MULTI-STEP MODIFIED DIFFERENTIAL TRANSFORM METHODS FOR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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MULTI-STEP MODIFIED DIFFERENTIAL TRANSFORM METHODS FOR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

by

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LIST OF SYMBOLS

L	second order linear differential operator
$\partial\Omega$	boundary domain
Ω	domain
t_0	initial time
t	variable time
J	integration operator
D	differentiation operator
α	order of fractional integral
$\mathbb{N}$	set of positive integers
$\mathbb{R}^+$	set of positive real numbers
n	order of the operator of the derivative
Γ	fractional integral
m	positive integer
δ	local error
τ	safety factor
N	order of the multi-step scheme
M	subintervals of multi-step scheme
T	variable time
h	step size of multi-step scheme
E_N	error at grid point t_k
h_{max}	maximum allowed step-size
$U_k(x)$	transformed function
$u(x,t)$	represents the original function

LIST OF ABBREVIATIONS

NLEEs	Nonlinear Evolution Equations
NLS	Nonlinear Schrödinger
FNLS	Fractional Nonlinear Schrödinger
NKdV	Nonlinear Korteweg-de Vries
NKG	Nonlinear Klein-Gordon
PDEs	Partial Differential Equations
FDEs	Fractional Differential Equations
DTM	Differential Transformation Method
IDTM	Improved Differential Transform Method
RDTM	Reduced Differential Transform Method
MRDTM	Modified Reduced Differential Transform Method
MMRDTM	Multi-step Modified Reduced Differential Transform Method
MFRDTM	Modified Fractional Reduced Differential Transform Method
MsFRDTM	Multi-step Fractional Reduced Differential Transform Method
MDTM	Modified Differential Transform Method
MsDTM	Multi-step Differential Transform Method
MsGDT	Multi-step Generalized Differential Transformation
MsRDTM	Multi-step Reduced Differential Transform Method
MMsDTM	Modified Multi-step Differential Transform Method
PEM	Parameter Expansion Method
VIM	Variational Iteration Method
HPM	Homotopy Perturbation Method
ADM	Adomian Decomposition Method
HAM	Homotopy Analysis Method
HATM	Homotopy Analysis Transform Method
RLW	Regularized Long Wave
IVPs	Initial Value Problems
TDM	Time Division Multiplexed System
WDM	Wavelength Division Multiplexing
SPM	Self-Phase Modulations
GVD	Group Velocity Dispersion

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KAEDAH PENJELMAAN PEMBEZAAN MULTI-LANGKAH DIUBAHSUAI UNTUK PERSAMAAN PEMBEZAAN SEPARA HIPERBOLIK

ABSTRAK

Dalam tesis ini, kami menggabungkan Adomian polinomial bersama pendekatan multi-langkah untuk mempersembahkan teknik baharu yang dikenali sebagai Kaedah Pengurangan Transformasi Pembezaan Multi-langkah Diubahsuai (KPTPMD). Teknik yang dicadangkan mempunyai kelebihan menghasilkan penghampiran analitik dalam urutan penumpuan yang pantas dengan bilangan pengiraan yang dikurangkan. KPTPMD dipersembahkan dengan beberapa pengubahsuaian Kaedah Pengurangan Transformasi Pembezaan (KPTP) dengan pendekatan berbilang dan ungkapan tak linear digantikan oleh polinomial Adomiannya. Oleh itu, masalah nilai awal tak linear dapat diselesaikan dengan mudah bersama pengurangan langkah pengiraan. Di samping itu, pendekatan multi-langkah menghasilkan penyelesaian dalam siri penumpuan yang pantas yang menghasilkan penyelesaian dalam kawasan masa yang luas. Dalam kajian ini, tiga jenis persamaan yang menghuraikan gelombang tunggal dipertimbangkan: persamaan tak linear Schrödinger (TLS), persamaan tak linear Korteweg-de Vries (TKdV) dan persamaan tak linear Klein-Gordon (TKG). Persamaan-persamaan ini diselesaikan dengan menggunakan KPTPMD. Selain itu, kami juga menyelidik kebolehlaksanaan penggunaan KPTPMD untuk persamaan pecahan TLS, persamaan pecahan TKdV dan persamaan pecahan TKG. Untuk menunjukkan kepersisan dan kejituan kaedah, kami menggunakan KPTPMD untuk menyelesaikan persamaan TLS, persamaan TKdV dan persamaan TKG dengan ketidaklinearan yang berbeza. Selain itu, kami menggunakan KPTPMD untuk menyelesaikan aplikasi persamaan TLS bersama daya yang memodelkan perambatan optik tunggal dalam gentian optik dan bersama-sama dengan kehilangan gentian. KPTPMD juga digunakan untuk mendapatkan penyelesaian persamaan TKdV yang mempunyai daya dalam gelombang air untuk yang diaplikasi bagi memahami Tsunami dengan ungkapan daya yang berlainan. Persamaan KdV yang mempunyai daya ini boleh digunakan

sebagai model matematik yang mudah yang dapat menggambarkan pemodelan gelombang Tsunami. Akhir sekali, aplikasi persamaan TKG yang mempunyai daya dalam teori bidang kuantum juga telah diselesaikan dengan menggunakan teknik baru ini. Hasil penghampiran analitik yang diperoleh untuk menyelesaikan beberapa masalah yang mempunyai pintalan, gelombang tunggal dan dua gelombang tunggal menunjukkan bahawa kaedah ini mempunyai ketepatan yang tinggi. Daripada keputusan yang diperoleh ditemui bahawa terdapat kemungkinan untuk mendapatkan keputusan yang tepat atau penyelesaian yang tepat dengan menggunakan KPTPMD. Untuk menggambarkan penyelesaian dan menunjukkan kesahihan dan ketepatan KPTPMD, hasil grafik dimasukkan dalam bab masing-masing. Kesimpulannya, KPTPMD yang dibentangkan dalam tesis ini untuk menyelesaikan persamaan pembezaan separa hiperbolik tak linear yang menggambarkan gelombang tunggal telah terbukti lebih jitu dan persis berbanding KPTP. Teknik baru ini juga tidak memerlukan penglinearan, pendiskritan atau pengusikan. KPTPMD adalah ringkas, mudah digunakan, dan urutan penumpuan yang pantas dengan bilangan pengiraan yang dikurangkan.

