MULTISTAGE APPROXIMATE ANALYTICAL SOLUTION OF NON-AUTONOMOUS NONLINEAR ORDINARY DIFFERENTIAL SYSTEMS

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by

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LIST OF ABBREVIATIONS

ADM	Adomian Decomposition Method
APM	Adomian Polynomial Method
DTM	Differential Transform Method
DEs	Differential Equations
EMsDTM	Efficient Multi-step Differential Transform Method
Eq	Equation
Eqs	Equations
FDTM	Fractional Differential Transform Method
HPM	Homotopy Perturbation Method
MDTM	Modified Differential Transform Method
MsDTM	Multi-stage Differential Transform Method
MSRM	Multistage Spectral Relaxation Method
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
RK4	The Fourth-Order Runge-Kutta
RKM	Runge-Kutta Method
SPIVPs	Singular Perturbation Initial Value Problems

LIST OF SYMBOLS

h	Step size
K	Order of approximation
N	The number of sub-interval
T	The end of interval time
ω	Angle in radians
f'	Derivative of f
$\int$	Integral
$\|\cdot\|$	Norm

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PENYELESEAIAN HAMPIRAN ANALITIK BERBILANG PERINGKAT BAGI SISTEM PERSAMAAN BIASA TAK LINEAR

ABSTRAK

Persamaan pembeza adalah alat penting untuk memodelkan banyak masalah dalam sains, kimia, fizik dan ekonomi. Sistem persamaan pembeza boleh dibahagikan kepada dua jenis: autonomi dan tak autonomi. Objektif penyelidikan ini adalah untuk membangunkan suatu teknik penyelesaian sistem pembezaan tak linear bukan autonomi yang berasaskan Transformasi Pembeza Piawai (DTM) dan Transformasi Pembeza Pelbagai Peringkat (MsDTM). Analisis perbandingan antara penyelesaian kaedah yang dibangunkan dan kaedah Runge-Kutta akan dibincangkan. Simulasi contoh-contoh berangka dijana menggunakan Perisian Maple 16 bagi menganalisis kaedah yang dicadangkan. Keputusan berangka menunjukkan bahawa kaedah Transformasi Pembeza Peringkat (MsDTM) memberikan anggaran yang tepat dibandingkan dengan kaedah berangka Runge-Kutta bagi penyelesaian sistem pembeza tak linear tidak autonomi. Di samping itu, batasan penumpuan pada selang masa yang besar dapat diatasi dengan teknik yang dicadangkan. Ini memastikan kebolehpercayaan dan kecekapan untuk skema ini. Selain itu, teknik yang dicadangkan akan menyediakan penyelesaian baru bagi pelbagai fenomena dalam kehidupan sebenar, serta mencari penyelesaian bagi pelbagai masalah sebenar yang dimodelkan sebagai sistem ODE tak linear bukan autonomi.

MULTISTAGE APPROXIMATE ANALYTICAL SOLUTION OF NON-AUTONOMOUS NONLINEAR ORDINARY DIFFERENTIAL SYSTEMS

ABSTRACT

Differential equations are an important tool for modeling many problems in science, chemistry, physics and economics. Differential equations system can be divided into two kinds: autonomous and non-autonomous system. The objective of this research is to develop a technique for solving non-autonomous nonlinear differential systems which is based on the standard Differential Transform Method (DTM) and Multistage Differential Transform method (MsDTM). A comparative analysis between the solutions that were obtained by the proposed methods and Runge-Kutta method will also be discussed. The numerical examples were simulated using the Maple 16 software to analyze the proposed methods. The numerical results showed that the MsDTM gives accurate approximation as compared to the Runge-Kutta numerical scheme for the solutions of non-autonomous nonlinear ordinary differential systems. In addition, the limitation of the convergence at large intervals is overcomed by the proposed technique. This ensures reliability and efficiency to the scheme. Moreover, the proposed technique will provide new solutions for many various phenomena in real life, and find solutions for many real problems which are modelled as non-autonomous systems of nonlinear ODEs.

CHAPTER 1

INTRODUCTION

1.1 Research Introduction

Differential Equations (DEs) have grown in parallel with the development of mathematics from the time of Newton until now (Grimshaw, 2017) due to their central role of the wide variety of their applications in different areas of physics, engineering, chemistry and other phenomena (El-Zahar, 2015; Gökdoğan, Merdan, & Yildirim, 2012c; Mohyud-Din, Usman, Wang, & Hamid, 2018). Ordinary Differential Equations (ODEs) can be classified into linear and nonlinear equations, and many linear and nonlinear equations can be divided into autonomous and non-autonomous groups. In this research, a new technique for solving non-autonomous nonlinear ODEs will be proposed.

