NUMERICAL SOLUTION OF NONLINEAR SCHRÖDINGER EQUATIONS BASED ON B-SPLINE GALERKIN FINITE ELEMENT METHOD

AZHAR IQBAL

UNIVERSITI SAINS MALAYSIA

2020

NUMERICAL SOLUTION OF NONLINEAR SCHRÖDINGER EQUATIONS BASED ON B-SPLINE GALERKIN FINITE ELEMENT METHOD

by

AZHAR IQBAL

Thesis submitted in fulfilment of the requirements
for the degree of
Doctor of Philosophy

November 2020

ACKNOWLEDGEMENT

First and foremost, I would like to sincerely thank Allah S.W.T for giving me this opportunity to pursue my research under the supervision of Professor Dr. Ahmad Izani Md. Ismail, School of Mathematical Sciences, Universiti Sains Malaysia. No words would suffice to express my highest gratitude and respect to my supervisor for his priceless advice, encouragement and consistent help during the period of my research. I am also grateful to him for appropriate, timely comments and valuable suggestions provided by him during this entire course of research at different challenging phases. This research would not have been possible without his support. I am also highly indebted to my co-supervisor Dr. Nur Nadiah Abd Hamid for her constant support, discussion and valuable comments. Further, I am also grateful to the Universiti Sains Malaysia and Institute of Postgraduate Studies for the facilities provided during this study. I wish to mention that whatever I have achieved in my life is because of the unconditional support and blessings of my beloved father Late. Din Muhammad, mother Hassan Bibi and dearest siblings. No words to express my gratitude to my wife Uzma Azhar for her unconditional love, patience and support through out the journey. Finally, I wish to thank all of my friends and relatives for their kindness and moral support.

TABLE OF CONTENTS

LIST OF TABLES

Page

LIST OF FIGURES

Page

LIST OF ABBREVIATIONS

CNLS	Coupled nonlinear Schrödinger
Cubic-BSGM	Cubic B-spline Galerkin method
FEM	Finite element method
FDM	Finite difference method
KDV	Korteweg-de Vries
NLS	Nonlinear Schrödinger
ODE	Ordinary differential equation
PDE	Partial differential equation
Quartic-BSGM	Quartic B-spline Galerkin method
Quintic-BSGM	Quintic B-spline Galerkin method
RLW	regularized long wave equation

PENYELESAIAN BERANGKA BAGI PERSAMAAN SCHRÖDINGER TAK LINEAR BERDASARKAN KAEDAH UNSUR TERHINGGA SPLIN B GALERKIN

ABSTRAK

Fungsi splin B telah digunakan sebagai alat untuk menghasilkan lengkung dan permukaan dalam Rekabentuk Geometri Bantuan Komputer dan grafik komputer. Kelebihan utama fungsi ini adalah sifat titik kawalan tempatan mereka, yang mana setiap titik kawalan disambungkan dengan fungsi asas tertentu. Tiap-tiap titik menentukan bentuk lengkung ke atas nilai-nilai julat parameter yang mana fungsi asas yang berkaitan adalah tidak sifar. Berdasarkan sifat-sifat ini, fungsi splin B dapat digunakan untuk menghasilkan penyelesaian anggaran kepada persamaan pembezaan separa. Terdapat pelbagai teknik berangka tersedia untuk mencari penyelesaian berangka bagi persamaan pembezaan separa tak linear. Dalam beberapa tahun kebelakangan ini, kaedah Galerkin telah mendapat perhatian para penyelidik kerana keupayaannya untuk memberikan penyelesaian berangka yang tepat dan cekap kepada masalah tak linear. Pilihan fungsi asas memainkan peranan utama dalam kaedah Galerkin. Pelbagai fungsi asas seperti fungsi interpolasi Lagrange dan sinus-kosinus telah digunakan untuk kaedah unsur terhingga Galerkin untuk penyelesaian berangka persamaan pembezaan tak linear. Fungsi asas ini mempunyai beberapa batasan. Bagi mendapatkan sifat sebenar lengkung penyelesaian persamaan pembezaan, fungsi licin dengan jumlah titik lengkok balas yang mencukupi diperlukan. Fungsi spline B sebagai fungsi asas dan berat memenuhi semua keperluan kerana sifat fungsi spline B Objektif utama penyelidikan ini adalah untuk menjalankan penggunaan splin B sebagai fungsi asas dalam kaedah Galerkin untuk menyelesaikan persamaan Schrödinger tak linear (NLS) dan persamaan Schrödinger tak linear berkait (CNLS). Latar belakang teori kepada persamaan NLS dibincangkan dan pendekatan pengiraan semasa dihuraikan. Kaedah unsur terhingga Galerkin dirumuskan berdasarkan fungsi splin B kubik untuk penyelesaian berangka persamaan NLS dan CNLS yang mana sebutan tak linear akan di linearkan. Pertama, kaedah Galerkin digunakan berdasarkan fungsi splin B untuk penyelesaian berangka persamaan NLS dan CNLS. Kedua, fungsi cubaan splin B kuartik dengan kaedah Galerkin digunakan. Kaedah berangka tersebut diuji melalui kajian pergerakan soliton, interaksi, dan pembangunan berikutan. Ketiga, kaedah yang dibentangkan adalah berdasarkan kaedah Galerkin menggunakan unsur terhingga splin B kuintik sebagai fungsi cubaan. Untuk menilai ketepatan dan kecekapan kaedah tersebut, beberapa contoh dibentangkan dan dibandingkan dengan hasil yang tepat dan hasil yang diterbitkan. Keputusan yang diperolehi oleh kaedah Galerkin berdasarkan unsur terhingga splin B menunjukkan bahawa skema ini memberikan hasil yang baik dan hukum-hukum keabadian dipatuhi dengan memadai. Kestabilan kaedah yang dicadangkan dianalisis oleh analisis kestabilan von Neumann. Skim ini telah disahkan stabil tanpa syarat. Sumbangan utama kajian ini adalah pembangunan kaedah unsur terhingga Galerkin dengan splin B sebagai fungsi asas untuk menyelesaikan persamaan NLS dan CNLS.

