# NEW ITERATIVE SCHEMES IN THE NUMERICAL SOLUTION OF TWO-DIMENSIONAL TIME-FRACTIONAL HYPERBOLIC PARTIAL DIFFERENTAIL EQUATIONS

**AJMAL ALI**

**UNIVERSITI SAINS MALAYSIA**

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**by**

**AJMAL ALI**

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# TABLE OF CONTENTS

# LIST OF TABLES

## LIST OF FIGURES

**LIST OF SYMBOLS**

| | |
|---|---|
| k | Time level |
| n | Number of iterations |
| m | Mesh size |
| p | Natural number |
| N | Set of natural numbers |
| h | Grid spacing in both x and y directions |
| $N_t$ | Natural number chosen for time |
| $N_x$ | Natural number chosen for space in x direction |
| $N_y$ | Natural number chosen for space in y direction |
| $\alpha$ | Order of fractional derivative |
| Ω | Solution domain |
| $z$ | Complex number |
| Γ | Gamma function |

## LIST OF ABBREVIATIONS

| | |
|---|---|
| 1D | One dimension |
| 2D | Two dimensions |
| CN | Crank Nicolson |
| PDE | Partial differential equation |
| FDE | Fractional differential equation |
| FDM | Finite difference method |
| FWDE | Fractional wave diffusion equation |
| FDWDE | Fractional damped wave diffusion equation |
| FHTDE | Fractional hyperbolic telegraph differential equation |
| SFGS | Standard fractional Gauss-Seidel |
| RFGS | Rotated fractional Gauss-Seidel |
| EG | Explicit group |
| EDG | Explicit de-coupled group |
| MEG | Modified explicit group |
| MEDG | Modified explicit de-coupled group |
| FEG | Fractional explicit group |
| FEDG | Fractional explicit de-coupled group |
| MFEG | Modified fractional explicit group |
| MFEDG | Modified fractional explicit de-coupled group |
| add | Addition |
| sub | Subtraction |
| div | Division |
| mul | Multiplication |
| w. r. t | With respect to |

## LIST OF APPENDICES

**SKEMA-SKEMA LELARAN BARU DALAM PENYELESAIAN BERANGKA PERSAMAAN PEMBEZAAN SEPARA HIPERBOLIK PECAHAN-MASA DUA DIMENSI**

## ABSTRAK

Dalam literatur, skema-skema berangka seperti kaedah perbezaan terhingga, kaedah unsur terhingga, kaedah isipadu terhingga, kaedah unsur sempadan dan kaedah spektral telah digunakan untuk pendiskretan pelbagai jenis persamaan pembezaan pecahan (PPP). Kaedah-kaedah sebegini mengalihkan PPP kepada suatu sistem persamaan linear serentak besar dan jarang yang boleh diselesaikan dengan kaedah-kaedah lelaran yang berdasarkan skema lelaran berorientasikan titik pada domain pendiskretan penuh $\Omega_h$, di mana $h$ ialah jarak grid dalam dua-dua arah $x$ dan $y$. Dalam semua jenis kaedah lelaran ini, operasi aritmetik yang besar diperlukan untuk penumpuan, kerana nilai-nilai sebelumnya perlu disimpan jika nilai-nilai baru ingin dikira. Sejak beberapa dekad yang lalu, ramai sarjana dan penyelidik telah membangunkan banyak algoritma cepat efisien untuk mengurangkan kos pengiraan. Permintaan yang meningkat bagi simulasi resolusi lanjutan dalam masa komputer yang singkat telah terus mencabar penyelidik untuk menghasilkan algoritma pengiraan yang pantas, teratur dan lebih efektif dalam menyelesaikan PPP. Salah satu cara untuk mencapai penumpuan pantas adalah dengan penggunaan kaedah lelaran berkumpulan yang berdasarkan skema lelaran berorientasikan kumpulan menggunakan kurang daripada $h^{-2}/2$ atau $h^{-2}/4$ (bergantung pada kaedah yang digunakan) titik lelaran pada domain ruang. Dalam

tesis ini, skema-skema lelaran kumpulan baru untuk menyelesaikan persamaan resapan gelombang pecahan dua dimensi kedua-dimensi, persamaan resapan gelombang lembap pecahan dan persamaan pembezaan telegraf hiperbolik pecahan, diterbitkan yang menggunakan keupayaan pengiraan yang kurang dan seterusnya mengurangkan masa pelaksanaan per lelaran tanpa menjejaskan ketepatan penyelesaian. Analisis kestabilan dan penumpuan bagi skema-skema lelaran titik dan kumpulan yang diterbitkan akan dibuktikan dengan menggunakan algoritma norma Fourier dan matriks masing-masing.

**NEW ITERATIVE SCHEMES IN THE NUMERICAL SOLUTION OF TWO-DIMENSIONAL TIME-FRACTIONAL HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS**

## ABSTRACT

In literature, numerical schemes such as finite difference method, finite element method, finite volume method, boundary element method and spectral methods have been utilized for the discretization of many types of Fractional Differential Equations (FDEs). Such types of methods lead FDEs into a large and sparse system of simultaneous linear equations which can be solved by iterative methods that are based on the point-oriented iteration schemes on the whole discretization domain $\Omega_h$, where $h$ is the grid spacing in both $x$ and $y$ directions. In all such type of iterative methods, large arithmetical operations are required for convergence, because previous values have to be stored if the recent value is to be calculated. Over the past decades, many scholars and researchers have established numerous proficient fast algorithms to reduce the computation cost. The increasing demand for advanced resolution simulations in less computer time have continuously challenged the researchers to come up with more effective, well-organized and fast computational algorithms in solving the FDEs. One of the ways to achieve faster convergence is by the utilization of group iterative methods which are based on group-oriented iteration schemes that utilize less than $h^{-2}/2$ or $h^{-2}/4$ (depending on the method used) iterative points of the spatial domain. In this thesis, new group iterative schemes for solving the second-order two-dimensional

fractional wave diffusion equation (FWDE), fractional damped wave diffusion equation (FDWDE) and fractional hyperbolic telegraph differential equation (FHTDE) are derived which utilize less computational efforts and hence reduce the execution of time per iteration without deteriorating the solution accuracies. The stability and convergence analysis of the derived points and group iterative schemes will be established by using the Fourier and matrix norm algorithms respectively.