1.2 Non-autonomous Nonlinear System of ODEs

A non-autonomous nonlinear system of ODEs can be defined in the following form:

$$\frac{du(t)}{dt} = f(u(t),t), \tag{1.1}$$

where $u \in R^n,\ t \in [a,b]$, with initial conditions,

$$u_1(t_0) = u_1(0), u_2(t_0) = u_2(0), ..., u_n(t_0) = u_n(0). \tag{1.2}$$

1.3 Motivation of Research

Efficient and reliable approximate techniques for solving initial value problems of non-autonomous nonlinear system had evolved in finding the exact solutions for some cases of DEs (Alvarez-Parrilla, Frías-Armenta, López-González, & Yee-Romero, 2012; Sun, Chen, & Nieto, 2012); nevertheless, when the DEs are nonlinear there are limited general techniques of solutions (Fatoorehchi & Abolghasemi, 2013; He, 2000). Therefore, many approximate solution methods have been developed to solve these equations (Ramdani, Meslem, & Candau, 2009; Tatari & Dehghan, 2009). Approximation technique can be divided into two categories, numerical and semi-analytical schemes. Numerical methods such as Runge-Kutta give numerical values of the solution at certain points only (Ahmadian, Salahshour, & Chan, 2015; Dehghan & Mohammadi, 2017; Hussain, Ismail, & Senu, 2016; Kalogiratou, Monovasilis, Psihoyios, & Simos, 2014; Yang & Shen, 2015), whereas semi-analytical methods such as Differential Transform Method (DTM), Homotopy Perturbation Method (HPM) (Ayati, Biazar, & Ebrahimi, 2014; Biazar, Asadi, & Salehi, 2015; Filobello-Nino et al., 2015; Liu, Adamu, Suleiman, & He, 2017; Rao & Begum, 2017) and Adomian Decomposition Method (ADM) (Hosseinzadeh, Jafari, Gholami, & Ganji, 2017; Rach, Wazwaz, & Duan, 2015; Wazwaz, Rach, & Duan, 2015) give approximate analytical solutions on sub intervals, which can be differentiated or integrated. Although analytical approximate methods have been widely applied to solve ODEs, many drawbacks of these methods were reported by several researchers. For example, some of methods require discretization or perturbation and linearization. For instance, DTM does not give a satisfactory approximation for a large time interval (Benhammouda & Vazquez-Leal, 2015; Khader & Megahed, 2014; Mohyud-Din et al., 2018; Nourifar, Sani, & Keyhani, 2017). These drawbacks are presented and discussed vastly in Chapter 2. Therefore, the motivation of this thesis is to overcome these disadvantages, by decreasing the size of computational work and make the computations easier and faster.

1.4 Problem Statement

The exact analytical solution of a non-autonomous system of ODEs for initial value problems is not available, especially for nonlinear systems because of its complexity. Therefore, many analytical approximate methods such as DTM and their modifications were utilized to provide analytical approximate solutions for this type of nonlinear systems of ODEs. However, this method still suffers from the drawbacks in convergence limitations which subsequently affect the accuracy and efficiency of the solutions (Nourifar et al., 2017; Odibat, Bertelle, Aziz-Alaoui, & Duchamp, 2010). Hence, the aim of this study is to develop a new technique which provides analytical approximate solutions for non-autonomous system of a large number of nonlinear ODEs, furthermore increasing convergence limits and thus reducing the number of arithmetic operations introduced by the standard method.

1.5 Research Objectives

The objectives of this study are:

1. To modify MsDTM for solving non-autonomous nonlinear ordinary differential system.

2. To perform convergence analysis to the proposed technique.

3. To compare the effectiveness and accuracy of the MsDTM with other analytical approximate methods.

1.6 Significance of The Study

The results of the study will be of great benefit to the following:

- The study will enhance new techniques for solving non-autonomous systems of high order nonlinear ODEs. Consequently, it will provide new solutions for many various phenomena in real life.
- Finding solutions for many real problems which are modelled as a non-autonomous system of nonlinear ODEs.
- The results of the study will help researchers to apply this method to include other areas, such as partial and fractional systems of ODEs in the future.

1.7 Methodology

In this research, standard DTM and MsDTM will be used to solve non-autonomous nonlinear ordinary differential systems. The two methods are investigated to analyze and understand the behavior of the approximate analytical solutions of non-autonomous nonlinear ordinary differential systems. The proposed solutions are then compared with exact solutions such as Runge-Kutta method. Convergence analysis is then performed by the proposed method. All computations are performed using Maple 16 software.