NUMERICAL SOLUTION OF NONLINEAR SCHRÖDINGER EQUATIONS BASED ON B-SPLINE GALERKIN FINITE ELEMENT METHOD

ABSTRACT

B-spline functions have been used as tools for generating curves and surfaces in Computer Aided Geometric Design and computer graphics. The main advantage of these functions are the properties of their local control points, where each control point is connected with a specific basis function. Every point determines the curve shape over a parameter range values where the basis function is non-zero. Because of these properties, B-spline functions can be used to produce the approximate solutions to partial differential equations (PDEs). Various numerical techniques are available to find the numerical solution of nonlinear PDEs. In recent years, the Galerkin method has gained much attention from researchers due to its ability to provide accurate and efficient numerical solutions to nonlinear problems. The choice of basis functions play a major role in the Galerkin method. Various basis function such as Lagrange interpolation polynomial and sine-cosine function have been applied to the Galerkin finite element method for the numerical solution of nonlinear differential equations. These basis functions have some limitations. To obtain the exact nature of the differential equation's solution curve, a smooth function with enough number of inflection points is required. The B-spline function as basis and weight functions fulfill all requirements due to the properties of the B-spline functions. The main objective of this research is to illustrate the use of B-spline as basis functions in the Galerkin method to solve nonlinear Schrödinger (NLS) and coupled nonlinear Schrödinger (CNLS) equations. The theoretical background to the NLS equations are discussed and current computational approaches are described. A Galerkin finite element procedure is formulated based on B-spline functions for the numerical solution of NLS equation and CNLS equation with the non-linear term is linearized. First, Galerkin method is used based on cubic B-spline function for the numerical solution of NLS and CNLS equations. Secondly, quartic B-spline trial function with Galerkin method is applied. The numerical method is tested through the study of soliton motion, interactions and subsequent development. Thirdly, the method presented is based on Galerkin method using quintic B-spline finite elements as a trial function. In order to assess the accuracy and efficiency of the method, some examples are presented and compared with exact results and published results. The results obtained by Galerkin method based on B-spline finite elements showed that the scheme presented good results and conservations laws are all adequately obeyed. The stability of the proposed method is analyzed by the von Neumann stability analysis. The schemes were verified to be unconditionally stable. The main contribution of this study is the development of the Galekrin finite element method with B-spline as basis functions to solve NLS and CNLS equations.

CHAPTER 1

INTRODUCTION

Researchers have often employed differential equations to structure and model physical problems. A soliton is a wave packet that is located in a region and maintain their shape while it propagates with constant speed for a certain period (Zabusky and Kruskal, 1965). Solitons also provide solutions for dynamical systems that are fully integrable. The stability of the solitons are achieved by delicately balancing "non-linearity" and "dispersion" in the equation (Infeld and Rowlands, 2000). John Scott Russell (1808–1882) first described the soliton phenomenon in 1834. He conducted experiments for solitons in a wave tank. It is known that (Bullough and Caudrey, 1980):

1. Solitons are of permanent shape,
2. They are localized in the region,
3. They can interact with other waves without any change in the shape and velocity.

In this thesis, soliton solutions of the nonlinear Schrödinger (NLS) and coupled nonlinear Schrödinger (CNLS) equations are presented. The NLS and CNLS equations describe how the quantum states of a physical system evolves in time and space. The NLS equation can be used, for example to describe the propagation of optical pulses and waves in water and plasmas (Karpman and Krushkal, 1969). Due to the nonlinearities and the complicated nature of the underlying problem, it is still a challenge for researchers to identifying the most appropriate method of solution for various problems which involve the NLS and CNLS equations. Many studies have been undertaken to overcome this difficulty. The analytical solution can be found for specific simple differential equations but because of nonlinear nature of the physical system, it is generally very difficult to find their solution analytically. Sometime analytical solutions can be obtained, but they can't be expressed in a convenient closed form. In some cases solutions are available in complex form or infinite series form which are sometime not easy to be determined. However, optical solitons can be represented generally by the NLS equation in an ideal fiber. But, due to the existence of birefringence which are double refractions, the real single mode optical fibers are bimodal (Porsezian and Kalithasan, 2007). To increase the transmitting efficiency of the telecommunication system, at least two optical fields need to be emitted, and a CNLS controls the pulse, where the signal passes at significantly opposite speeds around the both orthogonal polarization axes (Sun et al., 2004). The CNLS equation also plays an important role in many physical systems, including hydrodynamics, plasma physics and crystals.

Various numerical methods are available for solving the differential equations. Some recent and popular methods are the finite element method (FEM) (Aksan, 2006; Karakoc and Bhowmik, 2019b; Sharma and Sharma, 2015), finite difference method (FDM) (Gavete et al., 2017), finite volume method (Dumbser and Casulli, 2016), spectral method (Yin and Gan, 2015) and differential quadrature method (Qasim and AL-Rawi, 2018). The FEM and FDM are well developed and are the dominant numerical methods to approximate differential equations. FDM approximates the derivative of unknown functions at some discrete points and convert differential equation into algebraic equations. In FEM, the whole domain is divided into uniform elements that can be in different shapes. FEM with B-spline as basis functions are developed for structured grid that provide a continuous solution within the solution domain. There are many numerical studies in the literature on the solution of NLS and CNLS equations by many researchers. Different kinds of numerical methods such as FEM and FDM are used to solve the NLS and CNLS equations. However not much work has been carried out utilizing B-splines. In this thesis we study the solitons solution of the NLS and CNLS equations using B-splines as basis functions with one of the most popular FEM approaches which is called the Galerkin method.

1.1 Galerkin Finite Element Method

The Galerkin FEM is used to solve a different kind of physical boundary value problems. In the Galerkin FEM, the whole domain is divided into uniform finite elements that can be in different shapes. The choice of basis functions has an important role in the Galerkin FEM. The solution of the problems is extended in terms of basis functions.

The following are the basic steps to solve a boundary value problem by using Galerkin FEM (Thomée, 1984).

- First step: establish a strong form of our given equation and write out the strong form.
- Second step: the weak form is obtained. The weak form is also known as weighted residual.
- Third step: choose the approximate solution for our unknown functions.
- Fourth step: choose the basis functions.
- Fifth step: Crank Nicolson and finite difference schemes are used for nodal pa-

rameters and time derivatives respectively.

- Final step: system is solved by an appropriate method.

To explain the basic Galerkin FEM, consider the one-dimensional time dependent heat flow equation with the boundary conditions which is called a strong form:

$$u_t - ku_{xx} = 0, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \tag{1.2}$$

where u is temperature, k is real value and subscripts x and t represent differentiation with respect to space and time respectively.

Applying the Galerkin method to Equation (1.1) with the weight function $W(x)$ gives

$$\int_0^1 W(x)\,[u_t - ku_{xx}]\,dx = 0, \tag{1.3}$$

where $W(x)$ is the weight function over the domain. Integrating by parts,

$$\int_0^1 [W(x)u_t + ku_x W(x)_x]\,dx - k(W(1)u_x(1,t) + W(0)u_x(0,t)) = 0. \tag{1.4}$$

By using the boundary conditions (1.2), we obtain

$$\int_0^1 [W(x)u_t + ku_x W(x)]\,dx = 0. \tag{1.5}$$

Equation (1.5) is called the weak formulation of Equation (1.1).