MULTI-STEP MODIFIED DIFFERENTIAL TRANSFORM METHODS FOR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

ABSTRACT

In this thesis, we combined the Adomian polynomials with the multi-step approach to present a new technique called Multi-step Modified Reduced Differential Transform Method (MMRDTM). The proposed technique has the advantage of producing an analytical approximation in a fast converging sequence with a reduced number of calculated terms. The MMRDTM is presented with some modification of the Reduced Differential Transformation Method (RDTM) with multi-step approach and its nonlinear term is replaced by the Adomian polynomials. Therefore, the nonlinear initial value problem can easily be solved with less computational effort. Besides that, the multi-step approach produces a solution in fast converging series that converges the solution in a wide time area. In this study, three types of equations that describe solitary waves are considered: nonlinear Schrödinger (NLS) equation, nonlinear Korteweg-de Vries (NKdV) equation and nonlinear Klein-Gordon equation (NKG) equation. These equations are solved by using the MMRDTM. Besides that, we investigated the feasibility of applying the MMRDTM for the fractional NLS equations, fractional NKdV equations and fractional NKG equations. For demonstrating the precision and accuracy of the method, we applied the MMRDTM to solve the NLS equations, NKdV equations and NKG equations with different nonlinearity. Next, we implemented the MMRDTM to solve forced NLS equation that models the propagation of optical solitons in optical fiber and together with the fiber loss. Then, the MMRDTM is used to solve the forced NKdV equation with the application in water wave for understanding Tsunami with different forcing terms. Forced KdV equation can be used as a simple mathematical model that could describe the modelling of the Tsunami waves. Lastly, forced NKG equation application in quantum field theory also has been solved by using this new technique. Approximate analytical results that are obtained for solving several problems possessing kinks, single and double-soliton waves show that the method has

high accuracy. From the results, it was found that it is possible to obtain highly accurate results or exact solutions by using the MMRDTM. For illustrating the solution and to show the accuracy of the MMRDTM, graphical results are included in each chapters respectively. In conclusion, the proposed MMRDTM for solving nonlinear hyperbolic partial differential equations (PDEs) that describe solitary waves have been shown to be more accurate and precise than the MRDTM. There is no linearization, discretization or perturbation required in the proposed technique. The MMRDTM is conceptually simple, easy to use and a fast converging sequence with a reduced number of calculated terms.

CHAPTER 1

INTRODUCTION

1.1 Background

Nonlinear Evolution Equations (NLEEs), i.e. partial differential equations (PDEs) with time derivatives are useful tools to describe the science and engineering in natural phenomena. In the study of nonlinear physical phenomena, investigating traveling wave solutions of NLEEs plays a significant role. Studying NLEEs traveling wave solutions plays an important role in investigating the internal mechanism of complicated physical phenomena. Many physical phenomena such as fluid and quantum mechanics, optical fibers, electricity, plasma physics, chemical kinematics and propagation of shallow water waves can be modelled by nonlinear evolution equation, and the appearance of solitary wave solutions in nature is somewhat frequent (Islam et al., 2015).

However, nonlinear processes are one of the biggest challenges and not simple to control because the nonlinear characteristic of the system abruptly changes due to some small changes of parameters including time. The problem therefore becomes more complicated, wherever possible analytical solutions are desired. Thus, studying analytical solutions of NLEE plays a crucial role in understanding the physical mechanism of nonlinear phenomena. Exact analytical solution are often difficult to obtain and oftentimes, we need to resort to approximate analytical solution. Advanced nonlinear techniques are important in solving inherent nonlinear problems, especially in dynamical systems and related areas. Significant improvements have been made in recent years in finding analytical solutions for NLEEs (Islam et al., 2015).

There are many effective and powerful methods for approximate analytical solution that have been developed and improved. For instance, the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM), Variation Iteration Method (VIM), Hirota’s Bilinear Method, Balance Method, Inverse Scattering Method, and Differential Transform Method (DTM).

In this thesis, three types of equations that describe solitary waves such as the nonlinear Schrödinger (NLS), nonlinear Korteweg-de Vries (NKdV) and nonlinear Klein-Gordon (NKG) equations are considered. In mathematics and physics, a soliton is a self-reinforcing solitary wave. That means, it is, a wave packet or a pulse that retains its shape while traveling at constant velocity. Solitons are produced by an exact cancellation of nonlinear and dispersive effects in the medium. The term “dispersive effects” relates to a property of some systems where the velocity of the waves differs by frequency (Abazari, 2014). Solitons emerge as the solutions of a widespread class of weakly nonlinear dispersive partial differential equations (PDEs) that describe physical systems. John Scott Russell (1808-1882) first described the solitary phenomenon and observed a solitary wave at the Union Canal in Scotland. In a wave tank, he reproduced the phenomenon which is referred to as the "Great Wave of Translation" (Russell, 1844).

The NLS equation is an example of a universal nonlinear model describing various nonlinear physical systems. It can be used in the areas of hydrodynamics, nonlinear optics, nonlinear acoustics, quantum condensates, plasma physics, heat pulses in solids and various other nonlinear instability phenomena. In the propagation of optical pulses, the NLS equation was used to describe a variety of effects. In the context of optical communications, a pulse that propagates in an optical fibre with Kerr-law nonlinearity can form an envelope soliton. This pulse propagation behaviour offers the potential to understand pulse transmission over very long distances. The significance of studying optical solitons arises from the fact that they have potential applications in optical transmission and all-optical processing (Seadawy, 2012).

The Korteweg-de Vries (KdV) equation has gained significant attention in other physical contexts as ion-acoustic waves, plasma physics, collision-free hydromagnetic waves, lattice dynamics, stratified internal waves, etc. (Fung, 1997). The KdV model was applied in the field of quantum mechanics to explain certain theoretical physics phenomena. It is used as a model for shock wave formation, solitons, turbulence, boundary layer behaviour, and mass transport in fluid dynamics, aerodynamics, and continuum mechanics. The KdV equation also describes the shallow water waves (Akhmediev, 2018).