1.8 Thesis Outline

This thesis is divided into six chapters. Figure 1.1 presents the flow chart of the study. The literature reviews of previous studies are considered in Chapter 2. Chapter 3 offers the basic concepts and definitions, that are related to the study of non-autonomous systems of nonlinear ODEs. The chapter displays a brief depiction of the essential definitions of the approximate analytical methods that will be investigated in this study. Chapter 4 presents multistage analytical approximate solution of non-autonomous system of nonlinear ODEs. The numerical examples are tested in this chapter. The follows by Chapter 5 which considers the convergence analysis for the MsDTM method. The discussion of the numerical results and comparisons between DTM, MsDTM and RK4 are highlighted. Finally, Chapter 6 concludes the main results of the study and future work.

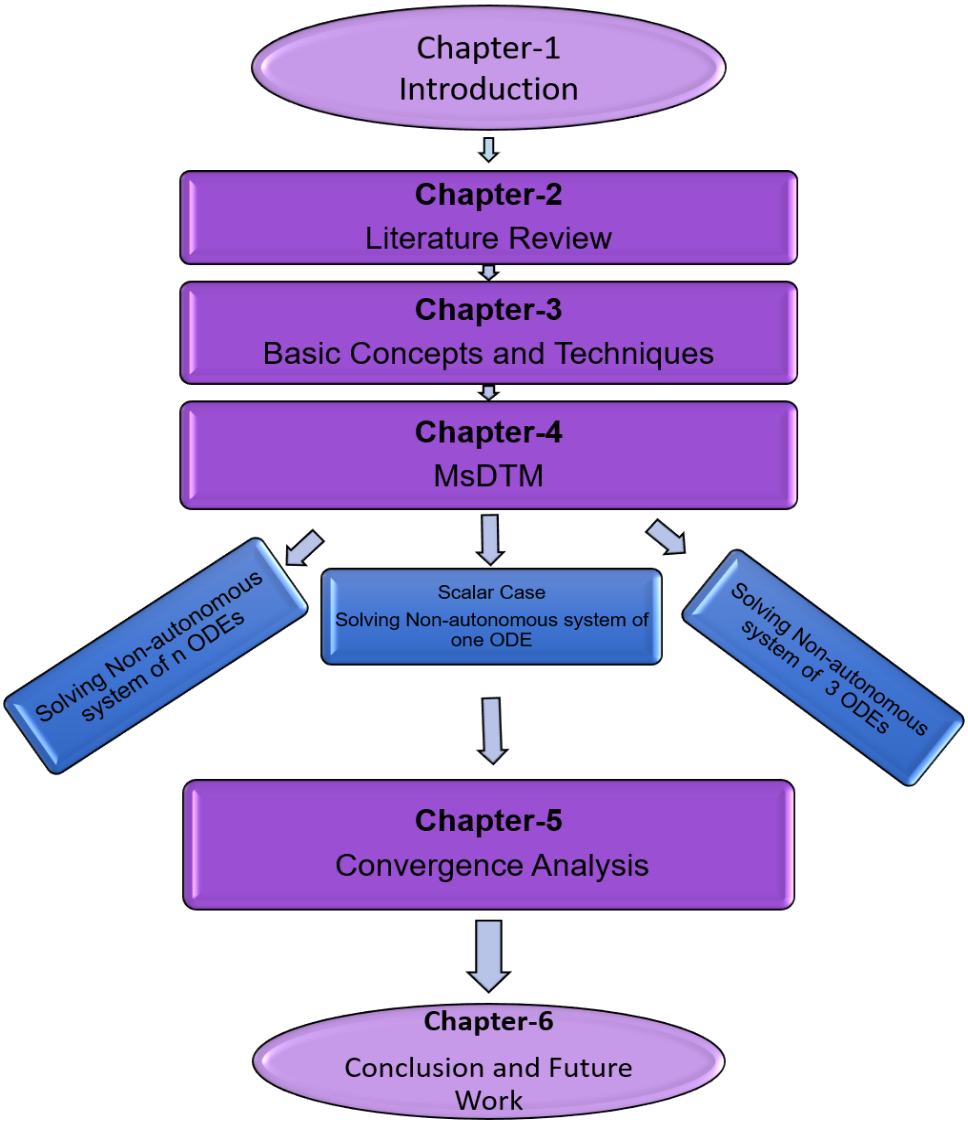

Figure 1.1: Flow Chart of the thesis

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Over the past two decades, linear and nonlinear system of differential equations have become a focus of interest for scientists and researchers. Many of the developed and presented analytical approximate and numerical methods are reviewed in this chapter. In this chapter, recent studies, that aim to find analytical approximate solutions for non-autonomous systems of nonlinear ODEs using standard DTM and MsDTM method, are briefly reviewed. Advantages and modifications of these methods are also presented in this chapter.