Third step, we choose the approximate solution as:

Let $u^e(x,t)$ be an approximate solution over finite element of Equation (1.1)

$$u^e(x,t) = \sum_{j=1}^{n} u_j^e(t) B_j^e(x). \tag{1.6}$$

In the Galerkin FEM, the weight function and shape function are chosen to be the same functions i.e. $W_j = B_j^e$.

Fourth Step: The splines functions are chosen as a weight and basis functions to find the approximate solution of Equation (1.1). Substituting (1.6) in (1.5), we obtain the j^{th} equation of element is given by

$$\int_0^1 \left[\sum_{j=1}^{n} B_j^{'} B_i \dot{u}_j + k \sum_{j=1}^{n} B_j^{'} B_i^{'} u_j \right] dx = 0. \tag{1.7}$$

The matrix form of the above Equation (1.7) can be written as

$$M_{ij} \dot{u}_j + k N_{ij} u_j = 0, \tag{1.8}$$

where $\dot{u}$ denotes time derivative and

$M_{ij} = \int_0^1 B_j^{'} B_i dx,$

$N_{ij} = \int_0^1 B_j^{'} B_i^{'} dx.$

Fifth step: Finite difference approximation $\dot{u} = \frac{u^{n+1} - u^n}{\Delta t}$ is used for time derivative and the Crank-Nicolson method $u = \frac{u^{n+1} + u^n}{2}$ is used for nodal parameters. we obtain the nonlinear recurrence relationship for time parameters as

$$(2M + k\Delta t N) u^{n+1} = (2M - k\Delta t N) u^n. \tag{1.9}$$

where $k = 1$.

First, we find the initial vectors from the initial conditions. With the initial vectors, the system can be solved at different time steps.

1.2 Short Review on B-Spline Basis Functions

In the Galerkin FEM, the choice of the basis function is very important and needs to be carefully selected because it directly affects the accuracy and computational efficiency. In this thesis, spline functions are used as basis functions for the solution of the partial differential equation (PDE) because of their flexibility and well-defined properties. A spline is a piecewise polynomial between each pair of points in organized form. The cubic splines are the most general splines that extend an interpolated function which is continuous through to the second derivative. In this section we discuss how spline interpolation works.

Consider the function $f(x) = \frac{1}{x^2+2}.\sin(x)$ over the interval $[-10, 10]$. The values of $f(x_m)$ are given as $x_m = -10, -9, \dots, 9, 10$. In other word $f_i = f(x_m),\ m = 0, \dots, 20$ and the locations are known as the "nodes", as we can see in Figure 1.1, the 21 nodes are represented by the red dots.

Let us start with the interval $(x_m,\ x_{m+1})$. The formula for the linear interpolation is (Mat Zin, 2016)

$$f = Af_m + Bf_{m+1}, \tag{1.10}$$

where $A = \frac{x_{m+1}-x}{x_{m+1}-x_m}$ and $B = 1 - A = \frac{x-x_m}{x_{m+1}-x_m}$. Figure 1.2 shows the piecewise interpolated function over the region $[-10, 10]$. Here it can be seen that this interpolation works quite well with larger values of $|x|$ but it is unsuccessful to obtain control over

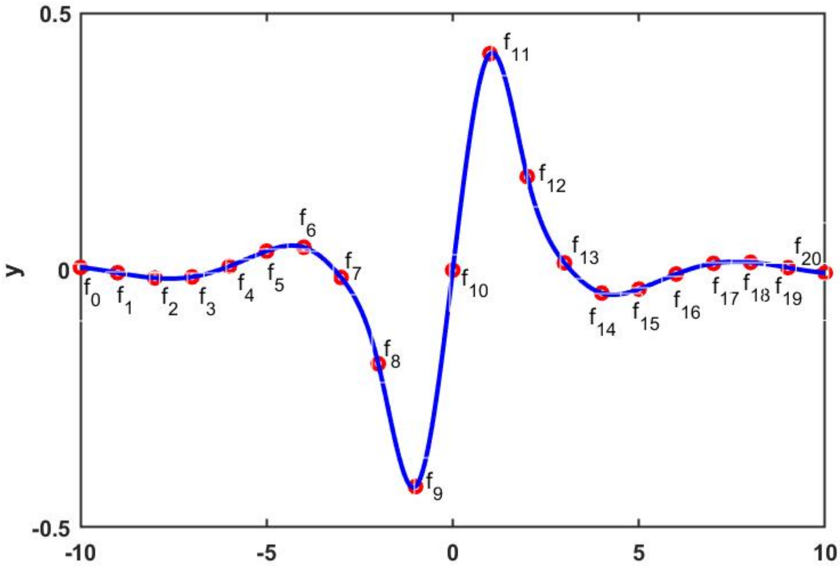

Figure 1.1: The function $f(x) = \frac{1}{x^2+2}.\sin(x)$ between $[-10, 10]$

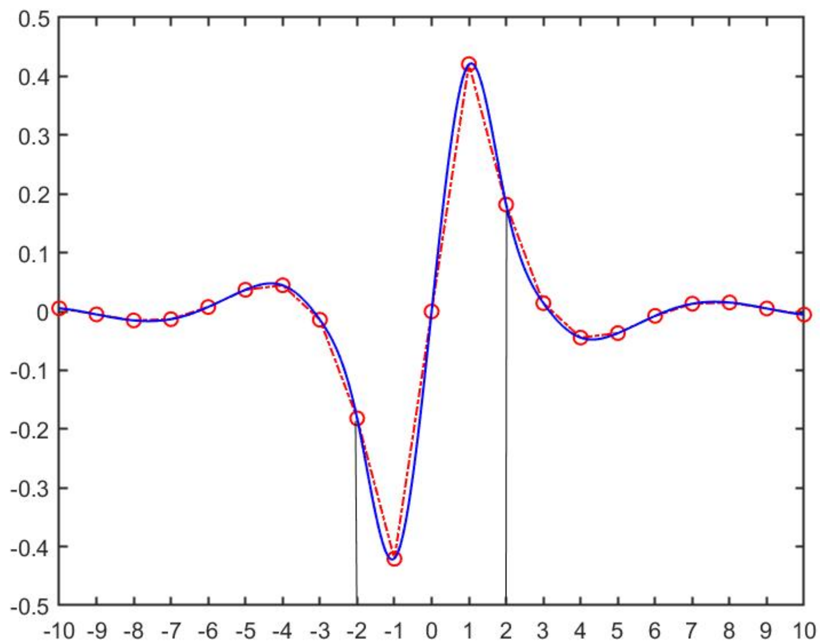

Figure 1.2: Piecewise linear interpolation with 21 nodes for the function between $[-10, 10]$.

