GENERALIZED LOGARITHMIC PENALTY FUNCTION APPROACH FOR INVEX NONLINEAR CONSTRAINED OPTIMIZATION AND ITS APPLICATION

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by

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CB	Courant-Beltrami
CG	Conjugate Gradient
CP	Constrained Problem
CQ	Constraint Qualification
GLP	Generalized Logarithmic Penalty Function
KKT	Karush-Kuhn-Tucker
LPF	Logarithmic Penalty Function
MCB	Modified Courant-Beltrami
MOPP	Multi-Objective Programming Problem
NLP	Non-Linear Programming
NPSOL	Nonlinear Programming System Optimization Laboratory
SD	Steepest Descent
SQP	Sequential Quadratic Programming
S*l*QP	Sequential Penalty Quadratical Programming
SUMT	Sequential Unconstrained Minimization Technique

LIST OF SYMBOLS

lim	limit
θ	angle in radians
ln	natural logarithm
F	set of feasible solution
$\Re$	set of real numbers
J	set of indices for equality constraints
J^+	set of indices of positive multiplier associated with equality constraints
J^-	set of indices of negative multiplier associated with equality constraints
I	set of indices for inequality constraints
A_k^*	sequence set of feasible solution
S_k	sequence of numbers
N	natural number
$\emptyset$	null set
ξ	lagrange multiplier associated with inequality constraints
μ	lagrange multiplier associated with equality constraints
α	step size
α_k	step size at *k*-th iteration
d_k	search direction
$f(x_k)$	objective function at iterate point x_k
g_k	gradient of $f(x_k)^T$
$q(x)$	quadratic function
β_k	CG coefficient

A	$n \times n$ symmetric positive definite matrix
$r(x)$	residual of the linear system
r_k	residual of the linear system at iterate k
ε	epsilon
$\Re^n$	real number space in n dimension
$\nabla^2 P$	Hessian of the penalized function
$\Re^m_+$	positive orthant of real number space in m dimension
G_{k+1}	approximate Hessian of f at iterate point x_{k+1}
∇^2	Hessian operator
∇	gradient operator
s_k	difference between two iterate points x_k and x_{k+1}
y_k	difference between g_k and g_{k+1}
H_{k+1}	inverse Hessian approximate
m_{k+1}	model function at x_{k+1}
$p(x)$	penalty function
$P_c(x)$	penalized function with respect to parameter c
c	penalty parameter
$\Psi(x)$	composite function representing proposed LPF
∇P	gradient of the penalized function
h_j	equality constraint with index j
g_i	inequality constraint with index i
$f(x)$	objective function
$\phi(x)$	composite function representing proposed MCB

c_k	parameter c_k to be updated at each iteration
β	scaler used for updating c_k
J_1	set of indexes for the equality constraint's feasible point
J_2	set of indexes for the equality constraint's infeasible point
I_1	set of indexes for the inequality constraint's feasible point
I_2	set of indexes for the inequality constraint's infeasible point
$\tilde{x}$	feasible point
$\overline{x}$	contradictory feasible point
$\tilde{\mu}$	multiplier associated with equality constraint at feasible point
$\tilde{\xi}$	multiplier associated with inequality constraint at feasible point
$\tilde{\mu}_j$	multiplier associated with *j*-th equality constraint at feasible point
$\tilde{\xi}_i$	multiplier associated with *i*-th inequality constraint at feasible point
X	subset of real number $\Re$
G	duality gap
P	primal optimal
D	dual optimal
L	Lagrangian function
$d(x)$	gradient projection vector
η	vector-value function
σ	Nie fixed penalty parameter
$\gamma(x)$	filter based approach penalty
Ψ	dual function
R_M	large real number
M	very large real number

PENDEKATAN FUNGSI PENALTI LOGARITMA TERITLAK BAGI KEKANGAN PENGOPTIMUMAN TERHADAP INVEX TIDAK LINEAR DAN APLIKASINYA

ABSTRAK

Pendekatan fungsi penalti digunakan secara meluas dalam bidang pengaturcaraan matematik, dan ia berfungsi sebagai alternatif kepada pendekatan pengoptimuman bukan linear tanpa kekangan konvensional. Dalam usaha untuk membuat kemajuan secara teoritis dan kemajuan secara praktikal, fungsi penalti yang berterusan diutarakan mengatasi masalah pengoptimuman yang tidak linear; ia disebut sebagai Kaedah Penalti Logaritmik (LPF). Dari sudut pandangan teori, penumpuannya telah diperiksa menggunakan teorem-teorem dan lemma-lemma yang relevan. Selanjutnya, fungsi penalti Courant-Beltrami (MCB) diubahsuai kepada bentuk logaritma untuk menjadikan LPF yang dicadangkan lebih umum. La dipanggil Fungsi Penalti Logaritma Umum (GLP). Bentuk umum (GLP) yang direkabentuk secara kate-gori untuk masalah pengoptimuman invex. Kesetaraan antara set penyelesaian yang optimum dalam masalah asal dan masalah berpenalti yang sepadan adalah mantap. Pengganda-pengganda Karush-Khun-Tucker (KKT) bagi kekangan jenis kesamaan danketidaksamaan diperolehi. Pengganda-pengganda tersebut digunakan untuk meneliti jurang dualiti sifar yang menggunakan teorem dualiti yang lemah dan kuat. Ini mem-bawa kepada penubuhan kriteria titik pelana. Dari sudut simulasi berangka, beber-apa masalah diselesaikan menggunakan algoritma quasi-Newton dengan fungsi rutin fminunc untuk mengesahkan penumpuan menggunakan beberapa contoh ujian. Didapati bahawa

