DYNAMICAL ANALYSIS OF FRACTIONAL-ORDER ECO-EPIDEMIOLOGICAL MODELS INCORPORATING HARVESTING

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by

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ANALISIS DINAMIK MODEL EKOLOGI-EPIDEMIOLOGI PERINGKAT PECAHAN MENGGABUNGKAN PENUAIAN

ABSTRAK

Dalam tesis ini, tujuh model eko-epidemiologi peringkat pecahan dirumuskan dan dianalisis: i) model eko-epidemiologi dengan mangsa yang dijangkiti digabungkan dengan penuaian; ii) model eko-epidemiologi dengan mangsa yang dijangkiti dan kadar pertumbuhan logistik yang merangkumi penuaian; iii) model eko-epidemiologi dengan mangsa yang dijangkiti dan kadar kejadian tidak linear yang menggabungkan penuaian; iv) model eko-epidemiologi dengan pemangsa yang dijangkiti dan tindak balas fungsi Holling tipe-II yang menggabungkan penuaian; v) model eko-epidemiologi dengan pemangsa yang dijangkiti dan tindak balas fungsional Holling tipe-IV yang menggabungkan penuaian; vi) model eko-epidemiologi dengan dua jenis penyakit pada populasi pemangsa yang digabungkan dengan penuaian; vii) model jangkitan Hantavirus yang merangkumi penuaian. Untuk memperjelas ciri-ciri model eko-epidemiologi peringkat pecahan yang dicadangkan, aspek keberadaan, keunikan, non-negatif dan batasan larutan dianalisis dan diperiksa. Keadaan kestabilan tempatan dan global dari semua titik keseimbangan yang mungkin secara biologi dari model eko-epidemiologi fraksional-pesanan yang dicadangkan disiasat oleh keadaan Matignon dan membina fungsi Lyapunov yang sesuai, masing-masing. Bukti kewujudan percabangan transkritikal diberi menggunakan teorem Sotomayor. Simulasi berangka dilakukan untuk menggambarkan hasil analisis. Model eko-epidemiologi peringkat pecahan yang dicadangkan terbukti mempunyai tingkah laku dinamik yang pelbagai termasuk fenomena bistabilitas, bifurkasi Hopf superkritical, dan percabangan transkritikal. Kesan peringkat pecahan, penyakit berjangkit dan penuaian terhadap kestabilan model eko-

epidemiologi peringkat pecahan yang dicadangkan telah diselidiki. Telah didapati bahawa peringkat pecahan, penyakit berjangkit, dan penuaian mempunyai pengaruh penting terhadap dinamika model eko-epidemiologi peringkat pecahan yang dicadangkan. Telah ditunjukkan bahawa memasukkan penuaian ke dalam model eko-epidemiologi peringkat pecahan yang dicadangkan mempunyai kesan penstabilan. Diperhatikan bahawa terbitan peringkat pecahan dapat meredam pergerakan pada populasi dan membantu kestabilan model eko-epidemiologi yang dicadangkan. Turut diperhatikan bahawa peringkat pecahan memainkan peranan penting dalam mengawal dinamika kacaubilau dan menyediakan ekosistem lestari di mana semua spesies dapat bertahan hidup. Selanjutnya, ditunjukkan bahawa domain kestabilan model peringkat integer lebih kecil daripada domain yang sesuai dari model pesanan pecahan yang dicadangkan.

DYNAMICAL ANALYSIS OF FRACTIONAL-ORDER ECO-EPIDEMIOLOGICAL MODELS INCORPORATING HARVESTING

ABSTRACT

In this thesis, seven fractional-order eco-epidemiological models are formulated and analyzed: i) an eco-epidemiological model with infected prey incorporating harvesting; ii) an eco-epidemiological model with infected prey and logistic growth rate incorporating harvesting; iii) an eco-epidemiological model with infected prey and nonlinear incidence rate incorporating harvesting; iv) an eco-epidemiological model with infected predator and Holling type-II functional response incorporating harvesting; v) an eco-epidemiological model with infected predator and Holling type-IV functional response incorporating harvesting; vi) an eco-epidemiological model with two disease strains in the predator population incorporating harvesting; vii) a Hantavirus infection model incorporating harvesting. In order to clarify the characteristics of the proposed fractional-order eco-epidemiological models, existence, uniqueness, non-negativity and boundedness of the solutions are analyzed. The local and global stability conditions of all biologically feasible equilibrium points of the proposed fractional-order eco-epidemiological models are investigated by the Matignon's condition and constructing suitable Lyapunov functions, respectively. The proof of the existence of transcritical bifurcation is given by using Sotomayor's theorem. Numerical simulations are conducted to illustrate the analytical results. The proposed fractional-order eco-epidemiological models are shown to have rich dynamical behavior including bistability phenomena, supercritical Hopf bifurcation and transcritical bifurcation. The effects of fractional-order, infectious disease and harvesting on the stability of the proposed fractional-order eco-epidemiological models are investigated. It was found that the

fractional-order, infectious disease and harvesting have a crucial effect on the dynamics of the proposed fractional-order eco-epidemiological models. It was shown that introducing harvesting to the proposed fractional-order eco-epidemiological models has a stabilizing effect. It was observed that the fractional-order derivative may damp out oscillations in the population and helps the stability of the proposed eco-epidemiological models. It was also observed that the fractional-order derivative may play important role in controlling chaotic dynamics and allow a sustained ecosystem where all species survive. Furthermore, it was shown that the stability domain of the integer-order model is smaller than the corresponding domain of the proposed fractional-order model.