2.2 Differential Transform Method (DTM)

Differential Equations (DEs) are basic tools to model many phenomena in various fields related to real life problems such as physics, chemistry, engineering and other areas. In spite of that, continuous development of mathematics in parallel with the ongoing growing of DEs over past decades, DEs have retained their basic role. The technological development helps in understanding and analysis of DEs, which are consequently represented to real life problems. In the literature, approximation techniques are classified into two types, numerical methods and semi analytical methods. DTM is a numerical method that depends on Taylor expansion. This method builds an analytical solution in the form of a polynomial. The concept of DTM was first presented and applied to find the approximate solution of linear and nonlinear initial problems in electric circuit analysis by Zhou (1986). Two boundary value problems were solved by Chen and Liu (1998) using the two-dimension DTM.

Ayaz (2004) successfully applied DTM for analytical solution of linear and nonlinear systems of Partial Differential Equations (PDEs). Four examples with initial conditions from the literature are presented. Furthermore, similar systems are solved by ADM method. The results obtained by these two methods are compatible. The DTM can then be applied to several complex linear and nonlinear PDEs and system of PDEs and does not need discretization, linearization or perturbation.

Arikoglu and Ozkol (2007), applied Fractional Differential Transform method (FDTM) for solving Fractional DEs. The numerical solutions obtained by the proposed technique showed that the DTM is easy to handle, but still suffers from drawbacks such as convergence limitation over a large time interval.

The DTM was then employed by Kanth and Aruna (2009) to solve KdV and mKdV equations. The DTM method is applied directly without using bilinear forms, Wronskian, or inverse scattering method. The obtained results by DTM method is demonstrated to show the reliability of the proposed technique and gives a vast applicability to nonlinear evolution equations.

In addition, Biazar and Eslami (2010) have used the DTM method for solving quadratic Riccati differential equation. The numerical solutions obtained by the presented technique indicated that the proposed method is easy to apply. The size of computations was small as compared to numerical methods

Despite the aforesaid advantages, DTM does not provide accurate results for time interval. Based on DTM literature, the secheme was utilized by authors to find analytical approximate solution for specific types of differential systems, such as fractional and partial systems, but not to ordinary differential systems and this happened at the level of system of consisting of two or three equations at most. To overcome these disadvantages, a new algorithm will be presented and discussed in Chapter 3 to solve non-autonomous system of nonlinear ODEs.

2.3 Multistage Differential Transform Method (MsDTM)

MsDTM was developed by Odibat (2010) to solve non-Chaotic or Chaotic systems. The new algorithm of DTM was named multistage DTM, which increased the interval of convergence for series solution. The proposed technique is then applied to solve Lotka-Volterra, Chen and Lorenz systems. The numerical results obtained by the presented technique indicate that the method is effective in enlarging convergence time interval and to minimize size of computations. Moreover, it is worth to mention that the method is applicable to many other nonlinear models and promising as compared with other methods.

Gökdoğan, Merdan, and Yildirim (2012a) has developed the multi-step method into a new method that it was named adaptive multi-step DTM. The proposed secheme is applied to a number of nonlinear DEs, such as Duffing equation and Quadratic Raccati equation. The numerical results obtained by adaptive MsDTM show that a remarkable improvement in convergence interval for the series solution, and, the computations time is decreased.

The multistage with Spectral method is applied by Motsa, Dlamini, and Khumalo (2013). They named this scheme Multistage Spectral Relaxation Method (MSRM) and has been applied to solve famous chaotic systems such as Lorenz, Chen, Liu, Gensesio-Tesi and Arneodo-Coullet chaotic systems. The MSRM results were in agreement with Runge-Kutta and Adams-Bashforth-Moulton results.

Khader and Megahed (2014) successfully utilized the DTM for solving PDEs, which were transformed into a system of coupled nonlinear ODEs with appropriate boundary conditions for various physical parameters, such as flow and heat of a Newtonian fluid. The results obtained by the DTM method show the DTM in its general form gives a reasonable calculation, easy to use, and can carry out the differential equation in general form.