LPF yang dicadangkan dapat mengatasi masalah yang mempunyai ciri-ciri yang tidak teratur kerana sifatnya yang boleh terbezakan. LPF yang dicadan-gkan tidak mempunyai sekatan untuk sama ada masalah dengan kesamaan atau ketidaksamaan. Selain itu, keserasian dengan masalah yang mempunyai klasifikasi yang berbeza adalah aspek penting lain untuk perlu diambil kira. Secara kesimpulan, LPF yang dicadangkan telah diuji pada masalah pengoptimuman proses kimia. Berdasarkan perbandingan hasil dengan beberapa kaedah terdahulu, Fungsi Penalti Logaritma Teritlak (GLP) ternyata lebih ekonomik untuk pembuat keputusan dalam industri petroleum.

GENERALIZED LOGARITHMIC PENALTY FUNCTION APPROACH FOR INVEX NONLINEAR CONSTRAINED OPTIMIZATION AND ITS APPLICATION

ABSTRACT

A penalty function approach is used widely in the field of mathematical programming, and it served as an alternative to conventional non-linear constrained optimization approach. In a quest to make an advancement theoretically and progress practically, we proposed a continuously differentiable penalty function to handle the nonlinear constrained optimization problem; it is called logarithmic penalty function (LPF) method. From the theoretical viewpoint, its convergence has been examined using the relevant theorems and lemmas. Further, we modified a Courant-Beltrami (MCB) penalty function into logarithmic form to make the proposed LPF more general. It is called Generalized Logarithmic Penalty Function (GLP). The general form of GLP was constructed categorically for the invex optimization problem.The equivalence between the sets of optimal solutions in the original problem and its correspondi-ng penalized problem is well-established. The Karush-Khun-Tucker (KKT) multipliers associated with both (equality and inequality) constraints were de-rived. Those multipliers were used to examine a zero duality gap employing weak and strong duality theorems; this leads to the establishment of saddle point criteria. From the numerical point of view, some problems were solved via quasi-newton's algorithmwith *f-minunc* routine function to validate its convergence utilizing some test examples. It was observed that the proposed LPF could handle the problems possessing irregular

features due to its differentiability nature. The proposed LPF has no restriction to either problem with equality or inequality constraints. Besides, its compatibility with problems bearing different classifications is another crucial aspect to reckon. Conclusively, the proposed LPF was tested on the chemical process optimization problem. Based on the comparisons of the results with some of the previous methods, Generalized Logarithmic Penalty Function (GLP) turns out to be more economical to decision-makers in the petroleum industries.

CHAPTER 1

INTRODUCTION

1.1 General Introduction

Optimization or mathematical programming has become a large research area with many branches, as described in Figure (1.1), it is the process or methodology of selecting the best of several possible decisions in a heterogenous real-life environment. Decision is a careful act of selection, by the mind, for choosing the favorite choice from the set of contend alternatives in expectation, anticipation, or belief that the selected alternative will accomplish the desire goals. It can also be described as a commitment to certain actions or in-actions.

Planning a decision mathematically is term as mathematical programming. In the study of rational decision making, mathematical programming constitutes a pivotal element in providing a sound theoretical basis for understanding managerial decision making. Generally, mathematical programming is a body of established algorithms that pursue the optimal value of an objective function without violating a set of constraints; it comprises of an objective function and the constraint functions, if all the functions constituting mathematical programming are linear, then we have a linear programming problem, but in the presence of at least a non-linear objective or one non-linear constraint function then, it is said to be a non-linear programming problem.

Mathematical programming is conventionally a single-value objective function f with n real variables $x_1, \ldots.., x_n$. The aim is to minimize (or maximize) the objective