# CHAPTER 1

# INTRODUCTION

## 1.1 Introduction

During the middle of last decade, Fractional Differential Equations (FDEs) have assumed more prominence over the Partial Differential Equations (PDEs) due to non-local property and its realistic natural phenomena, since FDEs describe values on each point continually and distinguished the gaps between the two integers. This was motivated by the fact that many significant applications of FDEs have been found in various and extensive fields of engineering (Bagley and Torvik, 1984; Mainardi, 1997), finance (Scalas et al., 2000; Raberto et al., 2002), hydrology (Benson et al., 2000; Liu et al., 2003), medical sciences (Santamaria et al., 2006; Henry et al., 2008; Margin et al., 2009; Hall and Barrick, 2008), physics (Metzler and Klafter, 2000; Saichev and Zaslavsky, 1997; Zaslavsky, 2002, 1994, 2005), bio sciences (Magin, 2006), chemistry (Yuste et al., 2004), and other sciences (including viscoelasticity (Mainardi, 2010), control system (Agrawal et al., 2010), signal analysis (Zhou et al., 2010), operator theory (Torbati and Hammond, 1998) etc.). The main benefit of FDEs is the formulation of the memory, hereditary characteristics of numerous substances and the procedure that are decreed by the anomalous diffusion in the fractional environment which can never be considered by PDEs (Torbati and Hammond, 1998).

As far as the geometrical and physical interpretation of integer-order differentials and integrals are concerned, they have the clear physical interpretation in calculus

(Oldham and Spanier, 1974). Fractional derivative and integral are more specific in description than the ordinary or integer-order derivative and integral in many cases and hence fractional derivative and integral give more deep result and realistic phenomena (Kiblas et al., 2006). Podlubny (2002) has developed innovative interpretation and he narrated the physical interpretation of fractional derivative to theory of relativity by introducing two type of time namely, individual and intergalactic time. Intergalactic time means the traveling at a speed closer to speed of light which is possible in theories only. Miller and Ross (1993) has also explored in detail the physical interpretation of fractional derivative. The geometrical interpretation of integer order differential and integration, can be seen in classical geometry, whereas some researchers seek the geometrical interpretation of fractional differential and integral in fractal geometry, as the classical geometry is the subclass of fractal geometry (Torbati and Hammond, 1998).

In different field of science, numerous problems have been successfully described by models using mathematical tools from FDEs in relation to the time-fractional, space- fractional and time-space fractional derivatives. Some types of FDEs can be solved analytically with the help of some particular types of transforms and special functions such as Laplace transform (Jumarie, 2009; Agrawal, 2002), Fourier transform (Chen et al., 2007; Orsingher and Beghin, 2004), Mellin transform (Liu et al., 2003), Fox functions (Wyss, 1986; Schneider and Wyss, 1989), Wright functions (Gorenflo et al., 2000) and Green functions (Momani and Odibat, 2007; Huang and Liu, 2005) etc. Most of these approaches are appropriate for the solution of linear FDEs but generally not applicable for non-linear FDEs. For such types non-linear FDEs it is quite difficult to obtain purely explicit analytic solution which usually contains special functions. Then, scholars and researchers are always looking forward to some suitable and excellent numerical approximations to the exact analytic solutions. For discretizing the research problems in term of PDEs, the most commonly used and easier to implement method is the finite difference method as compared to any other method in literature, since it is universal applicable for both linear and non-linear problems (Evans, 1997). Several types of finite difference methods related to group iterative approach such as standard and rotated five point, and standard and rotated seven point schemes have been utilized in the past few years (Othman and Abdullah, 2000b; Saeed and Ali, 2009, 2011; Teng and Ali, 2014; Kew and Ali, 2015). The rotated schemes can be obtained by rotating an angle of $45^\circ$ (clockwise) with respect to the standard mesh. On the basis of standard and rotated five point as well as seven point discretization techniques, a series of four-point and eight point explicit group methods have been introduced. Abdullah (1991) established four point Explicit Decoupled Group (EDG) method by describing the PDE on the basis of rotated (skewed) grids. Kadalbajoo and Rao (1997) introduced a new Explicit Group (EG) method for solving block tridiagonal linear system derived from standard five point finite difference discretization for parabolic PDE. Othman and Abdullah (2000a) derived EG iterative method and modified four point EG iterative method from standard five point approximations formula with $h$ and $2h$ grid spacing respectively. They also developed EDG iterative method from $h$-spaced rotated five point approximations formula and their results derived from modified four point EG method are compared which were much better than EG and EDG iterative methods. Since then, active investigation has been conducted to examine the capabilities of various group iterative methods by utilizing the rotated finite difference approximation rather than the traditional standard finite difference approximation in solving numerous types of PDEs. Tan et al. (2010) found the significant results from group iterative methods which were derived from standard and skewed five point meshes in solving the 2D convection-diffusion equation. Ali and Kew (2012) derived four point group iterative methods on the behalf of standard and rotated five point finite difference approximations considering *h* and *2h* grid spacing for the solution of 2D hyperbolic equation and established its consistency by applying Taylor's series and convergence by eigenvalue arguments. Ali and Saeed (2012) constructed a specific splitting-type preconditioner in block formulation applied to a class of group relaxation iterative methods derived from the centred and rotated finite difference approximations to increase the rate of convergence. Kew and Ali (2015) utilized both standard and rotated seven point finite difference approximations for solving the three dimensional hyperbolic telegraph PDE. Likewise the PDEs, finite difference schemes are also appropriate for discretization of FDEs. In this regard, recently Balasim (2017) discretized the 2D time-fractional diffusion, advection diffusion and cable equations by utilizing the standard and rotated five point finite difference schemes. On utilizing the basic idea of standard and skewed meshes (*h* or *2h* spacing), he developed Fractional Standard Point (FSP), Fractional Rotated Point (FRP), Fractional Explicit Group (FEG), Fractional Explicit De-coupled Group (FEDG), Modified Fractional Explicit Group (MFEG) and Modified Fractional Explicit De-coupled Group (MFEDG) iterative methods and found significant numerical results. In this thesis, our research work is continuation of Balasim (2017) research work for solving the second-order two-dimensional wave-diffusion, damped wave-diffusion and hyperbolic telegraphic equations of fractional order. In all types of these equations, $\alpha \in (1,2)$ are used, which made the case more complicated to achieve convergence due to the appearance of integer "2" in the second-order differential operator which we use in the definition of