CHAPTER 1

INTRODUCTION

1.1 Background

Over the last five decades or so humans have realized the importance of ecosystems and biodiversity and its relation to sustainable development. Aspects such as loss of biodiversity, sustainable and proper forest management, desertification, land degradation and sustainable ecosystem have attracted a great deal of concern as well as attention. In addition to these concerns about terrestrial ecosystems, there are also concerns about the marine ecosystem such as acidification and overfishing. All of these have been encapsulated in sustainable goals number 14 and 15 of the 17 sustainable goals for 2015-2030 promoted by the United Nations. To ensure the protection, restoration and well-being of ecosystems on earth, there needs to be an array of tools that can be used to study ecosystems. Various tools are available including mathematical modeling, statistical modeling, advanced data analysis, computer simulation, databases, empirical and laboratory studies. These tools can also complement and support each other. Biological and ecological problems can be explained, predicted and controlled by establishing mathematical models with scientific methods and reasonable assumptions (Bacaër, 2011). Mathematical modeling can be considered as a powerful tool and has been successful in describing reality. Generally, mathematical models that describe a system is based on a set of variables and a set of equations that establish the relationship between the variables. Mathematical models also play an essential role in describing and understanding the dynamical behavior of the interacting populations.

Recently, mathematical modeling has attracted much attention in epidemiology as well as ecology. Mathematical modeling may help to explain the real system and study the effects of different components. It can also be used to make predictions about the behavior of interacting populations (Turchin, 2003). Our focus in this thesis is in the use of mathematical modeling to study prey-predator models in the presence of infectious diseases either in the prey population or in the predator population incorporating harvesting.

In the natural world, predation describes a biological interaction where a predator feeds on its prey. These behaviors can be modeled mathematically by prey-predator models. The interaction between prey and predator was first studied by the famous mathematicians Lotka and Volterra. The Lotka-Volterra model consists of two coupled non-linear differential equations and illustrates the interactions of one prey and one predator population (Chauvet et al., 2002). Mathematicians and ecologists have significantly extended prey-predator models based on the Lotka-Volterra model. These extended models may contain more than two interacting species. Typical examples of predation are lions eating zebras and snakes eating mice. Holling (1959) introduced a common functional responses used to model the predation between a population of prey and predator. The common predator functional responses include four possible response forms: Holling type-I, Holling type-II, Holling type-III and Holling type-IV functional response. The prey-predator models will be further discussed in the Chapter 3.

Mathematical modeling can be used to analyze the spread of infectious diseases and to predict the future course of the outbreak (Ansari et al., 2015). The study of spreading infectious among the population can be considered as an important topic in mathematical biology and may help to predict the effect of those infectious (Biswas et al., 2017). Kermack and Mckendrick (1927) proposed the classical SIR (susceptible–infected–removed) disease model which has influenced the development of mathematical models for disease spread. Epidemiological models that modeled the spread of infectious diseases related to one species is a major issue in mathematical biology. It has been observed in nature that the species do not exist alone. So, the effect of an infectious disease on the ecological environment need to be taken into account. Infectious diseases may have an important role in the relationship between prey and predator population. Therefore, it is important to consider the impact of interacting species when epidemiological models are studied (Mukherjee, 2010). Eco-epidemiological models seek to study the impact of infectious diseases in the ecological environment (Saifuddin et al., 2016a). The study of eco-epidemiological models may help us understand the role of infectious on the interacting species (Saifuddin et al., 2016b). Over the last decades, eco-epidemiological models have become an active field of research in population biology. This is because both the ecological and epidemiological issues are considered in an eco-epidemiological model and therefore can better represent the natural system in many cases (Adak et al., 2020). Infectious diseases among prey and predator populations are disorders caused by bacteria or virus (Almeida et al., 2019). Infectious diseases that occur in animals such as avian influenza have posed a considerable threat to humans (Chen and Jiang, 2011). Introducing disease into the prey-predator model may help to stabilize prey-predator oscillations (Hilker and Schmitz, 2008). In the last few decades, some studies have been carried out on eco-epidemiological models with the disease either in the prey population only or in the predator population only. The eco-epidemiological models will be further discussed in the Chapter 3.