The adaptive MsDTM method is then employed by El-Zahar (2015) to find the analytical approximate solution for Singular Perturbation Initial Value Problems (SPIVPs). The proposed method was applied to four practical problems in various disciplines of science and engineering. The results obtained by presented method show that the accuracy of the method is independent of the perturbation parameter ε and the method works successfully in handling the SPIVPs with minimum size of computations and a wide interval of convergence.

Nourifar et al. (2017) presented a new modified version of DTM which, is named Efficient Multi-step Differential Transform Method (EMsDTM), to solve nonlinear differential equations of oscillatory systems such as Duffing (with and without damping), Van der Pol and Rayleigh equations. Based on the obtained results by EMsDTM, it is evident that EMsDTM solution in each example matches very well with Finite Difference Method (FDM) solution. Moreover, it is observed that the EMsDTM performance is significantly better than the MsDTM.

According to Ndii, Anggriani, and Supriatna (2018), they employed MsDTM method to solve system of nonlinear differential equation for dengue transmission mathematical model. the numerical results obtained by the proposed method show that the analytical and numerical solution of DTM are in good agreement with the RK4 method. Furthermore, the DTM allows to write analytical solutions of dengue mathematical model, which is not easily derived. It is worth to mention that the method can be alternative approach for solving nonlinear system including disease transmission models.

Based on a review of previous MsDTM and DTM literature, the following issues were observed. First, the MsDTM and DTM were widely implemented in solving several types of DEs. However, the concentration was only on specific types of fractional as well as partial systems such as Lorenz, Chen, Liu and Gensio system. In most cases the differential equations were autonomous nonlinear with initial conditions. Second, the MsDTM were used by several researchers (Gökdoğan et al., 2012c; Odibat et al., 2010) to enlarge the convergence interval of analytical approximate solutions of linear and nonlinear systems. Non-autonomous nonlinear of ODEs were not considered. In this context, the present study will highlight non-autonomous nonlinear systems of ODEs with initial conditions. Therefore, a new algorithm will be presented and discussed in Chapter 4 and Chapter 5 to solve non-autonomous system of a large number of nonlinear ODES with initial conditions.

2.4 Summary

In this chapter, the literature on DTM and MsDTM methods with their modifications, which were implemented to solve many types of differential equations and systems are reviewed. The current study will focus on the properties of DTM and its modifications to solve non-autonomous systems of ODEs. Depending on a review of previous searches in this chapter, specific issues are handled as follows:

- Many researchers presented the MsDTM to obtain solution for different types of fractional and partial differential equations with initial conditions as a system of nonlinear DEs. In this light, a new algorithm will be presented and discussed in Chapter 4 to solve non-autonomous system of a large number of nonlinear ODEs with initial value problems.
- MsDTM was used by many authors to solve special and famous systems of DEs such as chaotic systems (linear and nonlinear). To overcome this disadvantage in solving special systems, a new algorithm will be used in Chapter 4 to solve non-autonomous system of nonlinear ODEs, which will be more general technique.
- In most of cases, which many researchers studied, focus was given on solving system of two or three ODES (Gökdoğan et al., 2012c; Thongmoon & Pusjuso, 2010). Therefore, a new technique ,which is presented in chapter 4, will solve

several non-autonomous system of nonlinear ODES.

CHAPTER 3

BASIC CONCEPTS AND TECHNIQUES

3.1 Introduction

Numerical and semi-analytical methods are both used to solve non-autonomous nonlinear ordinary differential systems. Some researchers could benefit from the differences between numerical and semi-analytical methods for their goals and investigate these variations in their researches. In this chapter, some basic concepts between numerical and semi-analytical methods, autonomous and non-autonomous systems, and linear and nonlinear ordinary differential systems will be briefly explained. In addition, basic concepts of the DTM and the MsDTM will also be presented.

3.2 Basic Concepts

There are some basic definitions related to ODEs that should be stated.

3.2.1 Analytical and Numerical Methods

Many approximate solution methods have been developed to solve nonlinear ODEs (Chang & Chang, 2008; Duan & Rach, 2015; El-Zahar, 2013; Gökdoğan et al., 2012a). From these techniques, numerical methods and semi-analytical methods are identified. Numerical methods, such as Runge-Kutta, give numerical values of the solution at certain points only, whereas semi-analytical methods such as DTM, HPM and ADM give approximate analytical solutions on sub intervals, which can be differentiated or integrated (Ayati et al., 2014; Biazar et al., 2015; Filobello-Nino et al., 2015; Hosseinzadeh et al., 2017; Liu et al., 2017; Rach et al., 2015; Rao & Begum, 2017; Wazwaz et al., 2015).