the curvature of function at interval $(-2, 2)$. The accuracy can be improved by adding more interpolated nodes but note that the first derivative of the function is not continuous at the nodes. There are different classes of splines such as cubic, quartic and quintic spline. Cubic spline has also many classes of splines, for example cubic B-spline, cubic trigonometric B-spline, cubic Beta-spline and extended cubic B-spline. B-spline function has been used by several researchers for the differential equations. This function has very suitable properties for designing such as a convex hull and continuity properties (Abd Hamid, 2010). The following are the properties of B-spline function $B_k^m(x)$ (Carl de, 1978; Mat Zin, 2016):

1. $B_k^m(x)$ are positive for all m , k and x.
2. $B_k^m(x) > 0$ in knot span $[x_k, x_{k+1})$ and elsewhere $B_k^m(x) = 0$.
3. $\sum_{k=0}^{n} B_k^m(x) = 1$ for knot span $[x_{k-1}, x_{k+1}]$.
4. $B_k^m(x) = B_0^m(x - x_k)$ that is a B-spline basis function with identical order and interpreted to each other.

1.2.1 The Linear B-spline Basis Function

The linear B-spline basis function is given by Prenter (1975)

$$B_k(x) = \frac{1}{h} \begin{cases} (x_{k+1} - x) - 2(x_k - x), & [x_{k-1}, x_k), \\ (x_{k+1} - x), & [x_k, x_{k+1}), \\ 0, & \text{otherwise}. \end{cases} \tag{1.11}$$

The solution domain $[a, b]$ is partitioned into N finite elements of uniform length by the nodes x_k such that $a = x_0 < x_1 < \ldots x_N = b$ and $h = \frac{b-a}{N} = x_k - x_{k-1}$, $k = 1, \ldots, N$. Discussing only the interval elements, we see, from Equation (1.11), that each spline $B_k(x)$ covers 2 intervals $x_{k-1} < x < x_{k+1}$ so that 2 splines B_k, B_{k+1} cover each finite element $[x_k, x_{k+1}]$, all other splines are zero in this region. By using the local coordinate

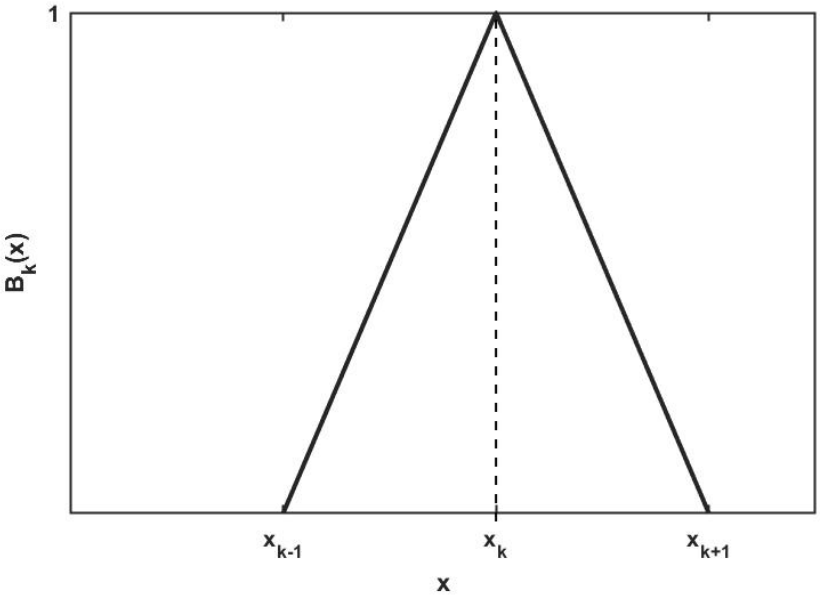

Figure 1.3: The linear B-spline basis function

transformation $\eta = x - x_k$, $\eta \in [0,h]$, linear B-spline shape functions are obtained in terms of the η over the interval $\in [0,h]$ as

$$(B_k, B_{k+1}) = \frac{1}{h}(h-\eta, \eta). \tag{1.12}$$

Figure 1.3 displays the graph of linear B-spline basis function of order 2.

1.2.2 The Quadratic B-spline Basis Function

The quadratic B-spline basis function is presented by Prenter (1975) as

$$B_k(x) = \frac{1}{h^2}\begin{cases} (x_{k+3}-x)^2 - 3(x_{k+2}-x)^2 + 3(x_{k+1}-x)^2, & [x_{k-1}, x_k), \\ (x_{k+3}-x)^2 - 3(x_{k+2}-x)^2, & [x_k, x_{k+1}), \\ (x_{k+3}-x)^2, & [x_{k+1}, x_{k+2}), \\ 0, & \text{otherwise}. \end{cases} \tag{1.13}$$

All other B-spline functions are zero over the finite element $[x_{k-1}\ ,\ x_{k+2}]$. A local coordinate transformation $\eta = x - x_k$, $\eta \in [0,h]$ is used and B-spline shape functions are obtained in terms of the η over the interval $[0,h]$ as

$$(B_{k-1}, B_k, B_{k+1}) = \frac{1}{h^2}(h^2 - 2\eta h + \eta^2, h^2 + 2\eta h - 2\eta^2, \eta^2). \tag{1.14}$$

Figure 1.4 shows the graph of quadratic B-spline function at $[x_{k-1}, x_{k+2}]$.