It is important to consider the impact of harvesting when the eco-epidemiological models are studied. Incorporating harvesting to an eco-epidemiological model is expected to provide a more realistic model since for a number of prey populations some form of harvesting in the ecosystem is known to be available. The harvesting of populations are commonly practiced in fishery, forestry and management of wildlife (Lee and Baek, 2017). Harvesting in the prey population affects the population of predators indirectly because it reduces the food population available in the area (Ávila-Vales et al., 2017).

Recently, with the advancement of fractional calculus, a new approach is appearing by replacing ordinary derivatives with fractional derivatives. Fractional-order differential equations can be considered as a generalization of ordinary differential equations to an arbitrary (non-integer) order and have been successfully applied in various fields of science and engineering to investigate the underlying dynamics of the systems (Kilbas et al., 2006). The fractional-order derivative is a non-local operator in the sense that the system at present states depends on the recent past states, therefor fractional derivative may be more suitable for long-time behavior studies (Ansari et al., 2015). The fractional-order differential equations exhibit richer dynamical behavior and this is because it incorporates the memory effect in the model (Santos et al., 2015). The definition of fractional derivative that will be used throughout this thesis includes memory effects, where the state of the model at each time depends on the previous history of the model, thus, better predict the interaction of population. The fractional-order differential equations also provide an effective description method for physical models (Ford et al., 2013), neural networks models (Zhang et al., 2018), system control models (Xiao et al., 2017), social networks models (Xu et al., 2019) and chaos models (Danca, 2017). The presence of fractional-order derivative in the eco-epidemiological models generalizes the results obtained in case of the standard derivative (Ghanbari and Djilali, 2020). Therefore, the dynamics of the relations between predators and their prey in the presence of infectious disease either in the prey population or in the predator population incorporating harvesting can be more accurately described by fractional-order models (Ahmed et al., 2007; Li et al., 2018). The fractional-order models will be further discussed in the Chapter 3.

In this thesis, a fractional-order prey-predator models in the presence of infectious disease either in the prey population or in the predator population incorporating harvesting are proposed and investigated. The focus will be on four populations which are susceptible prey, infected prey, susceptible predator and infected predator. The mathematical analysis and numerical simulations are performed to clarify the characteristics of proposed models. The existence, uniqueness, non-negativity and boundedness of the solutions of proposed models are investigated. The local and global stability of the equilibrium points of the proposed fractional-order models are studied. The proof of the existence of transcritical bifurcation is given. Moreover, the Adams-Bashforth-Moulton numerical method is conducted for the numerical simulation of the fractional-order eco-epidemiological models to indicate the rich dynamical behavior of proposed models which is in agreement with the theoretical analysis. The numerical simulations focus on the influences of fractional-order, infectious disease and harvesting on the population densities.

1.2 Motivation

So far as we are aware, no scholar has investigated the dynamics of the fractional-order eco-epidemiological models incorporating harvesting as presented in this thesis. This research, therefore, seeks to develop a fractional-order eco-epidemiological models in the presence of infectious disease either in the prey population or in the predator population incorporating harvesting.

The stability of interacting populations is one of the significant challenges to ecologists and it is one of the most important criteria to be considered for the purpose of understanding the behavior of the eco-epidemiological model. This study is focused on the effects of fractional-order and harvesting on the stability of eco-epidemiological models. It will be shown how introducing fractional-order and harvesting into the prey-predator model can stabilize ecosystems. The findings of this research can be applied to keep the coexistence of biological populations and maintain ecological balance. The findings of this research may also help to explain real eco-epidemiological situations, study the effects of different components and make predictions about the behavior of eco-epidemiological situations incorporating harvesting. Furthermore, it is also useful to ecologists who work on the effects of infectious disease on the ecological environment.

1.3 Research questions

The following research questions are relevant to this study.

1. What are the effects of fractional-order, infectious disease and harvesting on the

stability of equilibrium points of the eco-epidemiological models?

2. What are the effects of fractional-order, infectious disease and harvesting on the existences of populations?

1.4 Research objectives

The objectives of this thesis are as follows:

1.4.1 General objective

The general objective of this study is to formulate and study fractional-order eco-epidemiological models incorporating harvesting.

1.4.2 Specific objectives

The specific objectives of this study are:

1. To formulate and analyze:
   (a) a fractional-order eco-epidemiological model with infected prey incorporating harvesting.
   (b) a fractional-order eco-epidemiological model with infected prey and logistic growth rate incorporating harvesting.
   (c) a fractional-order eco-epidemiological model with infected prey and nonlinear incidence rate incorporating harvesting.
   (d) a fractional-order eco-epidemiological model with infected predator and Holling type-II functional response incorporating harvesting.

(e) a fractional-order eco-epidemiological model with infected predator and Holling type-IV functional response incorporating harvesting.