Caputo's fractional derivative. To apply suitable finite difference approximation on the time-fractional hyperbolic telegraph differential equation is not a straightforward task due to involvement of the double time-fractional derivatives in the equation. The crucial part is to find an appropriate finite difference approximation formula for the spatial derivative that works well with the time-fractional derivative, which is computationally inexpensive and easy to solve. Several two-dimensional higher-order FDEs are still left unsolved by group iterative methods and stability and convergence are the challenging tasks for interested scholars in this areas of research. Moreover, so far the grouping strategies have not been investigated on hyperbolic PDEs.

## 1.2 Motivation

To construct numerical approximation equations, most of the scholars discretized their proposed problems by utilizing various types of discretization techniques such as finite difference method (Zhang, 2009), finite element method (Zhu et al., 2017), boundary element method (Katsikadelis, 2011), finite volume method (Liu et al., 2014) and spectral methods (Bhrawy, 2016) which are further classified in three types of methods namely, collocation, Tau and Galerkin methods. The discretized schemes in this way generate a sparse system of simultaneous linear equations of the following form,

$$Au = b. \tag{1.1}$$

Here, iterative methods are very important in solving system of linear equations. In the literature, many iterative methods have been suggested for solving the linear system by various authors like Saad (1996), Hackbusch (1995) and Young (1971). To solve the linear system in Eq. (1.1), these types of iterative methods utilize all the grid points of solution domain $\Omega_h$, ($h$ is grid spacing in both $x$ and $y$ direction such that $h = \Delta x = \Delta y$) including the boundary points to achieve the convergence. Due to the involvement of all the grid points of solution domain $\Omega_h$, these iterative methods are forced to utilize a lot of arithmetic operations in the iterative loop that suffer into large computational complexities and hence, iterative methods consume more execution time per iteration.

To overcome this issue we suggest the use of quarter-sweep approach via the modified group explicit methods which are only possible for implementation of finite difference approximation with *2h* grid spacing. The main concept of quarter-sweep approach is based on time reduction techniques (the procedure that reduces the CPU-timings for attaining the convergence) that utilize quarter grid points of the solution domain and these points are treated as iterative points that take a part in the iterative process. Just by utilization of only the quarter grid points of the solution domain, iterative process reduces computational complexity of the algorithm and hence ultimately decrease in execution of time per iteration. The remaining grid points that do not take a part in the iterative process are called direct points which can directly be evaluated by fractional standard point method.

The quarter-sweep technique by using modified group explicit methods is an efficient technique to speed up the rate of convergence. On the basis of this concept, Othman and Abdullah (2000a) proposed Modified Explicit Group (MEG) iterative method for solving the 2D Poisson equation by utilizing the quarter grids stencil of the solution domain and it was found that it is much better and faster than the results obtained from EDG iterative method derived from utilizing the half grids stencil of the solution domain for the same 2D Poisson equation (Abdullah, 1991). Later, several researchers utilize this technique for other types of PDEs via MEG as well as Modified Explicit De-coupled Group (MEDG) iterative methods. Ali and Ng (2007) proposed MEDG iterative method using quarter-sweep approach and showed that this method is more rapid in convergence than EDG method which they derived by using half-sweep approach in solving 2D Elliptic PDE. Sulaiman et al. (2010) derived four point block MEGSOR iterative scheme utilizing quarter grid stencil and found that it better than four point block EDGSOR scheme derived from half grid stencil. Ali and Foo (2012) suggested four point MEDGAOR iterative scheme utilizing quarter grid stencil and found that it is more superior than four point EDGAOR scheme derived from half grid stencil in terms of CPU timings and number of iterations. Ali and Kew (2012) proposed an excellent comparison among the four types of group relaxation methods. They derived EG iterative method using the full grids stencil, EDG method iterative using the half grids stencil and, MEG and MEDG iterative methods by using $2h$-spaced quarter grids stencil (quarter grid points to be utilized as iterative points with $2h$-spacing in solution domain) for two-dimensional hyperbolic telegraph equation and find that MEDG method has the least number of iterations, total number of iterations and elapsed timings as compared to the other method tested. Ali and Aziz (2013) introduced MEDG (SOR) method on the basis of quarter-sweep technique for 2D Helmholz equation and find significant results in terms of least CPU-timings when compared to EDG (SOR) method which was derived from half-sweep technique. Recently, Balasim (2017) utilized this approach by using MEG and MEDG iterative methods for fractional parabolic PDEs and found significant results as compared to EG and EDG iterative methods.