3.2.2 Autonomous and Non-Autonomous Ordinary Differential Equations

An autonomous system or an autonomous differential equation is a system of ODEs which does not explicitly depend on the independent variable. Non-autonomous system is a system of ODEs which explicitly depends on the independent variable (Chicone, 2006). The autonomous system is represented by the following form (Kloeden & Schmalfuß, 1997):

$$\frac{du(t)}{dt} = f(u(t)). \tag{3.1}$$

The non-autonomous system is represented by the following form (Kloeden & Rasmussen, 2011):

$$\frac{du(t)}{dt} = f(u(t),t), \tag{3.2}$$

where $u \in R^n$.

An example for non-autonomous nonlinear system of three ordinary differential equations is given by (Biazar, Babolian, & Islam, 2004):

$$\begin{cases} \frac{du_1}{dt} = 2u_2^2, \\ \frac{du_2}{dt} = e^{-t}u_1, \\ \frac{du_3}{dt} = u_2 + u_3. \end{cases} \tag{3.3}$$

An example of an autonomous linear system of three ordinary differential equations is given by (He, 2000).

$$\begin{cases} \frac{du_1}{dt} = -u_1 - 8u_2, \\ \\ \frac{du_2}{dt} = 8u_1 - u_2. \end{cases} \tag{3.4}$$

The difference between equations (3.3) and (3.4) on the terms of the independent variable where they can be shown explicitly in the former but it is not shown explicitly in the latter. Furthermore, non-autonomous system represents a general form that it includes autonomous system. There are several applications for autonomous and non-autonomous as real problems in world such as Fabrikant system which is used to model waves in non-equilibrium substances (Kolebaje, Ojo, Ojo, & Omoliki, 2014), Hantavirus infection model (Gökdoğan et al., 2012c) and others.

3.2.3 Linear and Non-Linear Ordinary Differential Equations

A linear differential equation has only linear terms of unknown or dependent variable and its derivatives. It has no terms with the dependent variable of power higher than one or less than one and does not contain any multiple of its derivatives. It can not have nonlinear function such as trigonometric functions, exponential functions, and logarithmic functions with respect to the dependent variable. (Chicone, 2006).

In addition, nonlinear differential equation has nonlinear terms of unknown variable and its derivatives (Chicone, 2006).

An example for nonlinear system of three ODEs is shown as (Biazar et al., 2004):

$$\begin{cases} \frac{du_1}{dt} = 2u_2^2, \\ \frac{du_2}{dt} = e^{-t}u_1, \\ \frac{du_3}{dt} = u_2 + u_3. \end{cases} \tag{3.5}$$

An example of linear system of ODEs is shown below (Biazar et al., 2004):

$$\begin{cases} \frac{du_1}{dt} = u_3 - cos\, t, \\ \frac{du_2}{dt} = u_3 - e^t, \\ \frac{du_3}{dt} = u_1 - u_2. \end{cases} \tag{3.6}$$

3.2.4 Existence and Uniqueness of Solutions

In this section, the existence and uniqueness of solutions to a first order of nonlinear $n \times n$ system of differential equations is investigated given by

$$\frac{dx}{dt} = F(t,x), \tag{3.7}$$

and $x(t_0) = x_0$. Assuming F is bounded and continuous on $I \times \Omega$, where I is an open interval about t_0 and Ω is an open subset of R^n, containing x_0. Assume that F satisfies Lipschitz condition in x (Taylor, 2011):

$$\left\| F(t,x) - F(t,y) \right\| \leq L \left\| x - y \right\|, \tag{3.8}$$

for all $t \in I$, x, $y \in \Omega$, with $L \in (0,\infty)$ that is called Lipschitz's constant. Such an estimate holds if Ω is convex and F is C^1 in x and satisfies

$$\left\| D_x F(t,x) \right\| \leq L, \tag{3.9}$$

for all $t \in I$, $x \in \Omega$.

Proposition:

Assume $F : I \times \Omega \to R^n$ is bounded, continuous and satisfies the Lipschitz condition (3.8). Let $x_0 \in \Omega$, then there exists $T_0 > 0$ and a unique C^1 solution to (3.7) for $|t - t_0| < T_0$. The first step in proving this is to rewrite equation (3.7) as an integral equation (Taylor, 2011) as shown:

$$x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))ds. \tag{3.10}$$

The equivalence of equations (3.7) and (3.10) follows from the fundamental theorem of calculus (Taylor, 2011). It suffices to find a continuous solution x to (3.10) on $[t_0 - T_0, t_0 + T_0]$, since the right side of (3.10) will be C^1 in t. A technique known as Picard iteration will be applied to construct a solution to equation (3.10). By setting $x_0(t) = x_0$ and define $x_n(t)$ inductively then