The Klein-Gordon (KG) equation is significant NLEE that emerges in relativistic quantum mechanics and quantum field theory, which is also very important for the physics of high energy particles and is applied to model various forms of phenomena including the propagation of dislocations in crystals and the behaviour of elementary particles (Hafez et al., 2014). The NKG equation appears in many forms of nonlinearities. This equation has thrived in the study of solitons and condensed matter physics, in the investigation of solitons interaction in collision-less plasma, the recurrence of initial states, in lattice dynamics and in the examination of nonlinear wave equations. The KG equation plays an important part in many scientific applications, including solid state physics and nonlinear optics theory (Dehghan et al. 2009).

Apart from that, this study also considered the fractional NLS equation, fractional NKdV equation and fractional NKG equation. Due to their different applications in the fields of physics and engineering, considerable interest in fractional differential equations has been stimulated in recent years. The fractional calculus is used for modeling physical and engineering problems which are best described in the fractional differential equations (Ray, 2013).

This interest of fractional PDEs is due to various concepts of fractional derivatives and integrations involving these models such as Grunwald-Letnikov's definition, Riemann-Liouville's definition, Caputo's definition, and Riesz's definition. Fractional order derivatives and integrations include the whole function history in a weighted form, called the memory effect. Specifically, fractional PDEs have attention and popularity due to tremendous use in electrical circuits, quantum, viscoelasticity, electrochemistry and others. However, their non-local property is the main benefit of using fractional differential equations. It is well known that the differential operator for integer order is a local operator, but the differential operator for fractional order is non-local. This implies that the next state of a system depends not only on its present state, but also on all its historical states (Momani et al., 2016).

1.2 Problem Statement

Solving nonlinear problems can be difficult for researchers to find the exact analytical solution and therefore it can guide researchers to use numerous approximate analytical methods. Perturbation method is one of the most established methods of solving nonlinear equations, whereby it is based on the existence of a small parameter using the common perturbation method. Consequently, several new techniques have been recently introduced in order to eliminate the small parameter, such as Parameter Expansion Method (PEM), Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM), Adomian Decomposition Method (ADM), Differential Transformation Method (DTM) and Reduced Differential Transform Method (RDTM). These methods are called as approximate analytical methods (sometimes called as semi-analytical methods).

The RDTM provides an alternative approach to overcome the demerit of complicated classical DTM calculation. Although it is powerful, several equations remain difficult to solve by RDTM. One of the obtacles is to find a simple and effective way to obtain the differential transforms of nonlinear components. The RDTM expands these nonlinearities in an infinite series and would probably achieve its transformed nonlinear functions by one of its equivalent series. The computational difficulties in determining the transformed function of this infinity series will inevitably arise if using this approach.

On the other hand, the solution of existing methods also diverges in wide time area. That means, the solution of existing methods only converges in small time region. Besides that, the PDEs with complicated nonlinearity is hard to solve by the existing method due to its nonlinearity term. Therefore, we implement Multi-step Modified Reduced Differential Transform Method (MMRDTM) to overcome above mentioned shortcomings and to increase the interval of convergence for these series solutions.

In weak nonlinear systems, most of them have excellent convergences and effectiveness, but some of them are not good for strong nonlinear systems. For example, weak nonlinearity is seen in wave drift forces, which are the nonlinear effect by which waves produce constant forces on floating bodies (Rainey, 2007). Wave drift forces are usually proportional to the wave height square, and can be measured with a perturbation scheme (Stokes' expansion). Strong nonlinearity is most clearly in wave breakage and related effects of wave impact. These can not be analyzed by the Stokes perturbation scheme, as shown conclusively by the fact that the surface of the water overturns and therefore can not be described as a Fourier series (Rainey, 2007). Strong nonlinearities are known in acoustics, optics, mechanics and in quantum field theory.

1.3 Motivation of study

Ray (2013) proposed a modification on the fractional RDTM and implemented it to find solutions of fractional KdV equations. In this approach, the adjustment included the substitution of the nonlinear term by related Adomian polynomials. Therefore, the solutions of the nonlinear problem can be obtained in a simpler way with reduced calculated terms. Furthermore, Al-Smadi et al., (2017) introduced a multi-step approach for solving one-dimensional fractional heat equations. It produces the solution in a rapid convergent series which results in the solution converging in wide time area.

The principal benefit of the method highlights the fact that it elegantly provides explicit approximate analytical solution as well as numerical solution. There are two advantages of the new technique. Firstly, there is no discretization required. Secondly, linearization or small perturbation is not required. As a result, we executed the MMRDTM to solve NLS equations, NKdV equations and NKG equations which describe solitary waves.

Furthermore, the benefit of analytical approximation is that it allows a broad understanding of the nature and quality of nonlinear equations. Hence, analytical methods of these equations may not always be feasible. In such cases, approximate analytical solutions that yield series solutions are used. Approximate analytical methods are based on finding the other terms of the series for the problems being considered from the given initial conditions and boundary value conditions.

1.4 Aim

The aim of this study is to develop new approximate analytical methods based on modified fractional RDTM and multi-step fractional RDTM for solving nonlinear hyperbolic PDEs. We want to investigate the feasibility of MMRDTM for solving nonlinear hyperbolic partial differential equations. The word feasibility means capable of being done or carried out.

1.5 Objectives

The objectives of this study are as the following:

1) to develop approximate analytical solution of the MMRDTM,
2) to investigate the feasibility of applying MMRDTM for solving NLS equations, NKdV equations and NKG equations,
3) to investigate the feasibility of applying MMRDTM for the fractional NLS equations, fractional NKdV equations and fractional NKG equations,
4) to investigate MMRDTM for solving NLS equations, NKdV equations and NKG equations and fractional counterparts with different nonlinearities, and
5) to apply MMRDTM for solving application of forced NLS equations, forced NKdV equations and forced NKG equations.

1.6 Scope of Study

We consider one-dimensional PDEs problems such as one-dimensional NLS equations, NKdV equations and NKG equations. We study nonlinear and fractional of NLS equations, NKdV equations and NKG equations. Then, we study NLS equations, NKdV equations and NKG equations with different nonlinearities and also their fractional form. To show our new method is feasible, we also apply our method to solve forced NLS equations, forced NKdV equations and forced NKG equations. Besides that, we only consider PDEs with initial conditions in this study.

1.7 Methodology

- In this study, we solve the NLS equations, NKdV equations and NKG equations by using the MMRDTM. In the proposed scheme, we develop numerical algorithm of MMRDTM.