(f) a fractional-order eco-epidemiological model with two disease strains in the predator population incorporating harvesting.

(g) a fractional-order Hantavirus infection model incorporating harvesting.

2. To determine the effect of fractional-order, infectious disease and harvesting on the dynamic of the eco-epidemiological model and on the existence of populations.

1.5 Methodology

A fractional-order eco-epidemiological model is examined by extending the integer-order model.

- The existence and uniqueness of the solutions of eco-epidemiological models are studied by using the Lipschitz condition.
- The non-negativity and boundedness of the solutions of eco-epidemiological models are studied by using the positivity property, standard comparison theorem for fractional-order and the positivity of Mittag-Leffler function.
- The basic reproduction number of fractional-order of eco-epidemiological models is obtained by using the next-generation method.
- The existence conditions of the equilibrium points of fractional-order eco-epidemiological models are obtained.

- The local stability of the equilibrium points of fractional-order eco-epidemiological models is studied by the Matignon's condition.
- The global stability of the equilibrium points of fractional-order eco-epidemiological models is studied by constructing suitable Lyapunov functions.
- The proof of the existence of transcritical bifurcation is given by using Sotomayor's theorem.
- A Hopf bifurcation of eco-epidemiological models is shown by choosing the fractional-order and some other parameters as bifurcation parameters.
- The theoretical results of eco-epidemiological models are illustrated numerically by using MATLAB-R2019a and MATHEMATICA-11.2.
- The generalized Adams–Bashforth–Moulton method is applied for the numerical simulations.

The existence of transcritical bifurcation is not given in Chapters 5, 8 and 9 because these three chapters did not show transcritical bifurcations. The global stability of the equilibrium points is not studied in Chapters 8 and 10 because the governing equations of these two chapters are highly sensitive to the initial conditions as will be shown in the basin of attraction region.

1.6 Contribution

The main contributions of this study are as follows:

- Dynamical analysis of a fractional-order eco-epidemiological model with infected prey incorporating harvesting (Chapter 4).

- Dynamical analysis of a fractional-order eco-epidemiological model with infected prey and logistic growth rate incorporating harvesting (Chapter 5).
- Dynamical analysis of a fractional-order eco-epidemiological model with infected prey and nonlinear incidence rate incorporating harvesting (Chapter 6).
- Dynamical analysis of a fractional-order eco-epidemiological model with infected predator and Holling type-II functional response incorporating harvesting (Chapter 7).
- Dynamical analysis of a fractional-order eco-epidemiological model with infected predator and Holling type-IV functional response incorporating harvesting (Chapter 8).
- Dynamical analysis of a fractional-order eco-epidemiological model with two disease strains in the predator population incorporating harvesting (Chapter 9).
- Dynamical analysis of a fractional-order Hantavirus infection model incorporating harvesting (Chapter 10).

1.7 Scope of the research

This thesis focuses on formulating and analyzing a fractional-order eco-epidemiological models in the presence of infectious disease either in the prey population or in the predator population incorporating harvesting. The mathematical analysis and numerical simulations are performed to clarify the characteristics of the proposed models.

1.8 Limitation of the research

There are several limitations in this study. Because the deterministic eco-epidemiological models may be inadequate for capturing the exact variability in nature. Hence, stochastic models are required for an accurate approximation of the dynamics of prey-predator interactions in the presence of infectious diseases either in the prey population or in the predator population incorporating harvesting.

1.9 Structure of thesis

This thesis consists of eleven chapters. Chapter 1 gives an introduction, motivation, research questions, research objectives, methodology, contribution, scope and limitation of this study. Chapter 2 reviews the necessary concepts, basic definitions and known theory of stability analysis that will be used throughout this study. Chapter 3 presents the literature regarding the dynamics of fractional-order prey-predator models. In particular, those models which are related to eco-epidemiological models. Chapter 4 presents the dynamical analysis of a fractional-order eco-epidemiological model with infected prey incorporating harvesting. Chapter 5 presents the dynamical analysis of a fractional-order eco-epidemiological model with infected prey and logistic growth rate incorporating harvesting. Chapter 6 presents the dynamical analysis of a fractional-order eco-epidemiological model with infected prey and nonlinear incidence rate incorporating harvesting. Chapter 7 presents the dynamical analysis of a fractional-order eco-epidemiological model with infected predator and Holling type-II functional response incorporating harvesting. Chapter 8 presents the dynamical analysis of a fractional-order eco-epidemiological model with infected predator and Holling type-IV functional response incorporating harvesting. Chapter 9 presents the dynamical analysis of a fractional-order eco-epidemiological model with two disease strains in the predator population incorporating harvesting. Chapter 10 presents the dynamical analysis of a fractional-order Hantavirus infection model incorporating harvesting. In Chapters 4, 5, 6, 7, 8, 9 and 10 a variety of analytical methods and tools are used to study the existence, uniqueness, non-negativity, boundedness and local stability of the solutions of these eco-epidemiological models. The global dynamics of these eco-epidemiological models are investigated analytically as well as numerically. Numerical simulations conducted for theses eco-epidemiological models indicate rich dynamical behavior at the equilibrium points which is in agreement with theoretical analysis. The shown rich dynamical behavior includes bistability phenomena, supercritical Hopf bifurcation and transcritical bifurcation. Chapter 11 contains the general conclusions of the thesis and scope for future work.