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} F(s, x_n(s))ds. \tag{3.11}$$

It can be shown that equation (3.11) converges uniformly to a solution in equation (3.10), for $|t - t_0| \leq T_0$, if T_0 is taken small enough. To get this, some hypotheses has been made as shown in equations (3.8) and (3.9). Assume

$$\overline{B_R(x_0)} = \{x \in R^n : \left\| x - x_0 \right\| \leq R\} \subset \Omega, \tag{3.12}$$

where $\overline{B_R(x_0)}$ is an open ball around x_o of radius R, $M \in (0, \infty) < L$ and

$$\left\| F(s,x) \right\| \leq M, \qquad \forall s \in I, \quad x \in \overline{B_R(x_0)}. \tag{3.13}$$

Clearly, $x_0(t) = x_0$ takes values in $\overline{B_R(x_0)}$ for all t. Suppose that $x_n(t)$ has been constructed, by taking values in $\overline{B_R(x_0)}$, and $x_{x+1}(t)$ is defined by equation (3.11), then have

$$\left\| x_{n+1} - x_0 \right\| \leq \int_{t_0}^{t} \left\| F(s, x_n(s)) \right\| \leq M|t - t_0|, \tag{3.14}$$

so $x_{n+1}(t)$ also takes values in $\overline{B_R(x_0)}$ provided that $|t - t_0| \leq T_0$ and

$$T_0 \leq \frac{R}{M}. \tag{3.15}$$

As long as equation (3.15) holds and $[t_0 - T_0, t_0 + T_0] \subset I$, an infinite sequence of x_n, related to equation (3.11) will be obtained. By producing one more constraint on T_0, thus is guaranteed that convergence is achieved. Note that, for $n \geq 1$,

$$\left\|x_{n+1}-x_n\right\| = \left\|\int_{t_0}^{t}[F(s,x_n(s))-F(s,x_{n-1}(s))]ds\right\| \tag{3.16}$$

$$\leq \int_{t_0}^{t}\left\|F(s,x_n(s))-F(s,x_{n-1}(s))\right\|ds \tag{3.17}$$

$$\leq L\int_{t_0}^{t}\left\|x_n(s)-x_{n-1}(s)\right\|ds, \tag{3.18}$$

the last inequality by (3.8). Hence

$$max_{|t-t_0|\leq T_0}\left\|x_{n+1}(t)-x_n(t)\right\| \leq LT_0 \quad max_{|t-t_0|\leq T_0}\left\|x_n(s)-x_{n-1}(s)\right\|. \tag{3.19}$$

The additional constraint on T_0 is given by

$$T_0 \leq \frac{1}{2L}. \tag{3.20}$$

Noting that

$$max_{|t-t_0|\leq T_0}\left\|x_1(t)-x_0(t)\right\| \leq R, \tag{3.21}$$

it shows that

$$max_{|t-t_0|\leq T_0}\left\|x_{n+1}(t)-x_n(t)\right\| \leq 2^{-n}R. \tag{3.22}$$

Consequently, the infinite series are given by

$$x(t) = x_0 + \sum_{n=0}^{\infty} (x_{n+1}(t) - x_n(t)) \tag{3.23}$$

which is absolute and uniformly convergent for $|t - t_0| \leq T_0$, with a continuous sum, satisfying

$$max_{|t-t_0|\leq T_0} \left\| x(t) - x_n(t) \right\| \leq 2^{1-n} R. \tag{3.24}$$

It follows that

$$\int_{t_0}^{t} F(s, x_n(t)) ds \to \int_{t_0}^{t} F(s, x(s)) ds, \tag{3.25}$$

then equation (3.10) will follow equation (3.11) in the limit $n \to \infty$. To complete the proof of the proposition, condition of uniqueness is being established. Suppose $y(t)$ also satisfies equation (3.10) for $|t - t_0| \leq T_0$, then

$$\left\| x(t) - y(t) \right\| = \left\| \int_{t_0}^{t} [F(s, x(s)) - F(s, y(s))] ds \right\| \tag{3.26}$$

$$\leq \int_{t_0}^{t} \left\| F(s, x(s)) - F(s, y(s)) \right\| ds \tag{3.27}$$

$$\leq L \int_{t_0}^{t} \left\| x(s) - y(s) \right\| ds, \tag{3.28}$$

and hence

$$max_{|t-t_0|\leq T_0}\left\|x(t)-y(t)\right\| \leq T_0 L, \quad max_{|s-t_0|\leq T_0}\left\|x(s)-y(s)\right\|. \tag{3.29}$$

As long as (3.15) holds, $T_0 L \leq \frac{1}{2}$, thus equation (3.29) implies $max_{|t-t_0|\leq T_0}\left\|x(t)-y(t)\right\| = 0$, which gives the asserted uniqueness.