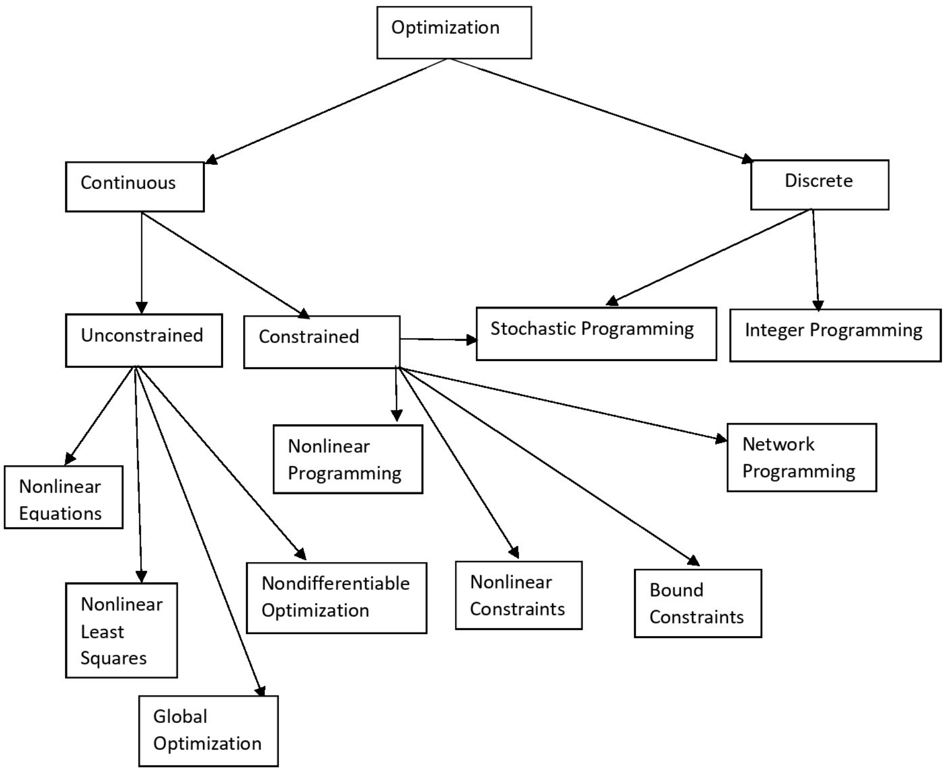

Figure 1.1: Optimization Chart.

function subject to a finite number of constraints in the form of equalities, inequalities or both. The general form of the problem is as follows: -

$$\begin{aligned} &\textit{minimize} \quad f(x) &(1.1)\\ &\text{subject to} \quad h_j(x) = 0, j \in J = \{1,2,........,s\},\\ &\qquad g_i(x) \leq 0, i \in I = \{1,2,.......,m\},\\ &\qquad x \in X, \end{aligned}$$

where $f : X \to \Re$ and $h_j : X \to \Re, j \in J, g_i : X \to \Re, i \in I$, are differentiable functions on a nonempty open subset of real number X.

The problem (1.1) is considered to be a nonlinear programming (i.e. at least one of the functions $f(x)$, $h_j(x)$, $g_i(x)$ is non-linear), and it possess various practical applications;

these includes but not limited to engineering, decision theory, economics, management science and in all other sciences (physical and natural). There exist a common teamwork between a theorist and practitioners (engineers), by which one can not stand without the help of the other.

Theorists are responsible for designing and applying the different approaches for solving the considered problem and suggests the more efficient and viable method. Practitioners are always focusing on the output and suggesting the areas that need theoretical advancement in their quest for problem-solving effectiveness. Theory or practical can never be thoroughly understood in segregation. Relatively, each of the two needs one another to achieve their common desire in the general field of mathematical programming.

The area has continued to receive much concern, and it is growing naturally in various directions, many researchers are working tirelessly to scrutinize different methods that might be advantageous and more powerful in contrast to the current ones in the literature. In recent years, an approach to solve a nonlinear optimization problem such as (1.1) is not limited to the conventional strategies. Several other methods come into existence in the last few decades. One of the attractive approaches is the use of the penalty function $P_c(x)$, and it is usually defined in terms of the constraints functions h_j and g_i as follows

$$P_c(x) = f(x) + cp(x), \tag{1.2}$$

where $p(x) = \psi(h_j(x)) + \phi(g_i(x))$, and both $\psi(h_j(x))$ and $\phi(g_i(x))$ are separate penalty functions for equality and inequality constraints respectively.

There are collections of penalty functions in the literature; the usual strategy is to transform a constrained non-linear optimization problem into a single or a sequence of an unconstrained problem. There are subclasses of penalty functions, which can also be subdivided into two main classes:

- Continuously differentiable exact penalty function.
- Non-differentiable exact penalty function.

The absolute value penalty function is one of the most popular non-differentiable penalty functions. The penalty term, which reflects the constraints set, was added to the merit function, which is the objective function of the original problem. The penalty term was constructed by multiplying penalty with a positive real number called penalty parameter c as in equation (1.2), and this penalty parameter is adjusted until the convergence is attained.

1.2 Problem Statement

In the last six decades, the penalty function approach for solving constrained optimization was introduced by Zangwill (1967) and Eremin (1967), the penalty function was popularly known as an absolute value penalty function that belongs to the class of non-differentiable.

The purpose of the penalty function is to transform the problem (1.1) from a constrained optimization problem into a single unconstrained problem called a penalized problem, so that the solution of the penalized problem coincides with the original problem, or at least approximates the optimal solution of the original problem. Primarily, the concentration of researchers in the literature was dedicated to ensuring that a local optimum of the original optimization problem is coinciding with local minimizer of the penalized optimization problem. The exact penalty function introduced by Zangwill (1967) for equality constrained was

$$p(x) = \sum_{i=1}^{m} g_i^{+}(x) + \sum_{j=1}^{s} \mid h_j(x) \mid . \tag{1.3}$$

However, the resulting unconstrained problem (1.3) is non-differentiable. Luenberger (1973) constructed the following unconstrained problem which works in $\Re^{n+m}$ space.

$$minimize \mid \bigtriangledown f(x) - \lambda^T \bigtriangledown h(x) \mid^2 + \mid h(x) \mid^2, \tag{1.4}$$

where λ^T is the transpose of Lagrange multiplier vector.