CHAPTER 2

PRELIMINARIES

2.1 Introduction

Eco-epidemiological models incorporating harvesting can be formulated and studied using fractional-order differential equations. To analyze these fractional-order eco-epidemiological models, various mathematical tools are required. This chapter is a review of the necessary concepts, basic definitions and established theories of fractional-order differential equations that will be used throughout this thesis.

2.2 Basic definitions

Fractional calculus is an extension of the ordinary calculus by considering integrals and derivatives of arbitrary order (Almeida et al., 2019; Li et al., 2016). In this section, some of special functions and the fundamental definitions of fractional calculus that will be utilized in this thesis are highlighted. Some theorems and facts related to fractional calculus that will be applied in this research are also discussed. The parameters presented in this chapter are given in Table 2.1.

Table 2.1: Description of parameters used.

Parameters	Description
β	Predation rate
a	Half saturation constant
r	Growth rate
k	Environmental carrying capacity.

2.2.1 Special functions

- **Gamma Function**

The Euler Gamma function is a generalized form of the factorial function and is given by (Kilbas et al., 2006)

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du, \quad Re(z) > 0. \tag{2.1}$$

The Gamma function can be reduced to the following factorial function

$$\Gamma(z+1) = z\Gamma(z) = z!, \quad Re(z) > 0. \tag{2.2}$$

The Gamma function will be used in the definition of the Mittag-Leffler function.

- **Mittag-Leffler function**

The Mittag-Leffler function plays a fundamental role in the research of fractional calculus and it is defined for one parameter by (Kilbas et al., 2006)

$$E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}, \quad q > 0, z \in \mathbb{C}. \tag{2.3}$$

The Mittag-Leffler function is also defined for two parameters by (Kilbas et al., 2006)

$$E_{q,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+\beta)}, \quad q > 0, \beta > 0. \tag{2.4}$$

Since the exponential function appears in general in the solution of the ordinary differential equation, the Mittag-Leffler function also appears naturally in the solution of fractional-order differential equations (Kilbas et al., 2006). The

Mittag-Leffler function will be used in Chapters 4, 5, 6, 7, 8, 9 and 10 when we study the boundedness of the solutions of the proposed fractional-order models.

2.2.2 Riemann-Liouville fractional integral

The Riemann-Liouville fractional integral operator of order q for a function f is defined by (Kilbas et al., 2006)

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau)\, d\tau, \quad q > 0. \tag{2.5}$$

The Riemann-Liouville fractional integral will be used in Subsection 2.2.3 when we substituting the memory kernel (2.10) into the model (2.9).

2.2.3 Caputo fractional derivative

The Caputo fractional derivative of order q for a function f is defined by (Kilbas et al., 2006)

$$^{c}D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-\tau)^{n-q-1} f^{(n)}(\tau)\, d\tau, \quad q \in (n-1, n), n \in \mathbb{N}, \tag{2.6}$$

where, $f^{(n)}(\tau) = \frac{d^n}{d\tau^n} f(\tau)$ and $f \in AC^n[0,T]$, the space of functions having nth derivatives absolutely continuous (Kilbas et al., 2006; Waheed et al., 2019). Particularly, when $q \in (0,1)$ such that $n = 1$, one has

$$^{c}D^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-\tau)^{-q} f'(\tau)\, d\tau. \tag{2.7}$$

Since integration is a non-local operator (as it is defined on an interval). The limits of the integration from 0 to t of definition (2.7) indicate the non-local structure of the fractional derivatives (Vyawahare and Nataraj, 2018). Calculating time-fractional derivative of a function $f(t)$ at some $t = t_1$ requires all the past history, i.e. all $f(t)$ from $t = 0$ to $t = t_1$ (Srivastava et al., 2015). The Caputo fractional derivative (2.7) has the following properties (Kilbas et al., 2006):

- ${}^cD^q(c) = 0$, where c is a constant.
- ${}^cD^1 f(t) = \frac{df}{dt}$.
- ${}^cD^0 f(t) = f(t)$.
- ${}^cD^q\left({}^cD^r(f(t))\right) = {}^cD^{q+r}(f(t)), \quad q, r \in \mathbb{R}_+$.
- ${}^cD^q(\lambda f(t) + \gamma g(t)) = \lambda\, {}^cD^q f(t) + \gamma\, {}^cD^q g(t), \quad \lambda, \gamma \in \mathbb{C}$.