- Then apply to hyperbolic wave type equations which arise in solitary wave such as NLS equations, NKdV equations and NKG equations. The differential equation and associated initial conditions are transformed into a recurrence relation that ultimately leads as coefficients of a power series solution to the solution of an algebraic equation system.

- The nonlinear term is replaced in this new approach by its Adomian polynomials. In addition, we apply multi-step scheme to transformed functions of differential equation and related initial conditions. The transformation of equation is based on theorem of the RDTM.

- Then we will analyse convergence and error estimate of MMRDTM. The solution of MMRDTM is in the form of polynomials in series solution. Then we make analysis on error with exact solutions and compare the results of nonlinear hyperbolic PDEs by MMRDTM, classical RDTM and exact solutions.

- We also apply MMRDTM for solving NLS equations, NKdV equations and NKG equations with different nonlinearity and solving fractional NLS equations, fractional NKdV equations and fractional NKG equations with different nonlinearity.

- Finally, we apply our method to problems of forced NLS equations, forced NKdV equations and forced NKG equations in order to proof our method is also feasible in solving real world problems.

- Mathematical software Maple 18 has been used as the main tool to carry out all the numerical tasks.

1.8 Thesis Organization

This thesis contains nine chapters. In Chapter 1, a brief introduction of the research background is given. Other than that, the problem statement, motivation, scopes of study, aims of the study, objectives, and methodology of the research are also described in this chapter.

In Chapter 2, the rudiments of PDE are discussed. Brief descriptions of NLS equation, NKdV equation and NKG equation are given. The basic theory of fractional calculus such as fractional derivatives and fractional integral is also described. Some methods such as Differential Transform Method (DTM), Two-dimensional DTM, RDTM, Modified Fractional Reduced Differential Transform Method (MFRDTM) and Multi-step Fractional Reduced Differential Transform Method (MsFRDTM) are also described.

In Chapter 3, some of related literatures are reviewed. The histories of Reduced Differential Transform Method (RDTM), Modified Differential Transform Method (MDTM), Multi-step Differential Transform Method (MsDTM), Multi-step Reduced Differential Transform Method (MsRDTM) and others related to RDTM. Some literatures of nonlinear hyperbolic wave type equations such as NLS equation, NKdV equation and NKG equation are also reviewed.

In Chapter 4, MMRDTM based on modified RDTM and multi-step RDTM is developed. In addition, fractional MMRDTM is also developed. In Chapter 5, we presented the application of MMRDTM to solve NLS equation and fractional NLS equation. Besides that, we also presented solutions of NLS equation of power law nonlinearity and cubic-quintic nonlinearity and solutions of fractional NLS equation of Power Law nonlinearity and cubic-quintic nonlinearity.

In Chapter 6, the application of MMRDTM to solve NKdV equation and fractional NKdV equation is presented. Other than that, solutions of NKdV equation with different nonlinearity and solutions of fractional NKdV equation with different nonlinearity are also presented. In Chapter 7, the application of MMRDTM to solve NKG equation and fractional NKG equation are performed. Next, solutions of NKG equation with different nonlinearity and solutions of fractional NKG equation with different nonlinearity are also performed.

In Chapter 8, we apply MMRDTM to solve forced NLS equation application in optical fibers, forced NKdV equation application in water wave for understanding Tsunami and forced NKG equation application in quantum field theory. In Chapter 9, we conclude this study and make remarks on possible future work. References, appendix and list of publications are provided at the end of this thesis.

CHAPTER 2

BASIC CONCEPTS, THEORY AND METHODS

2.1 Introduction

It is well known that PDEs can describe many phenomena that occur in mathematical physics and engineering fields. In physics for example, PDEs describe heat flow and wave propagation. In ecology, PDEs govern most population models. PDEs also characterize the dispersion of a chemically reactive material. Moreover, most physical phenomena of fluid dynamics, quantum mechanics, electricity, plasma physics, the propagation of shallow water waves and many other models are governed by PDEs (Wazwaz, 2009).

In recent years, fractional calculus has become more popular. This is because certain phenomena can better be modelled using fractional derivatives rather than the traditional integer derivative. Numerical methods in the form of approximate analytical methods such as ADM, VIM, HPM and DTM have been used widely. This chapter will review some basic concepts and theory related to PDEs, Fractional Calculus and Differential Transform Methods (DTMs).

2.2 Rudiments of Partial Differential Equation (PDE)

An equation that contains at least one partial derivative is a PDE. Examples include (Wazwaz, 2009),

$$u_{tt} = c^2 u_{xx}, \tag{2.1}$$

$$u_{tt} = c^2 \left(u_{xx} + u_{yy}\right), \tag{2.2}$$

$$u_{tt} = c^2 \left(u_{xx} + u_{yy} + u_{zz}\right), \tag{2.3}$$

describing the propagation of waves in one dimensional space, two-dimensional space, and three-dimensional space respectively. In addition, the unknown functions in Equation (2.1), Equation (2.2), and Equation (2.3) are defined by $u = u(x,t), u = u(x,y,t),$ and $u = u(x,y,z,t)$ respectively.

Any function which satisfies the equation is a PDE solution. It is frequently the core that solutions fulfilling certain boundary and initial conditions are sought. For example, the equation

$$4u_x + 3u_y + u = 0 \tag{2.4}$$

has the solution (O'Neil, 2008)

$$u(x, y) = e^{-\frac{x}{4}} f(3x - 4y),$$

in which f can be any differentiable functions of a single variable. This can be confirmed by replacing the PDE with $u(x, y)$.

A PDE is linear if it is linear in the unknown function and its partial derivatives. An equation that is not linear is nonlinear. For instance (O'Neil, 2008),

$$x^2 u_{xx} - y u_{xy} = u$$

is linear whereas

$$x^2 u_{xx} - y u_{xy} = u^2,$$

$$(u_{xx})^{\frac{1}{2}} - 4u_{yy} = xu$$

are nonlinear because of the u^2 and $(u_{xx})^{\frac{1}{2}}$ terms.