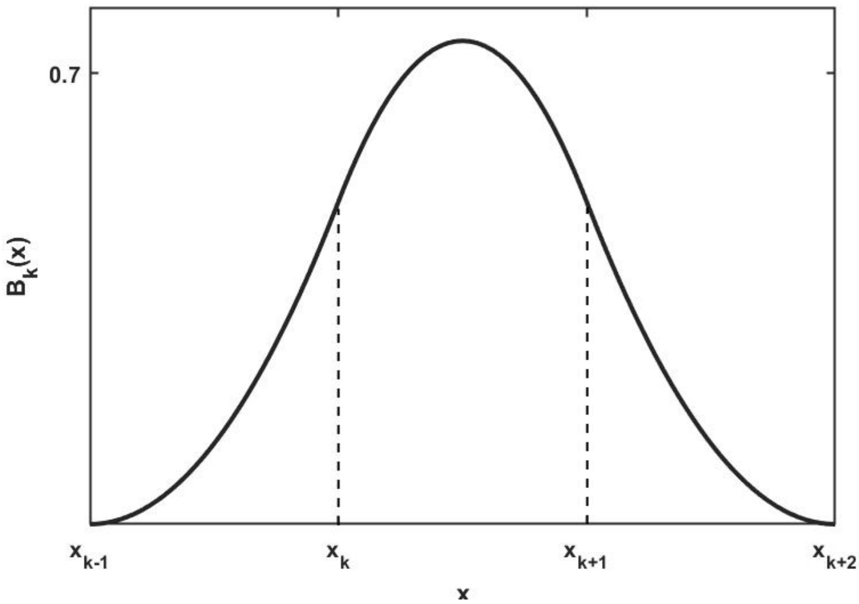

Figure 1.4: The quadratic B-spline basis function

1.2.3 The Cubic B-spline Basis Function

The cubic B-spline function $B_k(x)$, $k=-1,\ldots,N+1$ are defined over the interval $[a,b]$ as follows (Prenter, 1975):

$$B_k(x)=\frac{1}{h^3}\begin{cases}(x-x_{k-2})^3 & [x_{k-2},x_{k-1}],\\ h^3+3h^2(x-x_{k-1})+3h(x-x_{k-1})^2-3(x-x_{k-1})^3 & [x_{k-1},x_k],\\ h^3+3h^2(x_{k+1}-x)+3h(x_{k+1}-x)^2-3(x_{k+1}-x)^3 & [x_k,x_{k+1}],\\ (x_{k+2}-x)^3 & [x_{k+1},x_{k+2}],\\ 0 & \text{otherwise}.\end{cases} \tag{1.15}$$

Equation (1.15) can be stated using the local coordinate transformation $\eta=x-x_k$, $\eta\in[0,h]$. B-spline shape functions in terms of the η over the interval $[0,h]$ can be written as

$$\begin{aligned}B_{k-1}&=\frac{1}{h^3}(h-\eta)^3,\\ B_k&=\frac{1}{h^3}(4h^3-3h^2\eta+3h(h-\eta)^2-3(h-\eta)^3),\\ B_{k+1}&=\frac{1}{h^3}(h^3+3h^2\eta+3h\eta^2-3\eta^3),\\ B_{k+2}&=\frac{1}{h^3}(\eta^3).\end{aligned} \tag{1.16}$$

A graph of cubic B-spline basis function of order 4 is displayed in Figure 1.5.

1.2.4 The Quartic B-spline Basis Function

The solution domain $[a,b]$ is divided into N finite elements of uniform space by grid points x_k, such that $a=x_0<x_1<\ldots x_{N-1}<x_N=b$ and $h=x_k-x_{k-1}$, $k=$

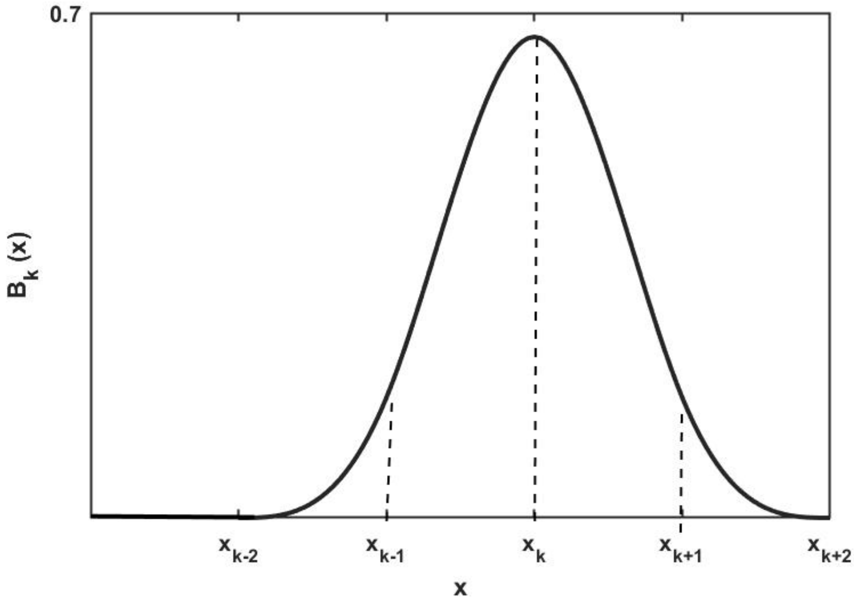

Figure 1.5: The cubic B-spline basis function

$1,\ldots,N$. The quartic B-splines $B_k(x)$, $k=-2,\ldots,N+1$ at the grid points x_k are defined over the interval $[a,b]$ as described by Prenter (1975):

$$B_k(x) \;=\frac{1}{h^4}\begin{cases} p_1{=}(x-x_{k-2})^4 & [x_{k-2},x_{k-1}],\\ p_2=p_1-5(x-x_{k-1})^4 & [x_{k-1},x_k],\\ p_3=p_2+10(x-x_k)^4 & [x_k,x_{k+1}],\\ (x_{k+3}-x)^4-5(x_{k+2}-x)^4 & [x_{k+1},x_{k+2}],\\ (x_{k+3}-x)^4 & [x_{k+2},x_{k+3}],\\ 0 & \text{otherwise.}\end{cases} \tag{1.17}$$

A local coordinate transformation $\eta = x - x_k$, $\eta \in [0, h]$ is used and quartic B-spline shape functions in terms of the η can be defined as

$$\begin{aligned}
B_{k-2} &= \frac{1}{h^4}(h^4 - 4\eta h^3 + 6\eta^2 h^2 - 4\eta^3 h + \eta^4), \\
B_{k-1} &= \frac{1}{h^4}(11h^4 - 12\eta h^3 - 6\eta^2 h^2 + 12\eta^3 h - \eta^4), \\
B_k &= \frac{1}{h^4}(11h^4 + 12\eta h^3 - 6\eta^2 h^2 - 12\eta^3 h + \eta^4), \\
B_{k+1} &= \frac{1}{h^4}(h^4 + 4\eta h^3 + 6\eta^2 h^2 + 4\eta^3 h - \eta^4), \\
B_{k+2} &= \frac{1}{h^4}(\eta^4).
\end{aligned} \tag{1.18}$$

The graph of quartic B-spline function can be seen in Figure 1.6.