The Caputo fractional derivative has been used to model real-world processes for different fields of science and engineering. There are other definitions for fractional derivatives; in this thesis, the Caputo fractional derivative is the one chosen. One of the advantages of Caputo's derivatives is that the initial conditions of fractional differential equations take on the same form as for integer-order ones and this has applications in modelling and analysis. Also, the Caputo fractional-order derivative for a constant is zero, which is not true for other fractional derivatives. Therefore the Caputo fractional-order derivative is useful in finding the equilibrium points since the derivative for a constant is zero (Podlubny, 1999).

The Caputo fractional derivative that will be used throughout this thesis may provide a more realistic eco-epidemiological model that considers memory effects. Consider the following non-linear ordinary differential equation model:

$$\begin{aligned}
\frac{du_1}{dt} &= f_1(u_1,u_2,u_3),\\
\frac{du_2}{dt} &= f_2(u_1,u_2,u_3),\\
\frac{du_3}{dt} &= f_3(u_1,u_2,u_3).
\end{aligned} \tag{2.8}$$

According to Saeedian et al. (2017), the influences of memory effects can be observed by introducing the memory kernel $(k(t-m))$ and taking the integral over a time period $[0,t]$ to model (2.8) as follows:

$$\begin{aligned}
\frac{du_1}{dt} &= \int_0^t f_1(u_1(m),u_2(m),u_3(m))k(t-m)dm,\\
\frac{du_2}{dt} &= \int_0^t f_2(u_1(m),u_2(m),u_3(m))k(t-m)dm,\\
\frac{du_3}{dt} &= \int_0^t f_3(u_1(m),u_2(m),u_3(m))k(t-m)dm,
\end{aligned} \tag{2.9}$$

in which $k(t-m)$ plays the role of a time-dependent kernel and is equal to a delta function $\delta(t-m)$ in a classical Markovian process (where the state of the model at each time does not depend on the previous history of the model). The power-law function for $k(t-m)$ can be considered as follows

$$k(t-m) = \frac{1}{\Gamma(q-1)}(t-m)^{q-2}, \quad q \in (0,1). \tag{2.10}$$

Utilizing the power-law function (2.10), the model (2.9) can be rewritten in the form of fractional differential equations with the Caputo fractional derivative. By substituting kernel (2.10) into the model (2.9), the right-hand side of the equations becomes fractional integrals of order $(q-1)$ on the interval $(0,t)$. Applying a fractional Caputo derivative of order $(q-1)$ on both sides of model (2.9) gives the following fractional differential equations with Caputo derivative

$$\begin{aligned} {}^cD^q u_1(t) &= f_1(u_1(t),u_2(t),u_3(t)), \\ {}^cD^q u_2(t) &= f_2(u_1(t),u_2(t),u_3(t)), \\ {}^cD^q u_3(t) &= f_3(u_1(t),u_2(t),u_3(t)). \end{aligned} \tag{2.11}$$

Hence, the fractional derivatives, when introducing a convolution integral with a power-law memory kernel, are useful to describe memory effects in dynamical models. The decaying rate of the memory kernel depends on fractional-order (q). A lower value of q corresponds to a more slowly decaying memory kernel (long memory). In a sense, the strength (through the "length") of the memory is controlled by fractional-order derivative (q) (Saeedian et al., 2017). As fractional-order derivative (q) tend to one, the influence of memory decreases: the model tends toward a memoryless model (Saeedian et al., 2017). Therefore, the definition of Caputo fractional derivative (2.7) that will be used throughout this thesis includes memory effects, where the state of the model at each time depends on the previous history of the model.

2.3 Important dynamical concepts

The dynamical system approach is used to explore the population dynamics of the eco-epidemiological models presented in this thesis. The main concepts are as follows.

2.3.1 Functional responses

Predator functional response used to model the predation between a population of prey and predators. The functional responses have a crucial part in modelling the prey-predator model (Jawad, 2018). The common functional responses include the following four possible response forms:

- Holling type I predator functional response as the following form (Denny, 2014)

$$f(x) = \beta x, \tag{2.12}$$

  In this functional response, prey consumption rises linearly with prey population density to a threshold level, therefore is called a linear functional response of the predator (Hurkova, 2013). The classical Lotka-Volterra model has used this type of predator functional response. The Holling type-I predator functional response is shown in Fig. 2.1. Fig. 2.1 shows that the Holling type-I capture rate rises linearly with prey density (blue curve), while the Holling type II capture rate approaches saturation gradually (red curve).