The disadvantages observed in the approach (1.4) above is dealing with higher dimension (i.e. $n+m$). Moreover, it is restricted to the problem with equality constraints only. The most recent proposed penalty function method is based on the projection matrix which was introduced by de Freitas Pinto and Ferreira (2014), the concept can be expressed in the following form

$$minimize \mid d(x) \mid^2 + \mid h(x) \mid^2$$

where $d(x) = P(x) \bigtriangledown^T f(x)$ (gradient projection vector used by Rosen (1960, 1961)) $\bigtriangledown^T f(x)$ is the transpose of $\bigtriangledown f(x)$ and $P(x) = I - \bigtriangledown^T h(x)[\bigtriangledown h(x) \bigtriangledown^T h(x)]^{-1} \bigtriangledown h(x)$ (projection matrix over the constrained tangent subspace of the considered problem).

Some of the observed setbacks of this approach include: all stationary points of the considered problem are local minima, and the difficulty of the matrix inversion required to compute $P(x)$. Moreover, none of the above-listed penalty functions approaches believe to be perfect on any formulations. Another critical point to be taking into consideration is the composition of any optimization problem may vary fromone form to another, apart fromthe notion of convex, invex, nonconvex optimization problems, there are also regularity, irregularity, linear objective with non-linear constraints functions, non-linearobjective with linear constraints functions, quadratic and polynomial functions.

This work will revisit the classical penalty function and propose another penalty function in accordance with existing penalty function methods. The proposed penalty function was constructed according to the general form of mathematical programming problem in equation (1.1). One of the interesting property of the proposed Logarithmic Penalty Function (LPF) is that it is not a barrier function (interior penalty function) and at the same time constructed in logarithmic form. Further, there is no restriction to either equality or inequality constraints and again, its differentiability nature is among the unusual property of the most widely used penalty functions. This will provide an avenue to address some of the setbacks for the existing approaches that are restricted either to equality constraints or inequality constraints only. Furthermore, it is among the category of the continuously differentiable penalty function. In a nutshell, the proposed LPF is intended to address the following setbacks:-

- Irregularity of the problem.
- Compatibility.
- Restrictions.

1.3 Objectives

The objectives of this research are listed as follows:

- To prove the convergence of the proposed LPF and establish the equivalence between the optimal solutions of the original optimization problem and its associated penalized problem.
- To derive the KKT Multipliers that could be used to investigate the saddle point and establish the duality gap between the primal and Lagrangian dual of the problem.
- To compare the proposed LPF with some of the existing penalty function approach on theoretical and practical problems.

1.4 Scope and Limitation

The focus of this research is on the general form of a non-linearly constrained optimization problem that possesses properties: Continuously differentiable, smooth objective and constraints functions, single objective function and invex optimization problem.

1.5 Preliminary Definitions

In this section, some useful notations and definitions that will frequently be used are presented. Consider the problem (1.1) with equality constraints, if $p(x)$ is a penalty function regarding the problem, then, the following conditions are satisfies: -

- $p(x)$ is continuous.
- $p(x) \geq 0, \forall x \in \Re^n$.
- $p(x) = 0$ if and only $h_j(x) = 0$, $g_i(x) \leq 0$.

Definition 1.1 (Zangwill, 1967) A function $p(x) : \Re^n \to \Re$ is said to be a penalty function for the problem in equation (1.1) with equality constraints, if $p(x)$ satisfies the following :

- $p(x) = 0$ if $h_j(x) = 0$.
- $p(x) > 0$ if $h_j(x) \neq 0$.

Definition 1.2 (Hassan & Baharum, 2019d) A feasible solution $\bar{x} \in F$ is said to be optimal to penalized optimization problem $P_{c_k}(\bar{x})$ if there exist no $x \in F$ such that $P_{c_k}(x) < P_{c_k}(\bar{x})$, where c_k is a positive penalty parameter.

Definition 1.3 (Antczak, 2009b) A function $p(x) : \Re^n \to \Re$ is said to be a penalty function for the problem (1.1) with inequality constraints, if $p(x)$ satisfies the following :

- $p(x) = 0$ if $g_i(x) \leq 0$.
- $p(x) > 0$ if $g_i(x) > 0$.

Definition 1.4 Problem (1.1) is said to be regular if the first and second derivatives of all the functions in the problem exist in the feasible region.

Definition 1.5 (Hanson, 1981) Let $f : X \to \Re$ be a differentiable function on $X \subset \Re^n$ and $u \in \Re^n$. If, there exists a vector-valued function $\eta : \Re^n \times \Re^n \to \Re^n$ such that, $\forall x \in X$, the following inequality

$$f(x) - f(u) \geq \bigtriangledown f(u)\eta(x,u)(>) \tag{1.5}$$

holds, then the function f is said to be an invex (strictly invex) function with respect to η at u on X. If in eq. (1.5) holds at each point $u \in \Re^n$, then f is said to be an invex (strictly invex) function with respect to η on $\Re^n$.

Definition 1.6 (Hanson, 1981) Let $f : X \to \Re$ be a differentiable function on $X \subset \Re^n$ and $u \in \Re^n$. If, there exists a vector-valued function $\eta : \Re^n \times \Re^n \to \Re^n$ such that, $\forall x \in X$, the following inequality

$$f(x) - f(u) \leq \bigtriangledown f(u)\eta(x,u)(<) \tag{1.6}$$

holds, then the function f is said to be an incave (strictly incave) function with respect to η at u on X. If equation (1.6) holds at each point $u \in \Re^n$, then f is said to be an incave (strictly incave) function with respect to η on $\Re^n$.