In addition, the group iterative method derived from $h$ grid-spacing is also very effective for solving the linear system (1.1) iteratively for many types of PDEs, where a group of 4-points behaves like single point in iterative process and this process continuously going on until a certain time level is achieved . The main advantage of this concept is also based on the computational reduction technique that reduces computational complexity of the complicated algorithm generated by corresponding approximation equations. In their earlier work, Evans and Abdullah (1983b) introduced the concept of group method in solving the heat equation by coupling two values of approximations, obtained from asymmetric schemes, in a group. The results obtained were in implicit form but they can easily be converted into explicit form with excellent accuracies. In the same year, Evans and Abdullah (1983a) proposed 4-point EG method for the solution of 2D parabolic equation where two level scheme were considered. Evans (1985) used 4-point block EG iterative method to solve the sparse system of simultaneous linear equations. Later, Evans and Yousif (1986) applied EG iterative method on solving the elliptic PDE in three-dimensional space. In recent years, Kew and Ali (2015) utilized EG iterative scheme on solving three dimensional telegraph equation and find efficient results in comparison with point based methods.

## 1.3 Research Objectives

In the past few years, after the implementation of grouping methods on several types of PDEs, for the first time, Balasim (2017) applied the grouping strategies on FDEs by using finite difference method. These types of FDEs include two-dimensional time-fractional parabolic differential equations i.e. the diffusion, advection and cable equations of fractional orders. Our main objective is to apply the same grouping strategies on two-dimensional time-fractional hyperbolic differential equations i.e. fractional wave-diffusion, damped wave-diffusion and telegraph equations. These types of hyperbolic PDEs of fractional order are solved by explicit group iterative methods derived from $h$ and $2h$-spaced standard and rotated five point Crank-Nicolson finite difference approximations. To deal with the fractional case, we use Caputo's fractional derivative. When we utilized $h$-spaced Crank-Nicolson finite difference discretization along with Caputo's fractional derivative we name the methods as FEG and FEDG, while we use $2h$-spaced Crank-Nicolson finite difference discretization along with Caputo's fractional derivative we name the methods as MFEG and MFEDG. The following points are the research objectives of our research work in this thesis.

- To formulate standard and rotated five-point Crank-Nicolson iterative schemes based on both $h$ and $2h$ grid spacing for solving 2D time-fractional hyperbolic PDEs.
- To formulate group iterative schemes by utilizing the standard and rotated five-point Crank-Nicolson iterative schemes based on both $h$ and $2h$ grid spacing for 2D time-fractional hyperbolic PDEs.
- To establish the stability and convergence for the point as well as group iterative schemes.
- To conduct the comparative studies, perform the numerical experiments on 2D time-fractional hyperbolic PDEs and analyze the results.

## 1.4 Methodology

This section presents the methodology for solving the 2D-FHTDE, while the procedure for two-dimensional FWD and FDWD equations are similar. Solving the 2D-

FHTDE is not straightforward and simple as two-dimensional FWDE and FDWDE because of the involvement of double time fractional derivatives in the equation. Suitable finite difference approximation formula for the spatial and time fractional derivatives at some appropriate point that works well need to be found. The following are the step by step procedure when 2D-FHTDE is solved by point and group iterative methods:

- Formulate two separate expressions for the time-fractional derivatives by utilizing the first and second differential operators in the Caputo's fractional derivative (Miller and Ross, 1993).
- For spatial derivatives, utilize standard Crank-Nicolson finite difference approximation by using both $h$ and $2h$-grid spacing.
- Utilize both approximations for space and time derivatives in 2D-FHTDE and collect the newly developed standard point finite difference scheme at point $(x_i, y_j, t_{k+1/2})$.
- Similarly a rotated point finite difference scheme at point $(x_i, y_j, t_{k+1/2})$ can be constructed by using rotated Crank-Nicolson finite difference approximation for both $h$ and $2h$-grid spacing in 2D-FHTDE.
- Apply both standard point finite difference schemes based on both $h$ and $2h$-spacing on a group of four points and find two group iterative schemes in matrix form.
- Apply both rotated point finite difference schemes based on both $h$ and $2h$-spacing on a group of four points and find two independently sets of group iterative schemes in matrix form.

- Study the stability and convergence analysis of rotated point finite difference scheme by Fourier analysis and group iterative schemes by matrix norm method.
- Perform numerical experiment to support the effectiveness of four group iterative schemes over the point iterative schemes with the help of some computer softwares.
- Compare the numerical results with any other method in the literature (if available) in terms of execution of time, total number of operations and number of iterations.

## 1.5 Organization of the Thesis

The organization of this thesis is comprised of the following chapters:

**Chapter 2** includes the fundamental mathematical concepts required for this thesis, introduction to fractional calculus and detailed literature review of hyperbolic differential equations of fractional order. The basic mathematical concepts includes fractional derivatives and integrals, finite difference approximations and iterative methods. In **Chapters 3-5** the derivations of point and group iterative schemes based on $h$ and $2h$-grid spacing for the solution of 2D time-fractional hyperbolic PDEs are explored. The stability and convergence of point and group iterative schemes for FWD, FDWD and FHTDEs are proven by Fourier and matrix norm respectively. The computational complexity, numerical experiments, discussion of results and graphical representations of all three types of FDEs are present. Finally, in **Chapters 6** overall conclusion of all the chapters of this thesis and related future work are given.