- Holling type-II predator functional response as the following form (Holling, 1959)

$$f(x) = \frac{\beta x}{a+x}, \tag{2.13}$$

  The Holling type-II predator functional response assumes that the prey population is a limited resource and that predation converges to a constant when the population of prey increases (Suryanto et al., 2019). The type-II predator functional response requires handling time for each individual of the prey species that

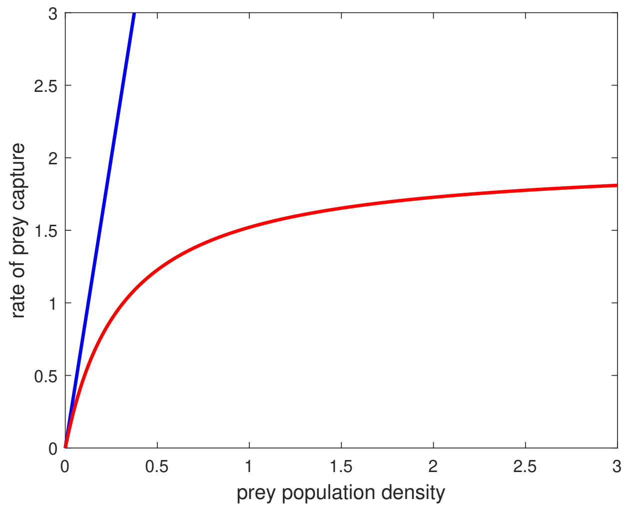

Figure 2.1: Holling type-I predator functional response (blue curve) and Holling type-II (red curve) (modified from (Holling, 1959)).

is consumed, which reduces the available time to search for more prey (Hurkova, 2013). This functional response is typical of predators that specialize on one or a few preys. The Holling type-II predator functional response is shown in Fig. 2.1.

- Holling type-III predator functional response as the following form (Holling, 1959)

$$f(x) = \frac{\beta x^2}{a + x^2}. \tag{2.14}$$

In the Holling type-III predator functional response, prey consumption remains low until a threshold density is reached. The predation rate then increases exponentially until levels out (Hurkova, 2013). Fig. 2.2 shows the Holling type-III predator function response where the predation rate reduces to zero with low levels of prey density.

- Holling type-IV predator functional response (Monod–Haldane) as the following

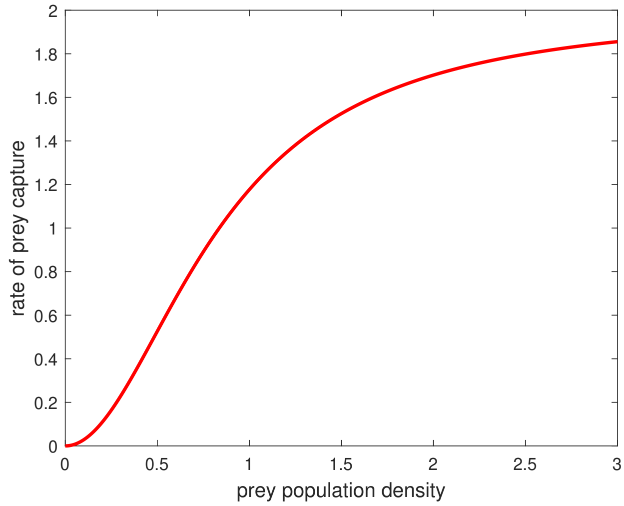

Figure 2.2: Holling type-III predator functional response (modified from (Holling, 1959)).

form (Sokol and Howell, 1981)

$$f(x) = \frac{\beta x}{a + x^2}. \tag{2.15}$$

The Holling type-I, type-II and type-III predator functional responses are monotonic in the first quadrant. Some experiments and observations indicate that non-monotonic response occurs in some prey-predator interactions. For example, when microorganisms are used for waste decomposition (Xiao and Ruan, 2001). To model such non-monotonic response, Sokol and Howell (1981) had proposed the Holling type-IV predator function of the form (2.15) and found that it fits their experimental about prey-predator interactions (Xiao and Ruan, 2001). The non-monotonic curve of Fig. 2.3 shows an upper bound on the rate of predation per individual predator of Holling type-IV predator functional response at some prey population density.

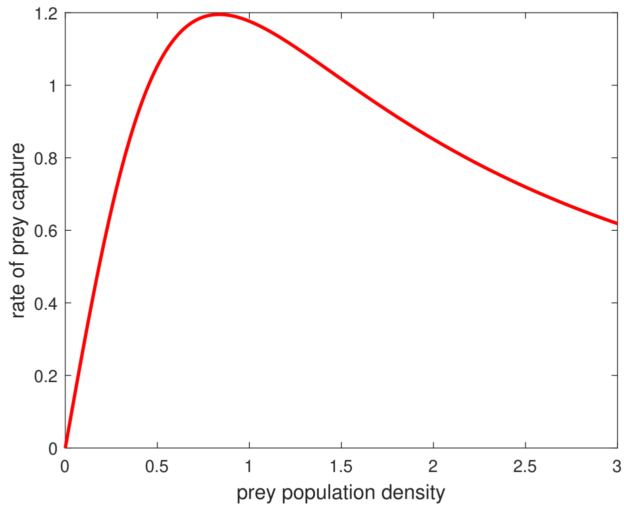

Figure 2.3: Holling type-IV predator functional response (modified from (Sokol and Howell, 1981)).