3.3 Differential Transform Method (DTM)

This section will present a review on DTM (Arikoglu & Ozkol, 2007; Ayaz, 2004; Benhammouda & Vazquez-Leal, 2015; Biazar & Eslami, 2010; Chang & Chang, 2008; Chen & Liu, 1998; El-Zahar, 2013; Gökdoğan et al., 2012a; Kangalgil & Ayaz, 2009; Kanth & Aruna, 2008, 2009; Lal & Ahlawat, 2015; Odibat, 2010; Zhou, 1986) to find solutions of the ODEs.

Definition 1 (Arikoglu & Ozkol, 2007; Zhou, 1986) If a function $u(t)$ is analytical with respect to t in the domain of interest, then

$$U(k)=\frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}, \tag{3.30}$$

is the transformed function of $u(t)$.

Definition 2 (Arikoglu & Ozkol, 2007; Zhou, 1986) The differential inverse transforms of the *set* $\{U(k)\}_{k=0}^{n}$ is defined by

$$u(t)=\sum_{k=0}^{\infty} U(k)(t-t_0)^k. \tag{3.31}$$

Substituting equation (3.30) into equation (3.31), it can be deduced that

$$u(t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{d^{k}u(t)}{dt^{k}}\right]_{t=t_0}(t-t_0)^{\mathrm{k}}. \tag{3.32}$$

From Definition (1) and Definition (2), it shows that the concept of the DTM is obtained from the power series expansion. To illustrate the application of the proposed DTM for solving systems of ordinary differential equations, consider the following nonlinear system

$$\frac{du(t)}{dt}=f(u(t),t),\ \ t\geq t_0, \tag{3.33}$$

where $f(u(t),t)$ is a nonlinear smooth function and initial condition

$$u(t_0)=u_0. \tag{3.34}$$

By using DTM, the solution of equation (3.33) can be written as

$$u(t)=\sum_{k=0}^{\infty}U(k)(t-t_0)^{k}, \tag{3.35}$$

where $U(0),U(1),U(2),\ldots$ are unknowns of equation (3.35) determined by the DTM. Applying the DTM to the initial conditions (3.34) and (3.33) respectively, the transformed initial conditions is obtained

$$U(0)=u_0, \tag{3.36}$$

with the recursion system

$$(1+k)\,U(k+1)=F\left(U(0),\ldots,U(k),k\right),\quad k=0,1,2,\ldots, \tag{3.37}$$

where $F(U(0),\ldots,U(k),k)$ is the differential of $f(u(t),t)$.

By using equations (3.36) and (3.37), the unknown $U(k), k=0,1,2,\ldots$ can be determined. The differential inverse transformation of the values $\{U(k)\}_{k=0}^{m}$ gives the approximate solution

$$u(t)=\sum_{k=0}^{m}U(k)\,(t-t_0)^k, \tag{3.38}$$

where m is the approximation order of the solution. Equation (3.35) gives the exact solution for problem (3.33)-(3.34).

If $U(k)$ and $V(k)$ are the differential transforms of $u(t)$ and $v(t)$ respectively, then the main operations of the DTM are shown in Table (3.1) (Benhammouda & Vazquez-Leal, 2015).

Table 3.1: Main Operation of DTM.

Function	Differential transform
$\alpha u(t)\pm\beta v(t)$	$\alpha U(k)\pm\beta V(k)$
$u(t)\,v(t)$	$\sum_{r=0}^{k}U(r)\,V(k-r)$
$u(t)\,v(t)\,w(t)$	$\sum_{r=0}^{k}\sum_{l=0}^{r}U(l)\,V(r-l)\,W(k-r)$
$\frac{d^n}{dt^n}[u(t)]$	$(k+1)\ldots(k+n)\,U(k+n)$
$e^{\lambda t}$	$\frac{\lambda e^{\lambda t_0}}{k!}$
$\sin(\omega t)$	$\frac{w^k}{k!}\sin\left(\omega t_0+\frac{\pi k}{2}\right)$
$\cos(\omega t)$	$\frac{w^k}{k!}\cos\left(\omega t_0+\frac{\pi k}{2}\right)$

By applying the differential transform to initial conditions (3.34) and (3.33) the recursion system for unknowns $U(0), U(1), U(2),\ldots$, the solution series can be obtained