In its general form, a second order linear PDE is given in two independent variables x and y (Wazwaz, 2009),

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{2.5}$$

where $A, B, C, D, E, F,$ and G are constants or functions of the variables x and y. A PDE of second order Equation (2.5) is usually classified into three basic equation classes, namely:

1. Parabolic equation is an equation that satisfies the property

$$B^2 - 4AC = 0.$$

Examples of parabolic equations are heat flow and diffusion processes equations. The heat transfer equation

$$u_{tt} = ku_{xx}.$$

2. Hyperbolic equation is an equation that satisfies the property

$$B^2 - 4AC > 0.$$

Examples of hyperbolic equations are wave propagation equations. The wave equation

$$u_{tt} = c^2 u_{xx}.$$

3. Elliptic equation is an equation that satisfies the property

$$B^2 - 4AC < 0.$$

Examples of elliptic equations are Laplace's equation. The Laplace equation in a two-dimensional space

$$u_{xx} + u_{yy} = 0.$$

2.3 Boundary and Initials Conditions

In general, there are infinitely many solutions to PDEs. To obtain a unique solution, additional conditions must be added to the equation. There are generally two types of conditions, boundary value conditions and initial conditions. Boundary conditions are function and space variable constraints, whereas initial conditions are constraints on the unknown function and time variable (Adzievski and Siddiqi, 2014).

Consider a second order linear PDE,

$$Lu = G(t, \mathbf{x}), \quad \mathbf{x} \in \Omega\,, \tag{2.6}$$

where L is a second order linear differential operator (partial derivatives are involved in Lu up to the second order). The following terms are presented in relation to the differential equation (2.6):

Boundary conditions are a set of constraints describing the nature of the unknown function $u(t, \mathbf{x}), \quad \mathbf{x} \in \Omega$, on the boundary $\partial\Omega$ of the domain Ω. There are three major types of boundary conditions (Adzievski and Siddiqi, 2014):

a) Dirichlet Conditions. These conditions determine prescribed values $f(t, \mathbf{x}), \quad \mathbf{x} \in \partial\Omega$, of the unknown function function $u(t, \mathbf{x})$ on the boundary $\partial\Omega$. Write these conditions in the form

$$u(t, \mathbf{x})|_{\partial\Omega} = f(t, \mathbf{x}). \tag{2.7}$$

b) Neumann Conditions. With these conditions, the value of the normal derivative $\frac{\partial u(t,\mathbf{x})}{\mathbf{n}}$ on the boundary $\partial\Omega$ is indicated. Symbolically, write as

$$\left.\frac{\partial u(t, \mathbf{x})}{\mathbf{n}}\right|_{\partial\Omega} = g(t, \mathbf{x}). \tag{2.8}$$

c) Robin (Mixed) Conditions. These conditions are linear combinations of the Dirichlet and Neumann Conditions:

$$a\, u(t, \mathbf{x})|_{\partial\Omega} + b\, \left.\frac{\partial u(t,\mathbf{x})}{\mathbf{n}}\right|_{\partial\Omega} = h(t, \mathbf{x}), \tag{2.9}$$

for some nonzero constants or functions a and b and a given function h, defined on $\partial\Omega$.

Note that different portions of the boundary $\partial\Omega$ can have different types of boundary conditions. In situations where PDE involves the time variable t, the initial (Cauchy) conditions must be considered. These conditions determine the value of the unknown function at the initial time $t = t_0$ and its higher-order derivatives. If in Equations (2.7), (2.8) and (2.9), the functions $f(t, x), g(t, x)$ and $h(t, x)$ are identically zero in the domain, then the boundary conditions are homogeneous; otherwise the boundary conditions are non-homogeneous.

The boundary and initial conditions decision depend on the given partial differential equation and the physical problem described by the equation. Consider a vibrating string, for example, described by the one space-dimensional wave equation (Adzievski and Siddiqi, 2014),

$$u_{tt} - u_{xx} = 0.$$

In the domain $(0, \infty) \times (0, l)$, the initial conditions are specified by specifying the initial position and speed of the string. The boundary conditions, can be for example (Adzievski and Siddiqi, 2014),

$$u(0) = u(l) = 0,$$

which implies that the two ends $x = 0$ and $x = l$ of the string are fixed.

2.4 Nonlinear Hyperbolic Wave Type Equations

Three main equations are considered in this thesis: nonlinear Schrödinger (NLS) equation, nonlinear Korteweg-de Vries (NKdV) equation and nonlinear Klein-Gordon (NKG) equation. These equations are chosen because they are nonlinear wave type equations that describe solitary waves.

2.4.1 Nonlinear Schrodinger Equation

In the propagation of optical pulses, the NLS equation can be used to explain a variety of effects. The balance between self-phase modulation and group velocity dispersion, as is well known, results in the so-called soliton solutions for the NLS equation. Solitary wave solutions in a variety of nonlinear and dispersive media have been known to exist for many years. A pulse propagating in an optical fiber with Kerr-law nonlinearity can form an envelope soliton in the context of optical communications. This pulse propagation behaviour offers the potential for pulse transmission over very long distances. Just as the balance between self-phase-modulation and group-velocity dispersion can lead to the formation of temporal solitons in single-mode fibers, diffraction and self-focusing can make up for one another and an analogous spatial soliton can also be found (Seadawy, 2012).

Studying optical solitons is important because they have potential applications in optical transmission and all-optical processing. Since analytical solutions are known for just a few cases, analyses of the properties of solutions are usually carried out numerically using such approaches. However, an analytical model that describes the dynamics of pulse propagation in a fiber is often desirable (Seadawy, 2012).

In this thesis, consider the NLS equation of the form (Seadawy, 2012),

$$iu_t + u_{xx} + \gamma|u|^2 = 0, \qquad i = \sqrt{-1},$$

$$u(x,0) = g(x),$$

where γ is a constant and $u(x,t)$ is a complex function.

2.4.2 Nonlinear Korteweg-de Vries Equations

The Korteweg-de Vries (KdV) equation used to model the evolution and interaction of nonlinear waves for a wide range of physics phenomena. The development of the KdV equation began in 1844 with the experiments of Scott Russell (Russell, 1844). It was derived as an evolution equation that governs a long-surface gravity wave of one-dimensional, small-amplitude, propagating in a shallow water channel.