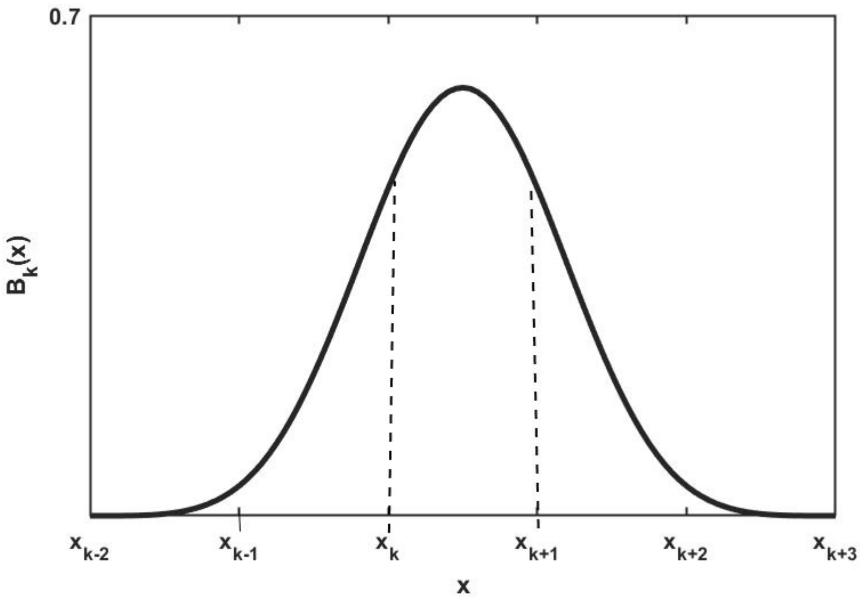

Figure 1.6: The quartic B-spline of basis function

1.2.5 The Quintic B-spline Basis Function

We consider a mesh Π over the finite domain $[a,b]$ divided uniformly by grid points x_k with $h = x_k - x_{k-1},\ k = 1,\ldots,N$. The quintic B-splines, $B_k(x)$, $k = -2,\ldots,N+2$ at the grid points x_k form a basis over the interval $[a,b]$ as follows (Saka et al., 2008):

$$B_k(x) = \frac{1}{h^5}\begin{cases} p_1=(x-x_{k-3})^5\,, & x\in[x_{k-3}\ ,x_{k-2}], \\ p_2=p_1-6(x-x_{k-2})^5\,, & x\in[x_{k-2}\ ,x_{k-1}], \\ p_3=p_2+15(x-x_{k-1})^5, & x\in[x_{k-1}\ ,\ x_k], \\ p_4=p_3-20(x-x_k)^5, & x\in[x_k\ ,x_{k+1}], \\ p_5=p_4+15(x-x_{k+1})^5, & x\in[x_{k+1}\ ,x_{k+2}], \\ p_6=p_5-6(x-x_{k+2})^5, & x\in[x_{k+2}\ ,x_{k+3}], \\ 0 & \text{otherwise.} \end{cases} \tag{1.19}$$

A local coordinate transformation $\eta = x - x_k$, $\eta \in [0,h]$ is used. The quintic B-spline shape functions in equation (1.19) can be defined in term of η,

$$\begin{aligned} B_{k-2} &= \frac{1}{h^5}(h^5 - 5h^4\eta + 10h^3\eta^2 - 10h\eta^4 - \eta^5), \\ B_{k-1} &= \frac{1}{h^5}(26h^5 - 50h^4\eta + 20h^3\eta^2 + 20h^2\eta^3 - 20h\eta^4 + 5\eta^5), \\ B_k &= \frac{1}{h^5}(66h^5 - 60h^3\eta^2 + 30h\eta^4 - 10\eta^5), \\ B_{k+1} &= \frac{1}{h^5}(26h^5 + 50h^4\eta + 20h^3\eta^2 - 20h^2\eta^3 - 20h\eta^4 + 10\eta^5), \\ B_{k+2} &= \frac{1}{h^5}(h^5 + 5h^4\eta + 10h^3\eta^2 + 10h^2\eta^3 + 5h\eta^4 - 5\eta^5), \\ B_{k+3} &= \frac{1}{h^5}(\eta^5). \end{aligned} \tag{1.20}$$

Figure 1.7 display the graph of quintic B-spline function at $[x_{k-3}, x_{x+3}]$.

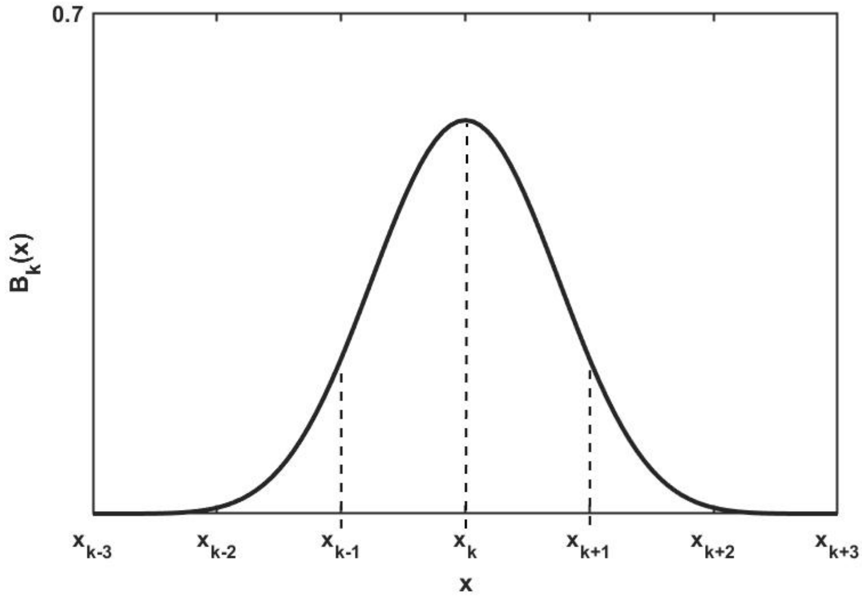

Figure 1.7: The quintic B-spline basis function

1.3 Finite Element Method and Splines

FEM is a well established method to approximate the solution to PDEs. Besides FEM, splines are also capable of approximating the solution of PDEs using piecewise polynomial approximation. A FEM with B-spline functions defines new approximate methods that has computational advantages of B-splines and finite elements. Splines have been used by various researcher's including Abd Hamid (2010) and Mat Zin (2016) to solve the PDEs.

