quad \Omega_5 = \frac{p^2 s}{2(phc-s)}, \quad \Omega_6 = -\frac{p^2 s}{phc-s},$$

The CuEBSM and its derivatives are the combination of V_{j-3}, V_{j-2} and V_{j-1} yet to be determined. A system of equation forming a matrix abbreviated as

$$\mathbf{E_1 V_j} = \mathbf{F}, \tag{3.35}$$

where

$$\mathbf{E_1} = \begin{pmatrix} \Omega_3 & 0 & \Omega_4 & & \cdots & & 0 \\ \Omega_1 & \Omega_2 & \Omega_1 & & \cdots & & 0 \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ 0 & & \cdots & & \Omega_1 & \Omega_2 & \Omega_1 \\ 0 & & \cdots & & \Omega_3 & 0 & \Omega_4 \end{pmatrix}_{(n+3)\times(n+3)},$$

$$\begin{pmatrix} V_{-3} \\ V_{-2} \\ \vdots \\ \vdots \\ V_{n-2} \\ V_{n-1} \end{pmatrix}_{(n+3)\times 1}, \quad \mathbf{F} = \begin{pmatrix} f_{-3} \\ f_{-2} \\ \vdots \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{pmatrix}_{(n+3)\times 1}.$$

Equation (3.35) is solved by finding the inverse of matrix $\mathbf{E_1}$. The following examples illustrate the use of CuEBSM.

Example 3.3 illustrates the interpolation function $\zeta(x)$ with polynomial, $P_i \in \mathbb{R}$ and

Example 3.4 polynomial, $P_i \in \mathbb{R}^2$.

Example 3.3. Taking an example of an interpolation of a few data points for $\zeta(x) \in \mathbb{R}$ where I_0, I_1,I_2, ... I_{20} be the interpolation points. Let $f(x) = e^{i(2x)}\,\mathrm{sech}\,(x)$ be the function that we interpolate where i is the imaginary number. The given data points are $(x_j, f(x_j))$, $j = 0, 1, 2, ..., 20$. Table 3.4 below is the given data points in tabulated form.

Table 3.4: Data points for Example 3.3, $(x_j, \mid f(x_j) \mid)$ for $j = 0, 1, 2, ..., 20$

j	x_j	$\mid f(x_j) \mid$
0	-10	9.1×10^{-5}
1	-9	2.5×10^{-4}
2	-8	6.7×10^{-3}
3	-7	1.8×10^{-3}
4	-6	5.0×10^{-3}
5	-5	1.3×10^{-2}
6	-4	3.7×10^{-2}
7	-3	1.0×10^{-1}
8	-2	2.7×10^{-1}
9	-1	6.5×10^{-1}
10	0	1.0×10^{0}
11	1	6.5×10^{-1}
12	2	2.7×10^{-1}
13	3	1.0×10^{-1}
14	4	3.7×10^{-2}
15	5	1.3×10^{-2}
16	6	5.0×10^{-3}
17	7	1.8×10^{-3}
18	8	6.7×10^{-3}
19	9	2.5×10^{-4}
20	10	9.1×10^{-5}

The following boundary conditions are as follows:

$$\frac{\partial \zeta}{\partial x}(x_0) = 0, \;\; \frac{\partial \zeta}{\partial x}(x_{20}) = 0.$$

By using the scheme from Equation (3.35), we obtain the following coefficients:

$$V_j = \begin{bmatrix} 0.000272918 \\ 0.000090800 \\ 0.000272918 \\ \vdots \\ \vdots \\ 0.000090800 \\ 0.000272918 \end{bmatrix}.$$

The coefficients obtained are used in the interpolating function from Equation (3.30) to form an interpolating polynomial

$$\zeta(x) = V_{j-3}BE^4_{j-3}(x) + V_{j-2}BE^4_{j-2}(x) + V_{j-1}BE^4_{j-1}(x) + V_jBE^4_j(x), \quad x \in [x_j, x_{j+1}].$$

$$= \begin{cases} -0.00112256 - 0.000114256x + 2.73425\cosh(x) & \\ +2.73425\sinh(x), & x \in [-10,-9], \\ 0.000888364 + 0.000109179x + 1.82899\cosh(x) & \\ +1.82899\sinh(x), & x \in [-9,-8], \\ 0.000147881 + 0.0000166186x + 1.96695\cosh(x) & \\ +1.96695\sinh(x), & x \in [-8,-7], \\ \vdots & \\ \vdots & \\ 0.000888364 - 0.000109179x + 1.82899\cosh(x) & \\ -1.82899\sinh(x), & x \in [8,9], \\ -0.00112256 + 0.000114256x + 2.73425\cosh(x) & \\ -2.73425\sinh(x), & x \in [9,10]. \end{cases} \tag{3.36}$$

The obtained piecewise polynomial in Equation (3.36) is used to plot the interpolation curve through the chosen interpolation points as in Figure 3.18.

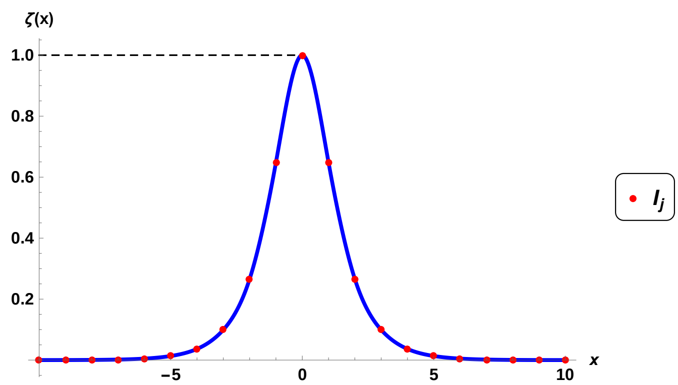

Figure 3.18: The interpolation curve through interpolation points, I_j for Example 3.3