In studying long waves, tides, solitary waves, and related phenomena the generalized KdV equation defined in Drazin (1989) is as follows:

$$u_t + (p+1)(p+2)u^p u_x + u_{xxx} = g(x,t), \tag{2.10}$$

where $g(x,t)$ is a given function and $p = 1,2,\ldots$ with $u, u_x, u_{xxx} \to 0$ as $|x| \to \infty$. If $p = 0, p = 1$ and $p = 2$, Equation (2.10) becomes linearized KdV, nonlinear KdV, and modified KdV equation, respectively (Kaya, 2002; Momani et. al., 2007). In Equation (2.10), the first term is the evolution term, the second term represents nonlinearity, and the last term is the third-order dispersion term. Also, x and t are the independent variables that represent the spatial and temporal variables, respectively, and $u(x,t)$ is the dependent variable describing the wave pattern (Houria et al., 2017). Equation (2.10) will be considered in this thesis.

2.4.3 Nonlinear Klein-Gordon Equations

Klein-Gordon (KG) equation has been used in quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics, nonlinear optics and applied physical sciences. The equation was named after the physicists Oskar Klein and Walter Gordon who suggested describing relativistic electrons in 1926.

The KG equations which are written in the following form with quadratic nonlinearity (Dehghan and Shokri, 2009)

$$u_{tt}(x,t) - \alpha u_{xx}(x,t) + \beta u(x,t) + \gamma u^2(x,t) = f(x,t),$$

and with cubic nonlinearity

$$u_{tt}(x,t) - \alpha u_{xx}(x,t) + \beta u(x,t) + \gamma u^3(x,t) = f(x,t),$$

where $u(x,t)$ denotes the particle wave profile at any varied instances and α, β and γ are known constants. The initial conditions

$$u(x,0) = f(x),$$

$$u_t(x,0) = g(x),$$

will be considered in this thesis. This KG equation is useful for analyzing the propagation of the particle wave in relativistic quantum mechanics and the theory of the quantum field, which is also very important for the physics of high energy particles. Besides that, explaining the propagation of dislocations in crystals and the behavior of elementary particles is also useful.

2.5 Fractional Calculus

Fractional calculus is a branch of mathematical analysis that studies real or even complex numbers, powers of the differential operator (Haubold and Mathai, 2017),

$$D = \frac{d}{dx},$$

and the integration operator J (usually J is used in favour of I to avoid with other I-like identities).

Powers in this context are referred to iterative application or composition, in the same sense as,

$$f^2 x = f(f(x)).$$

For example, one might ask the question of meaningful interpretation,

$$\sqrt{D} = D^{\frac{1}{2}},$$

as the square root of the differentiation operator (a half-iterate operator), that is, an expression for some operator which will have the same effect as differentiation when applied twice to a function. More generally, the question of defining can be considered,

$$D^S,$$

for real number values of s, the usual power of n-fold differentiation is recovered for $n > 0$, and the n-th power of J when for $n < 0$.

2.5.1 The Fractional Integral

Based on the Riemann-Liouville approach to fractional calculus, the notion of the fractional integral of order α $(\alpha > 0)$ is a natural consequence of the well-known formula (usually attributed to Cauchy) which reduces the computation of the $n-$fold primitive of function $f(t)$ to a single integral of the convolution type. The Cauchy formula reads in our notation (Carpinteri and Mainardi, 1997),

$$J^n f(t) = f_n(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau) d\tau \,, t > 0 \,, n \in \mathbb{N}, \qquad (2.11)$$

where $\mathbb{N}$ is the set of positive integers. We note that $f_n(t)$ vanishes at $t = 0$ with its derivatives of order $1, 2, \ldots, (n - 1)$ from this definition.

We require $f(t)$ and henceforth $f_n(t)$ to be a causal function for convention, i.e. to disappear identically for $t < 0$. By using the Gamma function, the above formula is naturally extended from positive integer index values to any positive real values. In fact, note that $(n -$

1)! $= \Gamma(n)$, and the introduction of an arbitrary positive real number α, the Fractional Integral of order $\alpha > 0$ is defined,

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \, , t > 0, \alpha \in \mathbb{R}^+, \tag{2.12}$$

where $\mathbb{R}^+$ is the set of positive real numbers. Define complementarity $J^0 = I$ (identity operator), whereby $J^0 f(t) = f(t)$. Furthermore, $J^{\alpha} f(0^+)$ is set as the limit (if it exists) of $J^{\alpha} f(t)$ for $t \to 0^+$; this limit may be infinite. Note the semi group property (Carpinteri and Mainardi, 1997)

$$J^{\alpha} J^{\beta} = J^{\alpha+\beta} for \ \alpha, \beta \geq 0, \tag{2.13}$$

which implies the commutative property $J^{\alpha} J^{\beta} = J^{\alpha+\beta}$, and the effect of our operators J^{α} on the power functions

$$J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma)}{\Gamma(\gamma+1+\alpha)} t^{\gamma+\alpha}, \qquad \alpha > 0, \gamma > -1, t > 0. \tag{2.14}$$

Naturally, when a positive integer is the order, the properties (2.13) and (2.14) are a natural generalization of those known. The evidence is based on the properties of Euler's two integrals, i.e. Gamma and Beta functions, (Carpinteri and Mainardi, 1997),

$$\Gamma(z) = \int_0^{\infty} e^{-u} u^{z-1} \, du, \Gamma(z+1) = z\Gamma(z), \qquad \mathrm{Re}\{z\} > 0, \tag{2.15}$$

$$\beta(p,q) = \int (1-u)^{p-1} u^{q-1} \, du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \beta(q,p), \ \ \mathrm{Re}\{p,q\} > 0. \tag{2.16}$$

The following causal function may be convenient

$$\Phi_{\alpha}(t) = \frac{t_+^{\alpha+1}}{\Gamma(\alpha)} \, , \alpha > 0, \tag{2.17}$$

where the suffix + only indicates that the $t < 0$ function disappears. This function is $\alpha > 0$; it turns out to be completely integrable locally in $\mathbb{R}^+$. Now, recall the notion of Laplace convolution, i.e. the integral convolution with two causal functions, read in a standard notation

$$f(t) * g(t) = \int_0^t f(t-\tau) g(\tau) d\tau = g(t) * f(t).$$

Then note from Equation (2.12) and Equation (2.17) that the fractional integral of order $\alpha > 0$ may be regarded as convolution of Laplace between $\Phi_\alpha(t)$ and $f(t)$, i.e.

$$J^\alpha \ f(t) = \Phi_\alpha(t) * f(t), \alpha > 0. \tag{2.18}$$

In addition, according to the Eulerian integrals, one proves the composition rule

$$\Phi_\alpha(t) * \Phi_\beta(t) = \Phi_{\alpha+\beta}(t), \alpha, \beta > 0, \tag{2.19}$$

which can be used to re-obtain Equation (2.13) and Equation (2.14).