Various researchers have worked on the solution of the PDEs by using collocation method based on splines. Dag et al. (2005) has studied the quadratic and cubic B-spline Galerkin (Cubic-BSGM) for numerical solution of the Burgers' equation. B-splines helped to resolve the unknowns in the equation. It was shown that the B-spline finite element is useful for the numerical solutions of Burgers' equation, particularly when continuity of the solutions was essential. A cubic B-spline based on collocation method for solving the one-dimensional Burgers' equation was presented by Idris et al. (2005). It was observed that a cubic B-spline collocation method is easy to implement and does not need any inner iteration to deal with the nonlinear term of the Burgers' equation and it can solve differential equation accurately.

Aksan (2006) used a quadratic B-spline FEM to approximate the numerical solution of Burgers' equation. The high accuracy of the proposed method was verified. Saka and Dag (2008) utilized a quartic B-splines Galerkin method for a numerical solution of the regularized long wave equation (RLW). In this paper the performance and the accuracy of the proposed method was observed. The method has a weakness because of its large matrix operations. It was noted that the quartic B-spline method is useful for finding the numerical solutions of the differential equation when higher continuity of the solutions occurs. Collocation method for the numerical solutions of one-dimensional heat and advection-diffusion equations have been set up based on cubic B-spline by Goh et al. (2012). In this paper the finite difference scheme is used for the time integration and the cubic spline function is used as an interpolation function. Zorsahin et al. (2016) solved the Burgers' equation using exponential B-spline Galerkin FEM. It was noted that the Galerkin method performed well compared with other methods for the numerical solution of the Burgers' equation. The B-spline Galerkin method and B-spline collocation method for differential equation have been setup by Dag et al. (2003), Karaagac et al. (2018) and Karakoc and Bhowmik (2019a). FEMs have been applied to the RLW equation using B-splines functions by Mittal and Rohila (2018) and Irk et al. (2019). B-spline can be used to solve nonlinear PDEs. This has been shown by various researchers.

From the previous work done by other researchers based on splines and FEM, most of them used collocation method or other techniques. Second, there are not many papers on NLS and CNLS equations together with B-splines and Galerkin method. There are also not many papers on Galerkin FEM together with splines. Thus, this gives the motivation for examining the use of B-splines Galerkin FEM for one-dimensional nonlinear PDEs. The main purpose of this thesis is to study the use of cubic B-splines, quartic B-splines and quintic B-splines associated with Galerkin method in the FEM. The spline method can be used to produce an accurate approximation at any point in the domain. As noticed from the previous work, B-spline can give much better result than standard FEM and FDM. The contributions of this thesis are the development of Galerkin FEM and the development of different B-spline methods to solve the NLS and CNLS equations.

1.4 Research Objective

The main goal of this thesis is to solve NLS and CNLS equations using Galerkin method based on B-splines function. The objectives of this research are:

1. to develop and apply Galerkin FEM based on cubic, quartic and quintic B-splines as shape and weight functions for solving NLS equation.
2. to develop and apply Galerkin FEM based on cubic, quartic and quintic B-splines as shape and weight functions for solving CNLS equation.
3. to investigate the stability and rate of convergence of the proposed scheme developed for NLS equation and CNLS equation.
4. to find the maximum error norms and conservative quantities for Cubic B-Spline

Galerkin scheme, quartic B-spline Galerkin scheme and quintic B-spline Galerkin scheme for NLS and CNLS equations.

5. to conduct a comparative study between the current results obtained by Galerkin method with cubic, quartic and quintic B-splines as basis functions and the published results.

1.5 Research Methodology

Different basis splines functions will be considered, namely cubic B-spline, quartic B-spline and quintic B-spline functions. These three functions will be developed. Figure 1.8 shows the flow chart of this research. Subsequently, to achieve objective 1 to

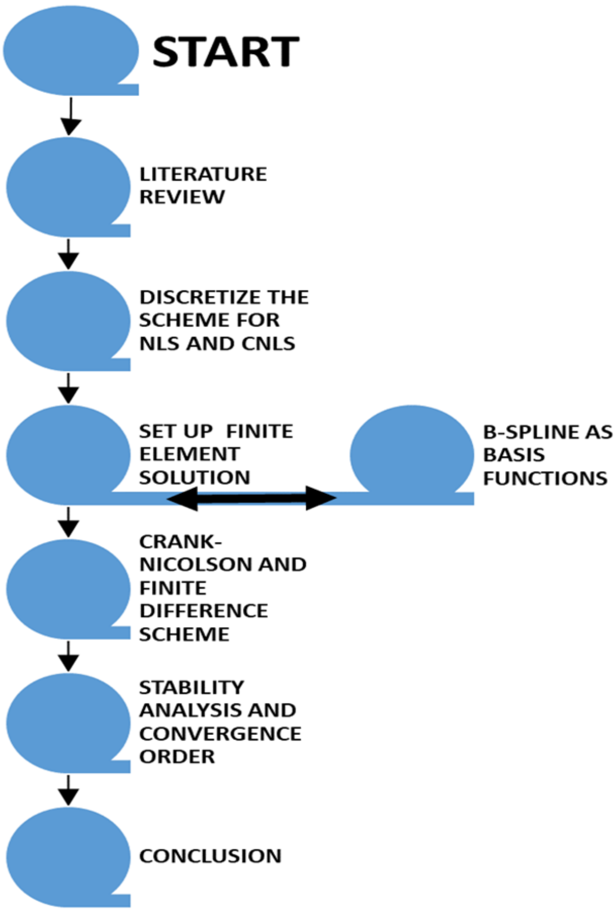

Figure 1.8: Flow chart of research methodology

4, we need to discretize the scheme for Galerkin FEM and Splines method. We set up the finite element solution using cubic B-spline, quartic B-spline and quintic B-spline as the element and weight function for NLS and CNLS equations. Crank–Nicolson scheme will be used for nodal parameters and a finite difference scheme will be applied for time derivative, the resulting system of the ODEs will be discretized to lead to system of algebraic equations. Galerkin B-spline FEM for the NLS and CNLS equations in which the nonlinear terms is locally linearized. The results will be compared with other published methods to demonstrate the capability of the proposed scheme. The stability will be analyzed by using von Neumann method.

1.6 Organization of Thesis

The contents of this thesis are discussed in nine chapters. Chapter 1 discusses the background and elements of splines. A brief literature review of B-splines with FEM, research objective and research methodology can be found in this chapter. Chapter 2 contains an in-depth literature review on the FEM for NLS equation and the B-splines to solve PDEs.