Definition 1.7 (Antczak, 2013) A continuous function $f(x)$ that is defined on $\Re^n$ is coercive if

$\lim_{\|x\| \to \infty} f(x) = +\infty$. That is, for any constant $M > 0 \; \exists \, R_M > 0$ such that $\|f(x)\| > M$ whenever $\|x\| > R_M$.

Definition 1.8 (Antczak, 2009b) A vector ξ is said to be a feasible solution for the dual problem, if $\phi(\xi) = \inf\{L(x,\xi) : x \in \Re^n\}_{\xi \geq 0} > -\infty$.

Definition 1.9 (Antczak, 2009b) If the set of all feasible solutions in the problem (1.1) with inequality constraints is not empty, we say that problem (1.1) is consistent .

Definition 1.10 (Antczak, 2009b) If there exists a strictly feasible solution $\tilde{x}$, that is, $g_i(\tilde{x}) < 0$, $i \in I$, then problem (1.1) is super-consistent, and the feasible point $\tilde{x}$ is called a Slater point for the problem (1.1).

Definition 1.11 (Bazaraa et al., 2013) $L\left(\tilde{x}, \widetilde{\xi}, \widetilde{\mu}\right) = f(\tilde{x}) + g(\tilde{x}) + \widetilde{\xi}\ \widetilde{\mu}\ h(\tilde{x})$ defines the Lagrange function for the problem (1.1).

Definition 1.12 (Antczak, 2009b) The duality gap between the primal problem and its associated dual problem can be defined by $G = P - D \geq 0$.

Definition 1.13 (Bazaraa et al., 2013) A point $(\tilde{x}, \widetilde{\xi}, \widetilde{\mu}) \in X \times \Re^m_+ \times \Re^s$ satisfying the following conditions:

(i) $L(\tilde{x}, \xi, \mu) \leq L\left(\tilde{x}, \widetilde{\xi}, \widetilde{\mu}\right)\ \forall\ \xi \in \Re^m_+,\ \mu \in \Re^s$,

(ii) $L\left(\tilde{x}, \widetilde{\xi}, \widetilde{\mu}\right) \leq L\left(x, \widetilde{\xi}, \widetilde{\mu}\right)\ \forall\ x \in X$,

is said to be a saddle point in the considered optimization problem (1.1).

1.6 Organization of the Thesis

The presentation of this thesis is organized as follows:

Chapter 1 provides a general introduction and an overview for the broad field of optimization, specifically regarding the notion of penalty function method. Further, the research problem and its objectives are presented.

A general review of the literature specifically for the concept of the penalty function approach has been presented in Chapter 2. The details of the earliest contributors to the general idea of penalty and some of the observed setbacks are also presented.

Chapter 3 presents some of the available techniques for the unconstrained optimization problem. The methods discussed are steepest descent (SD), Newton's method, conjugate gradient (CG), Quasi-Newton's method and sequential unconstrained minimization technique (SUMT).

The proposed logarithmic penalty function, modified Courant-Beltrami penalty function and their convergences are discussed in Chapter 4. In addition, KKT multipliers were derived in respect of the LPF and MCB.

Chapter 5 focuses on an invex optimization problem by hybridizing the LPF and MCB to have the generalized logarithmic penalty function. Furthermore, an equivalence between the optimal solution of the original problem and its associated penalized problem employing LPF has been established. Also, the theory of duality regarding the problem under consideration are presented.

Chapter 6 presents an application to alkylation process optimization utilizing the proposed LPF.

Chapter 7 provides the general discussions, contributions, conclusion and remarks for future research.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter discusses an overview of the literature applicable to the study of the penalty function method, a theoretical advancement and state of the art are provided. The method of penalty function is an approach precisely designed to solve a constrained optimization problem. The rationale behind all the penalty function methods is straight forward and easy to implement. The idea is achievable by replacing the original problem with its corresponding unconstrained problem, in such a way that their solution coincides or at least approximates the solution of the problem (1.1). Numerous research is conducted through proposing a different kinds of penalty function methods.

The notion of penalty function approach comes into existence in the early 1960s, the goal of the method is to make a constrained optimization problem more comfortable to handle, this is in line with the aspiration of both theorist and practitioners to achieve their common objectives. The motive for such an idea was due to availability of unconstrained optimization techniques, since the inception of the method, many researchers are trying to render it more positive to the field of optimization theory.

2.2 Exact Penalty Function Method

An exact penalty function was first suggested and introduced simultaneously by Zangwill (1967) and Eremin (1967), an algorithm that can be used to solve a non-linear programming problem were presented. However, the method appears to be more useful in the concave case. Conventionally, a penalty function is said to be exact if the solution of the penalized problem coincides with the solution of the original problem. Morrison (1968) proposed another exact penalty function of the form;

$$min| f(x) - M |^2 + | h(x) |^2, \tag{2.1}$$

where M is an estimated optimized objective function $f(\tilde{x})$ and $h(x)$ is an equality constraint. The Morrison function (equation (2.1)) can also be considered as an exact penalty function. The Morrison method was later revisited by Meng et al., (2013); Meng et al. (2004). Many authors continued to make advancement, especially regarding the differentiability and non-differentiablity of an exact penalty functions. Meng andYang (2015) studied the first-order and second-order necessary conditions for non-linear optimization problems. In their study, those conditions are considered from the viewpoint of exact penalty functions. The regular sub-differential of the penalty term is used to establish the necessary and sufficient conditions for a penalty term to be of KKT-type. Most of the result considered in the literature of exact penalization are generally concerned with finding conditions, under which the optimal solution for the transformed unconstrained problem is equivalent to the optimal of the original problem. Jane (2012) studied reverses property and provides the conditions under which the original constrained problem and transformed the exact penalized problem explicitly equivalent.