# CHAPTER 2

# BASIC CONCEPTS AND LITERATURE REVIEW

## 2.1 Introduction

This chapter comprised on the literature review and basic mathematical concepts which are relevant to our proposed point and group iterative schemes for the numerical solution of FDEs. To solve the fractional part of FDEs we need to utilize the basic definitions of fractional derivatives like Riemann-Liouville, Grünwald-Letnikov and Caputo's fractional derivatives. The solution of certain FDEs using the finite difference methods lead to large sparse linear system. Such type of system of linear equations can be solved by iterative methods. There are three main types of iterative methods in literature namely, Jacobi iterative method, Gauss-Seidel iterative method and Successive Over-Relaxation (SOR) iterative method.

## 2.2 Special Functions

This section briefly describes some special type of functions namely Euler's Gamma function, Beta function and Mittag-Leffler function which are useful for the numerical solution of several types of problems in fractional calculus where the order of derivative is not an integer.

### 2.2.1 Euler's Gamma Function

Euler was the first who discovered the gamma function in 1729 when he was exploring the interpolation problem for the factorial function. Gamma function is actually the generalization of factorial function in which the non-integers values are also considered. Euler's Gamma Function plays so much significant role for solving the many types of FDEs. There are numerous approaches leading to the comprehensive definition of gamma function $\Gamma(z)$ including the Euler limit defined by the following relation (Saedpanah, 2009),

$$\Gamma(z) = \lim_{N\to\infty} \frac{N!N^z}{z(z-1)(z-2)\cdots(z-N)}.$$

Another equivalent approach is Euler infinite product for all $z$ expect for non-positive integers Carpinteri and Maindardi (1997),

$$\Gamma(z) = \frac{1}{z}\prod_{p=1}^{\infty} \frac{(1+\frac{1}{p})^z}{1+\frac{z}{p}}, \quad p \in N$$

and some other are defined in terms of generalized Laguerre polynomials but the most preferred approach is from integral transfer definition which is so called Euler's integral of second kind (Samko et al., 1993),

$$\Gamma(z+1) = \int_0^{\infty} e^{-t}t^z dt, \quad R(z) > 0. \tag{2.1}$$

Integration by parts leads the recurrence formula $\Gamma(z+1) = z\Gamma(z)$ to

$$\Gamma(z+1) = z\Gamma(z) = z(z-1)\Gamma(z-1) = \cdots z(z-1)(z-2)\cdots 2.1.\Gamma(1) = z!,$$

since $\Gamma(1) = 1$ which give us $\Gamma(z+1) = z!$. Therefore, we have the following factorial function in the form of integral transform as (Oldham and Spanier, 1974),

$$z! = \int_0^\infty e^{-t} t^z dt, \quad R(z) > 0. \tag{2.2}$$

Euler's Gamma Function has the following properties,

- $\Gamma(z+1) = z\Gamma(z), \quad R(z) > 0$
- $\Gamma(p) = (p-1)!, \quad p \in N$
- $\Gamma(1-z) = -z\Gamma(-z). \quad R(z) > 0.$

### 2.2.2 Beta Function

The Beta function is Euler's integral of first kind and it is denoted by $\beta(z,w)$ and defined as (Oldham and Spanier, 1974; Samko et al., 1993; Podlubny, 1999),

$$\beta(z,w) = \int_0^\infty t^{z-1}(1-t)^{w-1} dt, \quad R(z), R(w) > 0. \tag{2.3}$$

The relationship between Gamma and Beta functions is defined by the following relation

$$\beta(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad R(z), R(w) > 0. \tag{2.4}$$

Beta function is appropriate to utilize rather than amalgamations of gamma function because it is analytically continuous to entire complex plane.

### 2.2.3 Mittag-Leffler function

A Swedish mathematician Gosta Mittag-Leffler introduced this function in 1902. This function is generalization of exponential function in a straightforward way. Several scholars often express analytical solution of their research problems in terms of Mittag-Leffler function in fractional calculus. The classical Mittag-Leffler function of one parameter $E_\alpha(z)$ is defined as (Rehman, 2011),

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad z \in C, R(\alpha) > 0 \tag{2.5}$$

where $\Gamma$ is the Gamma function. The one parameter Mittag-Leffler function always interpolates between exponential function $e^z$ and hypergeometric function $\frac{1}{z-1}$ for $0 < \alpha < 1$. The following are the calculated functions at particular values of $\alpha$ for one parameter Mittag-Leffler function

- $E_0(z) = \frac{1}{z-1}, \quad E_1(z) = e^z, \quad E_2(-z^2) = cos(x), \quad E_2(z^2) = cosh(x)$
- $E_3(z) = \frac{1}{2}[e^{z^{1/3}} + 2e^{-1/2z^{1/3}} cos(\frac{\sqrt{3}}{2} z^{1/3})]$
- $E_4(z) = \frac{1}{2}[cos(z^{1/4}) + cosh(z^{1/4})]$.

The two parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is generalization of the one parameter Mittag-Leffler function $E_\alpha(z)$ and it coincides with one parameter Mittag-Leffler function when $\beta = 1$. The two parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined as (Rehman, 2011),

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z \in C, R(\alpha), R(\beta) > 0. \tag{2.6}$$

The two parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ has the following properties (Kilbas et al., 2004),

- $\int\limits_0^\infty e^{-t}t^{\beta-1}E_{\alpha,\beta}(t^\alpha z)dt = \frac{1}{z-1}, \quad |z|< 1$
- $\int\limits_0^\infty e^{-zt}E_{\alpha,\beta}(z^\alpha)dt = \frac{1}{z-z^{1-\alpha}}. \quad |z|< 1.$

There are many generalization of two parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ in the form of generalized hyperbolic function, generalized trigonometric function and other can be seen in Gorenflo et al. (1998).