Chapters 3, 4 and 5 develop the Galerkin method based on cubic, quartic and quintic B-spline functions for one-dimensional cubic NLS equation, respectively. Chapters 6 and 7 develop the Galerkin method based on cubic and quartic B-spline functions for CNLS equation. A quintic B-spline Galerkin method (Quintic-BSGM) has been used to approximate the numerical solution of CNLS equation in Chapter 8. Lastly, the conclusion of the study and possibility of the future work have been discussed in Chapter 9.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, we review the literature on the B-spline methods for solving PDEs. This chapter has three sections. In the first section, we survey recent methods that used B-splines for the numerical solution of differential equations. The second and third sections present the literature on FEM and other recent numerical methods for solving NLS and CNLS equations.

2.2 B-Spline for Partial Differential Equation

Bickley (1968) proposed that the solution for a linear ODE can be produced with better results by using cubic splines compared to global high-order approximation. Fyfe (1969) worked on the same method that was used by Bickley (1968). Fyfe performed some analysis and found better approximation to the exact solution. In this study, the author concluded that the spline method is better than standard FDM with more accurate solution. By using splines method, the solution can be obtained at any point in the domain. Using the FDM, the solution is possible only at the specific knots.

Caglar et al. (1999) solved fifth order boundary value problems with sixth degree splines to get the unique solution and It was seen that the method is first order convergence. In this paper, two different problems were tested to asses the accuracy of the scheme. The authors concluded that the proposed method is a valid method and approximated the exact result accurately.

Dag and Ozer (2001) presented a numerical solution to study the RLW equation using cubic B-spline method with a focus on single solitary wave solution. A few years later, the same problem was solved by applying other types of B-splines, namely quartic B-spline and quintic B-spline (Saka and Dag, 2007; Saka et al., 2008), respectively. It was showed that the results were consistent with the best of existing B-splines solution of RLW equation. The disadvantages of quartic B-spline and quintic B-splines were the large number of matrix operation. However, both methods gave reasonable results and are candidates for the numerical solution of the differential equation when higher continuity of the solution is present.

In the paper by Wang et al. (2004), the one-dimensional parabolic PDE was solved based on a B-spline basis. Collocation method was employed for the spatial discretization and were defined by a high-order solver, adaptive in space and time as well. In this paper, a posteriori error estimate were considered at every time step. This paper was about a new mesh selection strategy for controlling the spatial error which were based on equidistributional principle. This study showed that the mesh adaptation technique is powerful and efficient for problems that have solutions with fast variation.

The study conducted by Caglar and Caglar (2006) was on a direct method based on B-splines for the numerical solution of boundary value problems. In this paper the authors modified the original differential equation at a singular point and applied the B-splines approximation for solving the boundary value problem. The authors have considered third-degree B-splines and three boundary value problems to demonstrate the efficiency of the proposed method. This study showed that the proposed method approximated the exact solution very well.

Jator and Sinkala (2007) proposed a high order B-spline method for applications to boundary value problems. The main objective was to calculate the numerical solution of the boundary value problems for dth order boundary value problem based on B-spline collocation technique with B-splines of order K. The authors compared the efficiency and accuracy of the proposed method with nodal and orthogonal collocation methods. They showed that the proposed method is more convenient than nodal or orthogonal collocation.

Numerical solution of Korteweg-de Vries (KDV) equation using Petrov-Galerkin method was presented by Ismail (2008). A cubic B-spline as a test function and linear B-splines as a basis functions have been used. Implicit midpoint method was applied for time integration. Ismail obtained a block non-linear Pentadiagonal system and solved it using Newton's approach. The stability of the proposed method was examined by the von Neumann's method and it was shown that the proposed scheme is unconditionally stable. The scheme was fourth order in space and second order in time. The author concluded that the resulting scheme is extremely precise and can simulate the KDV equation.

Lin et al. (2009) developed an efficient and practical numerical system for initial boundary value problems. The authors used the FEM based on linear B-spline basis functions to discretize the nonlinear PDE in space. The authors obtained a second order system which include only ordinary derivative. The authors showed that the coefficient matrix for the second order term is invertible. The numerical solutions were generated by using adaptive Runge-Kutta Verner method. With the help of this method, the authors showed that the numerical solution of the proposed problems are very close to the exact solution. Lin et al. (2009) concluded that this proposed scheme can be applied easily by using ODE solvers.

Kadalbajoo and Arora (2010) generated the numerical solution of the advection diffusion equation based on B-spline function. The authors conducted a Fourier accuracy and stability analysis for the given problem. Taylor Galerkin technique was used to get a good solution of differential equations over the standard Galerkin scheme. They have considered some examples to test the accuracy. The authors compared the obtained numerical results of high accuracy with standard polynomial interpolation. From this paper, it was concluded that the use of high order of B-spline function gives more accurate solutions.

Further, Abd Hamid et al. (2010) extended this work by using cubic trigonometric B-spline interpolation method to the numerical solution of two-point boundary value problems of order two. This is the extended form of cubic B-spline function. The trigonometric cubic interpolation method was examined on some problems and the results of this method was compared with cubic B-spline interpolation method. According to this study the trigonometric B-spline method approximated the solution with a slightly increased precision than cubic B-spline interpolation method. It is noted that the accuracy of the solutions depends on the types of the problem and the splines selected.

Cubic B-spline method for solving one-dimensional heat and wave equation was presented by Goh et al. (2011). In this study the standard FDM was used to discretize the time derivative and cubic B-spline was used to interpolate the solutions at each level of time. The truncation error and the stability of the proposed method were analyzed. This study concluded that the cubic B-spline method gives better solutions compared to the standard FDM for small space steps. Goh et al. (2011) concluded that the cubic B-spline method adjusted nicely and capable for the solution of one-dimensional equation accurately.

A new cubic trigonometric B-spline function for solving one-dimensional hyperbolic wave equation was presented by Abbas et al. (2014). The proposed scheme used as an interpolation function in the space dimension. In this study, the von Neumann stability method was applied for investigating the stability of the proposed method and it was shown that this method is unconditionally stable. The authors analyzed the accuracy of the proposed method by using several test problems. This study concluded that the proposed method approximated the results converges to its exact solution as compared to other finite difference scheme.

Ucar et al. (2015) have presented quadratic B-spline function with Galerkin FEM for the numerical solution of an improved Boussinesq type equation. In this study, they converted the original equation into coupled form and then applied fourth order Runge-Kutta technique. The authors considered two test problems to illustrate the efficiency and accuracy of the proposed method. They compared the result by using error norm L_2 and L_∞. According to the authors, the presented solutions are very close to the exact solution. The comparison showed that the B-spline method is very effective, efficient