Dolgopolik (2016) considered the exactness of linear penalty functions from which a unifying theory of exact linear penalty function was developed. Further, Dolgopolik (2017) established a general theory of exact parametric penalty functions for constrained problems. One of the advantages of the above mentioned approach is that, unlike non-parametric, the exact parametric penalty function can be both smooth and exact.

Recently, Dolgopolik (2018b) developed a unified method for the analysis of the global exactness of a different form of penalty and augmented Lagrangian function. The concept of global parametric accuracy in a finite-dimensional space was introduced. In the second part of the study, Dolgopolik (2018a) were able to present the idea of globally extended exactness, which helps in reducing the global exactness survey to a local analysis of a merit function.

Laptin (2016) utilized an exact penalty function by considering the approaches that allow estimating the values of penalty coefficients. In this approach, no auxiliary problems are required to solve the problem. Later, Laptin and Bardadym (2019) presents the results of computational experiments adopting a clear strategy for estimating coefficients in solving some classes of the problem.

2.3 Optimal Control Problems Via Exact Penalty Function Method

Li et al. (2011) considered a class of optimal control problems subject to terminal state (equality constraints) and continuous state and control (inequality constraints).

It was implemented through the control parameterization technique and time scaling transformation. An exact penalty function is used to construct a computational method to solve the described optimization problem. An optimal control problem is also considered by Jiang et al. (2012), especially with free terminal time and continuous inequality constraints. The problem is transformed into a penalized problem after the following steps:

- The problem has to be approximated by presenting the control functions as a piecewise-constant function.
- The inequality constraints have to be transformed into terminal equality constraint for an auxiliary differentiable system.

Then, the gradient-based optimization technique can be used to solve the problem.

Dolgopolik and Fominyh (2019) develop a general approach, particularly to the design and analysis of exact penalty functions; this can be applied to the various optimal control problem. For example:

- Problems with terminal and state constraints.
- Problems associated with differentiable inclusions.
- Optimal control problem.

This method grants one to remove some (or all) constraints of the problem by the use of exact penalty functions. Indeed, this will make the optimal control problem easier to handle.

Liu et al. (2016) apply the concept of exact penalty function approach with recently developed derivative-free global heuristic optimization algorithm for solving an unconstrained problem. It is called a differential search (DS) algorithm. A comparison study between the proposed algorithm and other evolutionary methods used universally is carried out on 24 benchmark problems.

2.4 Differentiable Exact Penalty Function

The work of Fletcher and Leyffer (2002) presents a continuously differentiable exact penalty function regarding the problem (1.1) for equality constraints. As reported by Fletcher and Leyffer (2002), it is possible to establish an exact penalty function which is sufficiently smooth to accept conventional techniques for solving the problem (1.1), the local minimum can be located. Other researchers further studied continuously differentiable and nondifferentiable exact penalty function (Bazaraa et al., 2013; Bertsekas & Koksal, 2000; Charalambous, 1978; Conn, 1973).

2.5 Convexity of the Problem

The concept of convexity plays a dominant role in almost all the collection of penalty function approaches (see, for example, Bazaraa et al. (2013); Charalambous (1978); Mangasarian (1985)). In the last few years, some numerous convex function generalizations have been derived which gives a room for extending optimality condition and some classical duality results, earlier restricted to convex programs to the larger classes of optimization problems. The notion of invexity introduced by Hanson (1981) and named by Craven (1981) was among the category. Hanson (1981) applied the extended theory of convex functions to prove optimality conditions and duality results for the non-linearly constrained optimization problem.

Antczak (2009a) establishes some new results on the exact penalty function methods; this work outline a differentiable nonconvex optimization problem with both (equality and inequality) constraints as in problem (1.1). It was realized via the following exact penalty function with $t = 1$;

$$p(x) = \sum_{i=1}^{m} [g_i^+(x)]^t + \sum_{j=1}^{s} \mid h_j(x) \mid^t, \tag{2.2}$$

where $g_i^+(x) = \max\{0, g(x)\}$, and t is a positive integer.

The equivalence between the sets of optimal solutions in the problem (1.1) and the following transformed unconstrained optimization problem under suitable invexity assumption is well-established:

$$P_c(x) = f(x) + c[\sum_{i=1}^{m} [g_i^+(x)] + \sum_{j=1}^{s} \mid h_j(x) \mid], \tag{2.3}$$

where c is a penalty parameter. Antczak (2011) introduced the l_1 exact exponential penalty function based on the classical penalty function constructed by Liu and Feng (2010); this was explicitly designed to solve an optimization problem (1.1) constituted by $r-invex$ functions (with respect to the function η). The l_1 exact exponential penalty function is of the following form:

$$p(x) = \sum_{i=1}^{m} \frac{1}{r}(e^{(rg_i^+(x))} - 1) + \sum_{j=1}^{s} \frac{1}{r}(e^{r|h_j(x)|} - 1), \tag{2.4}$$

where r is a finite real number not equal to 0. Note that, the function $\frac{1}{r}(e^{(rg_i^+(x))}-1)$ is defined by

$$\begin{cases} \frac{1}{r}(e^{(rg_i^+(x))}-1)=0, if\, g_i(x)\leq 0, \\ \\ \frac{1}{r}(e^{(rg_i^+(x))}-1), if\, g_i(x)>0 \end{cases} \tag{2.5}$$