## 2.3 Fractional Differential and Integral Operators

In this section we introduce some fundamental definitions of fractional order differentiation and integration, such as Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative, Grünwald-Letnikov fractional derivative and Caputo's fractional derivative.

### 2.3.1 Riemann-Liouville (R-L) Fractional Integral

The most well-known notation for fractional order integral of a function $f(x)$ are ${}_cD_x^{-\alpha}f(x)$ and ${}_cI_x^{\alpha}f(x)$ where, $c$ and $x$ are the lower and upper limits of the fractional integral operator respectively and these are generally called as the terminals of the fractional integral and $\alpha$ is the order of integration (Podlubny, 1999). There are several approaches to define fractional ordered differentiation and integration but the most important and easy to implement is the renowned Cauchy's integral formula for p-fold integral of any function $f(x)$ (Rehman, 2011),

$$ {}_cD_x^{-p} f(x) = {}_cI_x^p f(x) = \int_c^x \int_c^{x_{p-1}} \cdots \int_c^{x_1} f(x_0)dx_0 \cdots dx_{p-2}.dx_{p-1} \tag{2.7} $$

$$ = \frac{1}{(p-1)!} \int_c^x \frac{f(\xi)}{(x-\xi)^{1-p}} d\xi, \quad p \in N. \tag{2.8} $$

From equation (2.8), we can attain the fractional order integral by replacing the integer $p$ by real number $\alpha$ and discrete form of factorial $(p-1)!$ by Gamma function $\Gamma$ provided that the integral on the right hand side converges. The Riemann-Liouville integral is defined as follows (Diethelm, 2010),

$$ {}_cD_x^{-\alpha} f(x) = {}_cI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x \frac{f(\xi)}{(x-\xi)^{1-\alpha}} d\xi, \ \mathrm{R}(\alpha) > 0. \tag{2.9} $$

When $\alpha = 0$, then ${}_cI_x^0 = I$ is the identity operator. According to Riemann, lower limit in the fractional integral (2.9) may be any arbitrary number, but Liouville choose infinity as a lower limit i.e. $c = -\infty$ and when $c = 0$ then ${}_0I_x^{\alpha}$ is called the Riemann-Liouville fractional integral and is quite appropriate for further manipulations. Moreover, for the function of the type $f(x) = x^{\delta}, \quad \delta > -1$ and $\alpha > 0$, the fractional Riemann-Liouville integration is defined as (Kiblas et al., 2006),

$$ {}_cD_x^{-\alpha} x^{\delta} = {}_cI_x^{\alpha} x^{\delta} = \frac{\Gamma(\delta+1)}{\Gamma(\alpha+\delta+1)} x^{\delta+\alpha}. \tag{2.10} $$

### 2.3.2 Riemann-Liouville (R-L) Fractional Derivative

The famous notation for the fractional operator of a function $f(x)$ is denoted as ${}_cD_x^{\alpha} f(x)$, where $\alpha$ is the order of differentiation with $c$ and $x$ represent the two limits attached to the fractional differentiation operation. Riemann-Liouville calculate the fractional derivative by using Lagrange's rule for the differential operator and fractional integral by associating with a real function from the set of real numbers to itself for each value of the parameter $\alpha > 0$. The Riemann-Liouville fractional derivatives of order $\alpha$ of the function $f(x)$ can be defined as (Klages et al., 2008),

$$ {}_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(\xi)}{(x-\xi)^{\alpha}} d\xi, \quad 0 < \alpha < 1. \tag{2.11} $$

The generalization form of Eq. (2.11) is given by the following expression (Das, 2008),

$$ {}_cD_x^{\alpha} f(x) = \frac{1}{\Gamma(p-\alpha)} \frac{d^p}{dx^p} \int_c^x \frac{f(\xi)}{(x-\xi)^{\alpha+1-p}} d\xi, \quad (p-1) < \alpha < p. \tag{2.12} $$

### 2.3.3 Caputo Fractional Derivative

In 1967, Caputo introduced the idea of fractional derivative which he later utilize his definition in the theory of viscoelasticity. Both Caputo's fractional derivative and Riemann-Liouville fractional derivative utilize the definition of Riemann-Liouville fractional integral but the order of fractional integral with integer differential operator is interchanged. Caputo's fractional derivative is defined as as (Podlubny, 1999),

$$ {}_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f'(\xi)}{(x-\xi)^{\alpha}} d\xi, \quad 0 < \alpha < 1. \tag{2.13} $$

The most generalization form of Eq. (2.13) is the following expression (Miller and Ross, 1993),

$$ {}_cD_x^{\alpha} f(x) = \frac{1}{\Gamma(p-\alpha)} \int_0^x \frac{f^{(p)}(\xi)}{(x-\xi)^{\alpha+1-p}} d\xi, \quad p-1 < \alpha < p, \;\; p \in N. \qquad (2.14) $$

For the function of the type $f(x) = x^p, \;\; p \in N$. The Caputo's fractional derivative is defined as:

$$ {}_cD_x^{\alpha}(x^p) = \begin{cases} 0, & for \qquad p \in N, \;\; p < \lceil \alpha \rceil \\ \\ \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha}, & for \qquad p \in N, \;\; p \geq \lceil \alpha \rceil. \end{cases} $$