Introducing the transformation of the Laplace through notation,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \tilde{f}(s),$$

where $s \in \mathbb{C}$, and using the sign ÷ to denote a Laplace transform pair, i.e. $f(t) \div \tilde{f}(s)$.

Note the following rule for the transformation of the fractional integral in Laplace (Carpinteri and Mainardi, 1997),

$$J^\alpha f(t) \div \frac{\tilde{f}(s)}{s^\alpha}, \tag{2.20}$$

which is simple generalization of the case with a repeated integral n-fold ($\alpha = n$). In order to prove this, it is sufficient to recall the convolution theorem for Laplace transforms and note the pair $\Phi_\alpha(t) \div {}^1/_{s^\alpha}$, with $\alpha > 0$.

2.5.2 The Fractional Derivatives

The fractional derivative of order α ($\alpha > 0$) becomes a natural requirement after the notion of fractional integral, and one is attempted to replace α with $-\alpha$ in the above formulas. This generalization, however, needs some care to ensure integral convergence and preserve the well-known properties of the ordinary integer order derivative (Carpinteri and Mainardi, 1997).

Denoting by D^n with $n \in \mathbb{N}$, the operator of the derivative of order n , first note that

$$D^n J^n = I,\ J^n D^n \neq I,\ n \in \mathbb{N}\,, \tag{2.21}$$

i.e. D^n is left-inverse (and not right-inverse) to the corresponding integral operator from J^n. In fact, from Equation (2.21),

$$J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{t^k}{k!}, t > 0. \tag{2.22}$$

Consequently, expect that D^α is defined as left-inverse to J^α. For this purpose, introducing the positive integer m such that $m-1\ < \alpha \leq m$.

Define the Fractional Derivative of order $\alpha > 0$:

$$D^\alpha f(t) = D^m J^{m-\alpha}\ f(t)\ ,$$

namely

$$D^\alpha f(t) = \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(\mathrm{m}-\mathrm{\alpha})}\displaystyle\int_0^t \dfrac{f(t)}{(t-\tau)^{\alpha+1-m}}\ d\tau\right]; m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t);\ \alpha = m. \end{cases} \tag{2.23}$$

Defining for complementation $D^0 = J^0 = I$, then easily recognize that

$$D^\alpha J^\alpha = I, \alpha\ \geq\ 0, \tag{2.24}$$

and

$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}\ , \alpha > 0, \gamma > -1, t > 0. \tag{2.25}$$

Of course, when the order is a positive integer, the properties Equation (2.24) and Equation (2.25) are a natural generalization of those known. Since in Equation (2.25) the argument of the Gamma function in the denominator may be negative, it is necessary to consider the analytical continuation of Γ(z) in Equation (2.15) to the left half plane.

The remarkable fact that the fractional derivative $D^\alpha f$ is not zero for the constant function $f(t) \equiv 1\ if\ \alpha \notin \mathbb{N}$. In fact, Equation (2.25) with $\gamma = 0$ teaches that (Carpinteri and Mainardi, 1997),

$$D^{\alpha}1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\ ;\ \alpha \geq\ 0, t\ > 0. \tag{2.26}$$

Of course, this is $f(t) \equiv 0$ for $\alpha \in \mathbb{N}$, due to the poles of the gamma function in the points $0, -1, -2, ...$

Now observe that the so-called Caputo Fractional Derivative of order $\alpha > 0$ is an alternative definition of fractional derivative, originally introduced by Caputo and Mainardi in the theory of Linear Viscoelasticity,

$$D^{\alpha} f(t) = J^{m-\alpha} D^{m}\, f(t),$$

with $m - 1 < \alpha \leq m,$ namely

$$D^{\alpha} * f(t) = \begin{cases} \left[\dfrac{1}{\Gamma(\mathrm{m}-\alpha)} \displaystyle\int_{0}^{t} \dfrac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\right]; m-1 < \alpha < m \\ \dfrac{d^{m}}{dt^{m}} f(t)\ ;\ \alpha = m. \end{cases} \tag{2.27}$$

Naturally, this definition is more restrictive than Equation (2.23), in that the derivative of order m requires absolute integrability. We assume that this condition is satisfied if we use the operator D^{α}. It is easy to recognize that in general,

$$\left.D^{\alpha} f(t) \begin{array}{l} = D^{m} J^{m-\alpha} f(t) \\ \neq J^{m-\alpha} D^{m} f(t) \\ = D^{\alpha} * f(t) \end{array}\right\}, \tag{2.28}$$

unless the function $f(t)$ along with its first $m - 1$ derivatives vanishes at $t = 0^{+}$. In fact, assuming that the passage of the m-derivative under the integral is legitimate, one recognizes that, for $m - 1 < \alpha < m$ and $t > 0$,

$$D^{\alpha} f(t) = D^{\alpha} * f(t) + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0^{+}), \tag{2.29}$$

and therefore, recalling the fractional derivative of the power functions Equation (2.25) (Carpinteri and Mainardi, 1997),

$$D^{\alpha}\left(f(t) - \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0^{+})\right) = D^{\alpha} * f(t). \tag{2.30}$$

For the fractional derivative, the alternative definition Equation (2.27) thus incorporates the initial values of the function and its lower-order integer derivatives. The Taylor polynomial subtraction of degree $m-1$ at $t = 0^+$ from $f(t)$ means a kind of regularization of the fractional derivative. In particular, according to this definition, it is easy to recognize the relevant property for which the fractional derivative of a constant remains zero , i.e.

$$D_*^\alpha 1 \equiv 0, \qquad \alpha > 0. \tag{2.31}$$

Now examine the most significant differences between the two fractional derivatives of Equations (2.23) and (2.27). To distinguish this from the standard Riemann-Liouville fractional derivative Equation (2.23), agree to denote Equation (2.27) as the Caputo fractional derivative. Looking again at Equation (2.25), we observe that

$$D^\alpha t^{\alpha-1} \equiv 0, \qquad \alpha > 0, \quad t > 0. \tag{2.32}$$

From Equation (2.32) and Equation (2.31) we thus recognize the following statements about functions which for $t > 0$ admit the same fractional derivative of order α , with $m - 1 < \alpha \leq m, m \in \mathbb{N}$.