Certainly, equation (2.5) has the penalty features relative to a single constraint function $g_i(x) \leq 0$, that is 0 for all values of x that satisfy the constraint and the outcomes of a large values for any infeasible point. The penalty function in equation (2.4) is considered to be a classical, if $r = 0$ that was defined by Pietrzykowski (1969) and also by Han and Mangasarian (1979). Further, the results have been proved through the classical l_1 exact penalty function method under $r-invexity$ assumption by Antczak (2010) for inequality constraints. The work of Antczak (2016) demonstrated that the particular sort of minimizers in nonconvex nonsmooth optimization problems with both (equality and inequality) constraints could be identified using the exact absolute value penalty function method. Antczak (2018a) introduced a new vector exponential penalty function method, specifically for nondifferentiable multiobjective programming problems, and established its convergence restricted to inequality constraints. Further, a vector exact penalty function method's exactness property is defined and analyzed.

Echebest et al. (2016) applied the exponential penalty function to prove global convergence results of an augmented Lagrangian method. This can be achieved using the constant positive generator Constraints Qualification (CQ) if the sub-problem is solved in an approximate form.

2.6 Multiobjective Optimization Problem

The idea of a penalty function approach has been extended to multiobjective programming problem (MOPP). Liu and Feng (2010) constructed a classical exponential penalty function method for multiobjective programming problems (MOPP) and its convergence have been investigated. Further, an approach was used to solve a finite min-max MOPP. Jayswal and Choudhury (2014) were able to extend the work of Antczak (2011) and Liu and Feng (2010) to multiobjective fractional programming problems and examine the convergence of the method.

2.7 Filter Based Approach

Filter based approach for solving the same constrained optimization problem (1.1) with equality constraint were introduced by Fletcher and Leyffer (2002). The concept is achievable by minimizing two functions $f(x)$ & $\gamma(x)$ simultaneously. The function $\gamma(x)$ possess the basic properties of penalty function. That is:

- $\gamma(x) > 0$, if x is infeasible.
- $\gamma(x) = 0$, if x is feasible.

It is a list of pairs (f^l ,γ^l) whereas no pair will be allowed to influence another. Nie (2007) modified the original filter method specifically for the equality constraint problem, the process is implemented by combining the advantages of penalty function techniques and the sequential quadratic programming (SQP) approaches. The approach performed better than that of sequential penalty quadratical programming (SlQP). This approach replaced the objective function by the penalized function of the form;

$$f(x) + \sigma\gamma(x), \tag{2.6}$$

where σ is a fixed parameter that does not need to be updated at each step. According to Nie (2007), this approach is advantageous compared to the original filter method. Luenberger (1973) studied the same problem considered by Morrison (1968) and explored an unconstrained problem that works in the space $\Re^{m+n}$ with respect to the objective and constraint functions in equation (2.1) as in equation (1.4), this approach does not require successive minimization solution. Nevertheless, the approach admits disadvantage of higher dimension. In the same manner, de Freitas Pinto and Ferreira (2014) proposes an exact penalty function based on matrix projection, and the constructed unconstrained problem is of the form;

$$min \mid d(x) \mid^2 + \mid h(x) \mid^2,$$

where $d(x) = P(x) \bigtriangledown^T f(x)$ (gradient projection vector used by Rosen (1960, 1961)) $\bigtriangledown^T f(x)$ is the transpose of $\bigtriangledown f(x)$ and $P(x) = I - \bigtriangledown^T h(x)[\bigtriangledown h(x) \bigtriangledown^T h(x)]^{-1} \bigtriangledown h(x)$ (projection matrix over the constrained tangent subspace of the considered problem). Chen et al. (2019) suggested a new penalty-free method to solve non-linearequality constrained optimization. This method is established as an alternative to penalty function or a filter methods. Under standard assumptions, a super-linear and global convergence are well-established.

Baba et al. (2015) study and addressed the accuracy of the predicted response employing a penalty function method via the dual response surface optimization approach. The primary objective of the proposed plan is to reduce the variance influence for the predicted response through minimizing variability corresponding to the quality characteristics of interest. Zhou et al. (2017) used an exact penalty function method to optimize quadratic assignment problem formulation in locating the facility layout problem to increase a system's operating efficiency. Zhou et al. (2017) developed an improved backtracking search algorithm. The symbolic organism search algorithm is formulated and combined with an adaptive penalty function to solve the multiobjective problem with both (equality and inequality) constraints by Panda and Pani (2016). The approach appeared to be more useful in metaheuristic optimization algorithm, especially for engineering applications. Kong et al. (2018) described and established the iteration-complexity for linearly constraints nonconvex minimization problem using a quadratic penalty accelerated inexact proximal point method.