Here $\lceil \alpha \rceil$ denotes the ceiling value of $\alpha$. It is the smallest integer greater than or equal to $\alpha$. Moreover if $f(x) = c$ is any constant function then its the Caputo's fractional derivative will be zero similar the integer order derivative i.e. ${}_cD_x^{\alpha} c = 0$. The relationship between Riemann-Liouville fractional integral and Caputo's fractional derivative is to be outlined as follows (Klages et al., 2008),

$$ {}_cD_x^{\alpha} f(x) = {}_cD_x^{p} {}_0I_x^{p-\alpha} f(x), p-1 < \alpha < p, \quad p \in N, \qquad (2.15) $$

$$ {}_cD_x^{\alpha} f(x) = {}_cI_x^{p-\alpha} {}_cD_x^{p} f(x), p-1 < \alpha < p, \quad p \in N. \qquad (2.16) $$

The fractional derivative of the composition of two analytical functions $f$ and $g$ on the interval $(c-h, c+h)$ for $0 < \alpha < 1$ is defined as (Kiblas et al., 2006),

$$ {}_cD_x^{\alpha}[fg](x) = \frac{(x-c)^{-\alpha}}{\Gamma(1-\alpha)} g(c)(f(x) - f(c)) + ({}_cD_x^{\alpha}(g(x)) f(x) $$

$$+\sum_{p=1}^{\infty}\binom{\alpha}{p}({}_cI_x^{p-\alpha}g(x))_cD_x^pf(x). \quad (2.17)$$

Another property of Caputo's fractional derivative is the linearity property Hilfer (2000) i.e.

$${}_cD_x^{\alpha}(\lambda f(x)+\mu g(x))=\lambda_cD_x^{\alpha}f(x)+\mu_cD_x^{\alpha}g(x), \quad (2.18)$$

where $\lambda$ and $\mu$ are constants. The Caputo's derivative utilized the physical boundary conditions where as Riemann-Louiville derivative required fractional order boundary conditions. Furthermore, Riemann-Liouville fractional derivative exists for a class of integrable function while existence of Caputo's fractional derivative depends on the integrability of $p$ times the differentiable functions.

### 2.3.4 Grünwald-Letnikov Fractional Derivative

Grünwald and Letnikov independently worked on fractional differentiation approximately at the same time when Riemann and Liouville established Riemann-Liouville non-integer differentiation for solving many types of FDEs. Later, many other authors utilized Grünwald-Letnikov non-integer differentiation operator to build numerical techniques for FDEs. Grünwald-Letnikov define the definition of fractional derivative of a function as follows (Kiblas et al., 2006),

$${}_cD_x^{\alpha}f(x)=\lim_{\Delta x\to 0}\frac{1}{(\Delta x)^{\alpha}}\sum_{p=0}^{\lceil\frac{x-c}{\Delta x}\rceil}\omega_{\alpha}^{p}f(x-p\Delta x),\ x\geq 0 \quad (2.19)$$

where $\omega_p^{\alpha}=(-1)^p\binom{\alpha}{p}=(-1)^p\frac{\alpha(\alpha-1)\cdots(\alpha-p+1)}{p!}=\frac{\Gamma(p-\alpha)}{\Gamma(-\alpha)\Gamma(p+1)}$. Here the binomial coefficient $\binom{\alpha}{p}$ is calculated by the utilization of the Gamma function $\Gamma$. To compute the coefficients $\omega_p^{\alpha}$, where $\alpha$ is fractional order differentiation, in Eq. (2.19), we need the following recurrence formula so that our calculation can easily be calculated by some computer software. The recurrence formula is given by,

$$\omega_0^\alpha = 1 \ and \ \omega_p^\alpha = (1 - \frac{1+\alpha}{p})\omega_{p-1}^\alpha, \quad p = 1, 2, 3, \cdots, \lceil \frac{x-c}{\Delta x} \rceil.$$

The shifted Grünwald-Letnikov formula can be written as (Oldham and Spanier, 1974),

$$_cD_x^\alpha f(x) = \frac{1}{(\Delta x)^\alpha} \sum_{p=0}^{[\frac{x-c}{\Delta x}]} \omega_p^\alpha f(x - \Delta x p) + O((\Delta x)^q), \ x \geq 0. \quad (2.20)$$

Grünwald-Letnikov formula is frequently used in discretization of FDEs. For this purpose we need the shifted Grünwald-Letnikov schemes, since unshifted scheme always generate unstable numerical methods (Ali et al., 2017).

## 2.4 Finite Difference Method

Finite difference method (FDM) is utmost common, efficient, frequent and universally applicable method for the solution of various types of PDEs. The numerical solutions obtained from FDM are actually the values of discrete points in the solution domain which we are called them grid points as shown in Figure 2.1. We usually prefer the space between the grid points in both $x$ and $y$ directions should be uniform and grid spacing between the points in x-direction is denoted by $\Delta x$ or $h_x$, likewise space between the grid points in y-direction is denoted by $\Delta y$ or $h_y$. One can also be utilized the unequal (non-uniform) grid spacing in both coordinate directions but the difference between successive pairs of grid points in each direction should be the same. If $(i, j)$ represents the coordinates of the grid point P in solution domain as shown in the Figure