$$D^\alpha f(t) = D^\alpha g(t) \Leftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{\alpha-j}, \tag{2.33}$$

$$D_*^\alpha f(t) = D_*^\alpha g(t) \Leftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{m-j}. \tag{2.34}$$

The c_j coefficients in these formulae are arbitrary constants. Note that, incidentally, Equation (2.32) provides an instructive example of how D^α is not right-inverse to J^α, since

$$J^\alpha D^\alpha t^{\alpha-1} \equiv 0, \text{ but } D^\alpha J^\alpha t^{\alpha-1} = t^{\alpha-1}, \qquad \alpha > 0, \quad t > 0. \tag{2.35}$$

We also note a difference with respect to the formal limit as $\alpha \rightarrow (m-1)^+$ for the two definitions. We obtain from Equations (2.23) and (2.27), respectively,

$$\alpha \rightarrow (m-1)^+ \Rightarrow D^\alpha f(t) \longrightarrow D^m J f(t) = D^{m-1} f(t), \tag{2.36}$$

$$\alpha \rightarrow (m-1)^+ \Rightarrow D_*^\alpha f(t) \longrightarrow J D^m f(t) = D^{m-1} f(t) - f^{(m-1)}(0)^+. \tag{2.37}$$

We now consider the Laplace transform of the two fractional derivatives. For the standard fractional derivative D^{α} the Laplace transform, assumed to exist, requires the knowledge of the (bounded) initial values of the fractional integral $J^{m-\alpha}$ and of its integer derivatives of order $k = 1, 2, \dots, (m - 1)$. The corresponding rule reads, in our notation,

$$D^{\alpha} f(t) \div s^{\alpha} \tilde{f}(s) - \sum_{k=0}^{m-1} D^{k} J^{m-\alpha} f(0^{+}) s^{m-1-k}, \quad (m-1) \le \alpha \le m. \tag{2.38}$$

The Caputo fractional derivative appears more suitable to be treated by the Laplace transform technique in that it requires the knowledge of the (bounded) initial values of the function and of its integer derivatives of order $k = 1, 2, \dots, (m - 1)$, in analogy with the case when $\alpha = m$. In fact, by using Equation (2.20) and noting that

$$J^{\alpha} D_{*}^{\alpha} f(t) = J^{\alpha} J^{m-\alpha} D^{m} f(t) = J^{m} D^{m} f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^{+}) \frac{t^{k}}{k!}. \tag{2.39}$$

We easily prove the following rule for the Laplace transform,

$$D_{*}^{\alpha} f(t) \div s^{\alpha} \tilde{f}(s) - f^{(k)}(0^{+}) s^{\alpha-1-k}, \quad (m-1) \le \alpha \le m. \tag{2.30}$$

In fact, the outcome (2.30) first reported by Caputo with the theorem Fubini-Tonelli in 1969 appears to be the most "natural" generalization of the corresponding result well known for $\alpha = m$.

It now appears that both (2.23) and (2.27) definitions for the fractional derivative of $f(t)$ can at least formally be derived by the convolution of $\Phi_{-\alpha}(t)$ with $f(t)$, in a kind of fractional integral analogy with Equation (2.18). For this reason, consider that (with proper interpretation of the quotient as a limit if $t = 0$) from the treatise on generalized functions by Gel'fand and Shilov.

$$\Phi_{-n}(t) := \frac{t_{+}^{-n-1}}{\Gamma(-n)} = \delta^{(n)}(t), \quad n = 0,1, \dots \tag{2.31}$$

where $\delta^{(n)}(t)$ denotes the generalized derivative of order n of the Dirac delta distribution.

The Equation (2.31) offers an interesting (not very well known) representation of $\delta^{(n)}(t)$, which is useful for our subsequent treatment of fractional derivatives. Indeed, notice that the derivative of order n of a causal function $f(t)$ can be formally obtained through the (generalized) convolution between Φ_{-n} and f,

$$\frac{d^n}{dt^n}f(t) = f^{(n)}(t) = \Phi_{-n}(t) * f(t) = \int_{0^-}^{t^+} f(\tau)\delta^{(n)}(t-\tau)d\tau\,, \quad t > 0, \tag{2.32}$$

based on the well-known properties

$$\int_{0^-}^{t^+} f(\tau)\delta^{(n)}(\tau - t)d\tau = (-1)^n f^{(n)}(t),\ \delta^{(n)}(t-\tau) = (-1)^n \delta^{(n)}(\tau - t). \tag{2.33}$$

According to a common convention, the limits of integration are extended in Equations (2.32-2.33) to take into account the possibility of extreme-centered impulse functions. Then, the fractional derivative of order α could be formally defined;

$$\Phi_{-\alpha}(t) * f(t) = \frac{1}{\Gamma(-\alpha)}\int_{0^-}^{t^+} \frac{f(\tau)}{(t-\tau)^{1+\alpha}}d\tau\,, \quad \alpha \in \mathbb{R}^+.$$

The formal character is obvious in that it turns out that the kernel $\Phi_{-\alpha}(t)$ is not completely integrable locally and therefore the integral is generally divergent. To obtain a definition that is still valid for classical functions, the divergent integral must be regularized in some way. Consider the integer $m \in \mathbb{N}$ such that $(m-1) < \alpha < m$ and write $-\alpha = -m + (m-\alpha)$ or $-\alpha = (m-\alpha) - m$. Then, we obtain

$$[\Phi_{-m}(t) * \Phi_{m-\alpha}(t)] * f(t) = \Phi_{-m}(t) * [\Phi_{m-\alpha}(t) * f(t)] = D^m J^{m-\alpha} f(t), \tag{2.34}$$

or

$$[\Phi_{m-\alpha}(t) * \Phi_{-m}(t)] * f(t) = \Phi_{m-\alpha}(t) * [\Phi_{-m}(t) * f(t)] = J^{m-\alpha} D^m f(t). \tag{2.35}$$

Therefore, derive two alternative definitions corresponding to (1.13) and (1.17), respectively, for the fractional derivative. In these formulas, the singular behavior of $\Phi_{-m}(t)$ is reflected in the non-commutativity of convolution.