2.8 Exterior and Interior Penalty Function Method

There are two different approaches to the penalty function method. The first is called an exterior penalty function method; in this method, the constraints are incorporated into the objective function by adding a penalty term that penalized any violations of the single unconstrained optimization problem. This method generates a sequence of infeasible points whose limit is optimal to the original problem. The approach guaranteed that the optimal could be found through an unconstrained minimization technique. The second is called an interior penalty function (or barrier), in this method, a barrier term is added to the objective function with the aim to prevents the generated point from leaving the feasible region. This method generates a sequence of feasible points whose limit is optimal to the original problem.

Wang et al. (2014) proposed a new class of smooth exact penalty function as a special case for both interior-type and exterior-type penalty functions. Further, Wang et al. (2014) establish necessary and sufficient conditions for exact penalty property and inverse proposition of exact penalization, respectively. The numerical results were reported to validate the proposed algorithm of a feasible penalty functionand its convergence analysis.

2.9 Summary

This chapter reviewed some of the penalty function approaches for solving a constrained optimization problem. Substantially, some important points should be taken into consideration. In summary, some of the penalty functions are designed specifically for inequality constrained optimization, some are for equality constrained optimization problem, and some are generally constructed to accommodate the generic form of an optimization problem in equation (1.1). Even though most of the penalty function that possesses the features of the general form (problem (1.1) with mixed constraints) are non-differentiable as depicted in the Figure 2.1, this is what makes it impossible to use conventional unconstrained minimization techniques. Many researchers are trying to make advancement to the existing penalty function methods, while at the same time, working interminably to devise an alternative to the penalty function method. For example, Filter based approach and Penalty-free method.

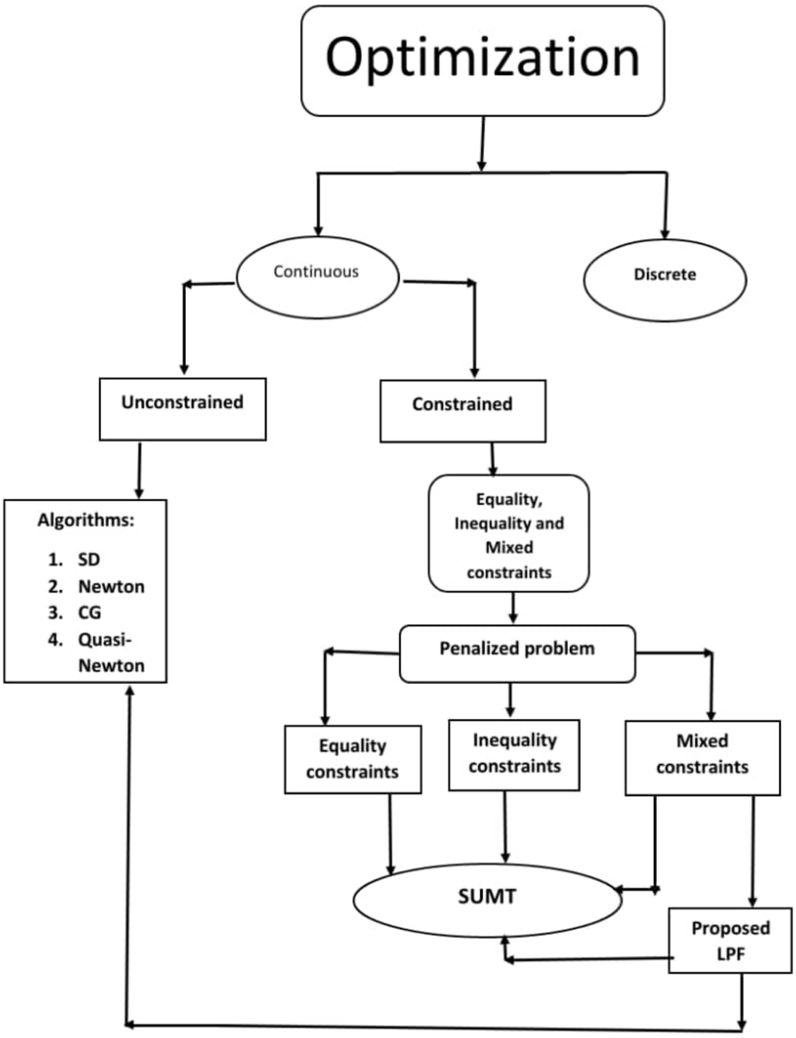

Figure 2.1: LPF Chart.

CHAPTER 3

UNCONSTRAINED OPTIMIZATION TECHNIQUES

3.1 Introduction

This chapter discusses the leading techniques for solving unconstrained optimization problems. The focus will be on the study of some of the available algorithms for unconstrained problems. The methods are Steepest Descent (SD), Newton's method, Conjugate Gradient (CG), Quasi-Newton's method, and Sequential Unconstrained Minimization Technique (SUMT).

3.2 General Design of Optimization Methods

The optimization method is the science of an iterative procedure of selecting the best of many possible outcomes to decide in a real situation. The concept of an iterative strategy is to start with the initial guess point, which will be used to generate the next point. The newly obtained solution is then taken to the next iteration as the initial point: this process will lead to the required optimal solution if convergence is certain. The iterative method obtained can be expressed as

$$x_{k+1} = x_k + \alpha_k d_k,$$

where x_{k+1} is known as the new iterate point while x_k is the present iterate point, also, α_k is the step size while d_k denoted the search direction.