Second Semester Examination
2020/2021 Academic Session

July/August 2021

EMT 212 – Computational Engineering

Duration: 2 hours

Please check that this examination paper consists of **SIX (6)** pages including appendices before you begin the examination.

Instructions : Answer ALL **FIVE (5)** questions.

Answer to each question must begin from a new page.

1. With the help of a sketch, use the divergence theorem to derive the continuity equation

$$\nabla \cdot (\rho \boldsymbol{v}) = -\frac{d\rho}{dt}$$

where ρ is the density of the fluid, t is the time and $\boldsymbol{v}$ is the velocity of the fluid. Explain the principle of conservation that is fulfilled by the divergence theorem.

(18 marks)

2. Consider an inverted cone that floats in water such that ONE THIRD of its height, h is in water as depicted in Figure 2.

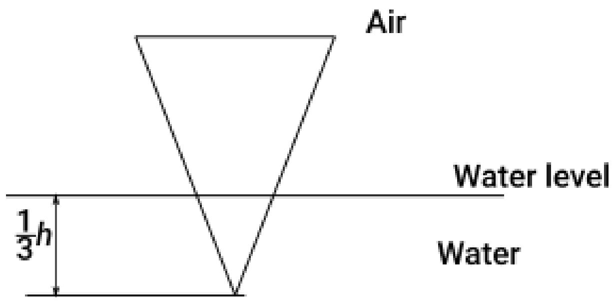

Figure 2: The side view of the cone partially in water.

[a] State the scalar field and the vector field that are relevant in modeling the buoyancy on the cone.

(4 marks)

[b] Express the magnitude of the total force $\mathbf{F}_f$ on the cone due to the water pressure in terms of the surface integral over the affected area. DO NOT evaluate the integral.

(10 marks)

[c] Use the divergence theorem to express the magnitude of the buoyancy in terms of the base diameter, D of the cone, the height of the cone, and the density of water.

(10 marks)

3. Figure 3 is a part of the complete MATLAB code for function minimization with the golden section method. The partial code has the following descriptions:

- variables {a,b,c,d} are points in the x-coordinate where (b-a) < (d-c)
- u(x) is the MATLAB function that returns the value of the mathematical function at x
- the values returned by u(x) are stored in uc and ud
- R is the golden ratio

The code uses the while loop that contains THREE coding errors. Rewrite the code by correcting the errors so that the iteration will terminate successfully.

```
k = 0;
while((abs(d-c))/(b-a)< 1*10^-5 && (k == max_k)
  if (uc<ud)
    b=d;
    d=c;
    c=a+(1-R)*(b-a);
    uc=u(c);
    ud=u(d);
  else
    a=c;
    c=d;
    d=a+R*(b-a);
    uc=u(c);
    ud=u(d);
  end
end
```

Figure 3

(8 marks)

4. [a] Derive the backward difference for second-order derivative:

$$u''(x_i) = \frac{2u(x_i) - 5u(x_{i-1}) + 4u(x_{i-2}) - u(x_{i-3})}{(\Delta x)^2} + O(\Delta x)^2$$

(15 marks)

[b] For a function $\boldsymbol{u}$ that is given by

$$u(x) = 0.1x^3 + 2.1\sqrt{x}$$

(i) Calculate the $u''(x)$ at $x = 2$ using backward difference scheme given in part [a].

(ii) Calculate the difference for $u''(x)$ using exact solution and compare the answer in part [b] (i).

(10 marks)

5. For a system with the length of 5, use the finite difference method to solve one-dimensional transient problem governed by

$$\frac{\partial u}{\partial t} = -k\frac{\partial u}{\partial x}$$

where coefficient k is given to be 0.2. The initial and boundary conditions of the system are given by

$$u(x,0) = 10$$
$$u(0,t) = 50, \qquad u(5,t) = 40$$

Calculate the profile u over the range between 0 and 5, at time $t = 3$ for this system based on discretization given in Figure 5. The Δt is given to be 1.

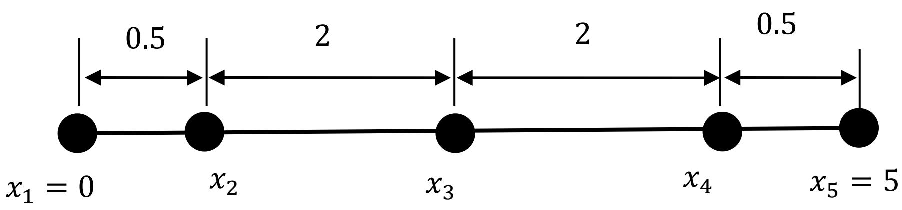

Figure 5

The discretization schemes are given as

Temporal discretization: $$\left[\frac{\partial u}{\partial t}\right]_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

Spatial discretization: $$\left[k\frac{\partial u}{\partial x}\right]_i^n = k\frac{u_i^n - u_{i-1}^n}{\Delta x}$$

(25 marks)

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APPENDIX 1

USEFUL FORMULAS

1. Newton's Method

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)}$$

2. Formulas for first finite differences

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + O(h)$$
$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$
$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2)$$

3. Formulas for second finite differences

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h)$$
$$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2} + O(h)$$
$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + O(h^2)$$

4. Heat equation

$$-\alpha\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = Q(x,t)$$

5. Convective boundary condition

$$hu + ku' = hu_\infty$$

6. Discrete form of 1D Poisson's equation

$$-k\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = f_i$$

7. Explicit and implicit methods for heat equation

$$-\lambda(u_{i+1}^l - 2u_i^l + u_{i-1}^l) = u_i^{l+1} - u_i^l - sf_i^{l+1}$$

$$-\lambda u_{i+1}^{l+1} + (1 + 2\lambda)u_i^{l+1} - \lambda u_{i-1}^{l+1} = u_i^l + sf_i^{l+1}$$

$$\lambda = \frac{\alpha s}{h^2}$$

8. Integrals of sine and cosine

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax + C$$
$$\int \cos ax\, dx = \frac{1}{a}\sin ax + C$$

9. Spherical coordinates

$$x = \rho \sin\varphi \cos\theta\,;\ \ y = \rho \sin\varphi \sin\theta\,;\ x = \rho\cos\varphi$$
$$\rho \geq 0; 0 \leq\ \varphi \leq\ \pi$$
$$dV = \rho^2 \sin\varphi\ \ d\rho d\theta d\varphi$$

10. Cylindrical coordinates

$$x = r\cos\theta\,;\ \ y = r\sin\theta\,;\ z = z$$
$$dV = rdzdrd\theta$$

11. Taylor series at point a

$$u(x) = u(a) + u'(a)(x-a) + u''(a)\frac{(x-a)^2}{2!} + u'''(a)\frac{(x-a)^3}{3!} + \cdots$$
$$\ldots + u^{(n)}(a)\frac{(x-a)^n}{n!} + \cdots$$

12. Miscellaneous

$$\nabla \times \mathbf{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}$$

$$\oint_C M(x,y)dx + N(x,y)dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$\oint_{\partial S} \mathbf{F}(x,y,z)\cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F})\, dS$$

$$\oint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \iiint_V \nabla\cdot\mathbf{F}\, dV$$

$$\text{Volume of a sphere} = \frac{4}{3}\pi r^3$$

$$\text{Volume of a cone} = \frac{1}{3}Ah$$