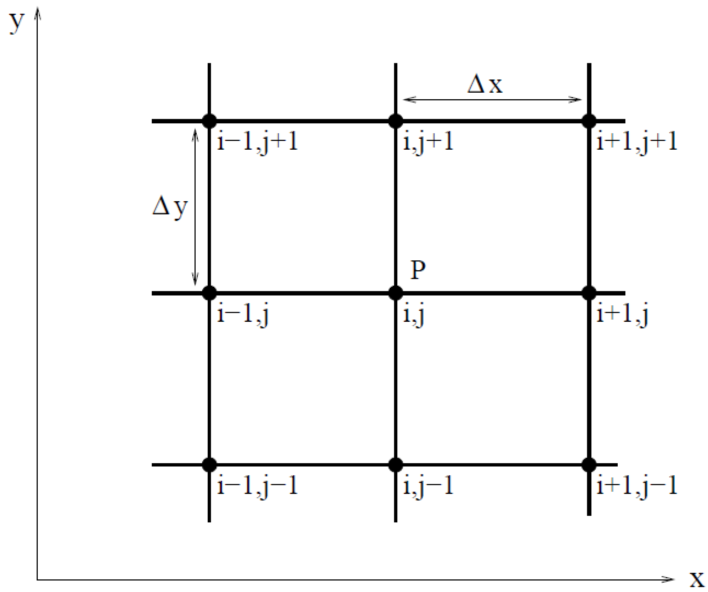


Figure 2.1: Discretization of discrete grid points

2.1, then the grid point $(i+1, j)$ will show its position is immediately to the right of the grid point $(i, j)$ in positive x-direction and likewise the grid point $(i-1, j)$ will show its position is immediately to the left of the point $(i, j)$ in negative x-direction. Similarly, the grid point $(i, j+1)$ will move immediately one step up in positive y-direction and grid point $(i, j-1)$ will move immediately one step down in negative y-direction. Finite difference approximation techniques are basically applied on as an alternative source of the derivatives to find out the approximate solution by converting the desire research problem in the form of PDEs into the easily solvable algebraic difference equations (Atkinson and Han, 2009).

### 2.4.1 Taylor Series Expansion Applied to Finite Difference Method

The partial derivatives in PDEs are replaced by the finite difference approximations at each grid point which are approximated by the neighboring values utilizing the Taylor series expansion. The general interpretation of Taylor's series expansion says that if we know the value of a function and its derivatives at some particular point, say $(x_i, y_j, t_k)$ then we can easily find the values of function at its nearby points $(x_i + h_x, y_j, t_k)$ and $(x_i - h_x, y_j, t_k)$. The Taylor series expansions about the point $(x_i, y_j, t_k)$, the exact expression for $f(x_i + h_x, y_j, t_k)$ and $f(x_i - h_x, y_j, t_k)$ are defined as follows (Smith, 1985),

$$f(x_i + h_x, y_j, t_k) = f(x_i, y_j, t_k) + \frac{(h_x)}{1!} f_x(x_i, y_j, t_k) + \frac{(h_x)^2}{2!} f_{xx}(x_i, y_j, t_k) + \frac{(h_x)^3}{3!} f_{xxx}(x_i, y_j, t_k) + ..., \tag{2.21}$$

and

$$f(x_i - h_x, y_j, t_k) = f(x_i, y_j, t_k) - \frac{(h_x)}{1!} f_x(x_i, y_j, t_k) + \frac{(h_x)^2}{2!} f_{xx}(x_i, y_j, t_k) - \frac{(h_x)^3}{3!} f_{xxx}(x_i, y_j, t_k) + \cdots. \tag{2.22}$$

Since all the three points $(x_i, y_j, t_k)$, $(x_i + h_x, y_j, t_k)$ and $(x_i - h_x, y_j, t_k)$ are the grid points and with the help of Taylor series expansions, we are able to find out the values of the $(x_i + h_x, y_j, t_k)$ and $(x_i - h_x, y_j, t_k)$ that allows us to rearrange the Eqs. (2.21) and (2.22) to finite difference approximations to the derivatives. In particular, if $h_x$ is very small and any higher order term of $h_x$ is smaller than $h_x$, then for any function $f(x_i, y_j, t_k)$, Eqs. (2.21) and (2.22) can be truncated after a finite number of terms. For example, if the term of magnitude $(h_x)^3$ and higher order are neglected, then the Eqs. (2.21) and (2.22) become,

$$f(x_i + h_x, y_j, t_k) \approx f(x_i, y_j, t_k) + \frac{(h_x)}{1!} f_x(x_i, y_j, t_k) + \frac{(h_x)^2}{2!} f_{xx}(x_i, y_j, t_k), \tag{2.23}$$

and

$$f(x_i - h_x, y_j, t_k) \approx f(x_i, y_j, t_k) - \frac{(h_x)}{1!} f_x(x_i, y_j, t_k) + \frac{(h_x)^2}{2!} f_{xx}(x_i, y_j, t_k). \qquad (2.24)$$

We have neglected the $(h_x)^3$ and its higher order terms, therefore both Eqs. (2.23) and (2.24) are second order accurate. If the terms of order $(h_x)^2$ and higher order terms are neglected, then Eqs. (2.21) and (2.22) are reduced to the following equations,

$$f(x_i + h_x, y_j, t_k) \approx f(x_i, y_j, t_k) + \frac{h_x}{1!} f_x(x_i, y_j, t_k), \qquad (2.25)$$

and

$$f(x_i - h_x, y_j, t_k) \approx f(x_i, y_j, t_k) - \frac{h_x}{1!} f_x(x_i, y_j, t_k). \qquad (2.26)$$

The Eqs. (2.25) and (2.26) are first order accurate because we have neglected $(h_x)^2$ and its higher order in the Eqs. (2.21) and (2.22). The truncation error is the amount of quantity by which the solution of a PDE fails to satisfy the approximate solution at some grid point. The truncation error can be reduced by retaining more terms in the Taylor series expansion of the corresponding derivatives and hence reduced the magnitude of $(h_x)$ (Mattheij et al., 2005).

### 2.4.2 Simple Finite Difference Approximation to a Derivative

In this subsection, we will derive some simple finite difference approximation to first and second order derivatives in both $x$ and $y$ directions. For this purpose, let us