2.3.2 Incidence rate function

In mathematical models, the transmission of a disease is known as incidence rate function. This function plays a key role in eco-epidemiological models and divided into (Khan et al., 2020):

- The bilinear incidence rate (βxy).
- The nonlinear incidence rate $(\frac{\beta xy}{1+\eta x})$.

Where βxy means infection rate and $\frac{1}{1+\eta x}$ measures the inhibition effect from the behavioral change of the susceptible population when their number increases or from the crowding effect of the infected population. The nonlinear incidence rate is more appropriate than the bilinear incidence rate because it includes the behavioral change and crowding effect of the infected population Peng et al. (2018).

2.3.3 Harvesting of populations

The harvesting of populations are commonly practiced in fishery, forestry and management of wildlife and are divided into (Gupta et al., 2015):

- linear harvesting, $H(x) = H$, where a constant number of individuals are harvested per unit of time.
- Proportional harvesting, $H(x) = Ex$.
- Holling type-II harvesting (Michaelis-Menten), $H(x) = \frac{qEx}{m_1 E + m_2 x}$.

Incorporating harvesting to an eco-epidemiological model is expected to provide a more realistic model since for a number of prey populations some form of harvesting in the ecosystem is known to be available.

2.3.4 Logistic growth

Logistic growth is one of the most common components of population-growth and has been used to model many different biological models (Pearl and Slobodkin, 1976). Logistic growth can be described as the self-limitation of a population x as follows

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right), \quad x(0) = x_0, \tag{2.16}$$

where r is the growth rate and k is the carrying capacity. The solution of the logistic growth differential equation (2.16) shows the exponential growth rate until a population reaches carrying capacity (k) as indicated in Fig. 2.4.

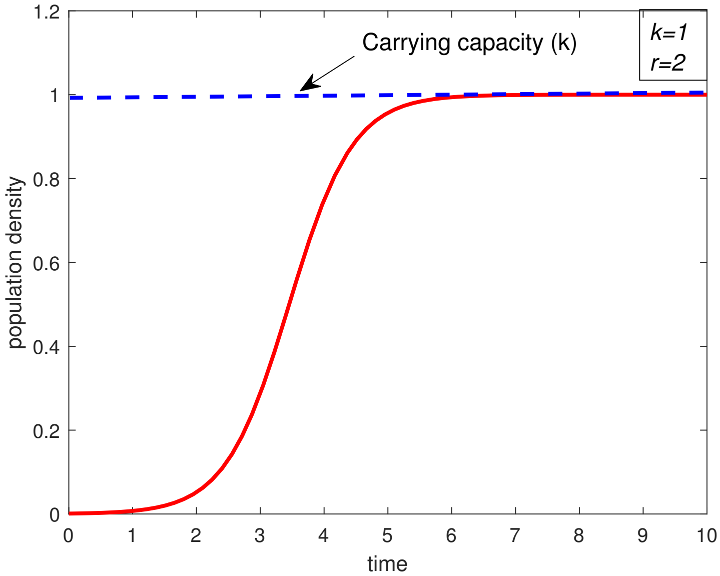

Figure 2.4: Logistic growth curve (modified from (Pearl and Slobodkin, 1976)).

2.3.5 Fractional-order model

In this subsection, some lemmas and theorems of fractional-order dynamical models that will be used in Chapters 4, 5, 6, 7, 8, 9 and 10 are given.

To study the existence and uniqueness of the solutions of the fractional-order eco-epidemiological models the following lemma is needed.

Lemma 2.1. *(Li et al., 2010) Consider the following Caputo fractional differential model:*

$$^{c}D^{q}x(t) = f(t,x) \quad t > 0, x(0) = x_0, \tag{2.17}$$

where $q \in [0,1)$, $f : (0,\infty) \times \Psi \to \mathbb{R}^n$, $\Psi \in \mathbb{R}^n$, *if* $f(t,x)$ *satisfies the locally Lipschitz condition with respect to x, then there exists a unique solution of* (2.17) *on* $[0,\infty) \times \Psi$.

To investigate the non-negativity of the solutions of the fractional order eco-epidemiological models the following two theorems are needed.

Positivity property (Cresson and Szafrańska, 2017): A fractional differential model of the form (2.17) satisfies the positivity property if for all initial conditions (x_0), the solution passing through x_0 at time $t = 0$ remains positive for all $t > 0$.

Theorem 2.1. *(Cresson and Szafrańska, 2017) A model of the form* (2.17) *with* $q = 1$ *satisfies the positivity property if and only if for all* $i = 1, \ldots, m,\ f_i(x) \geq 0$ *for all* $x \in \mathbb{R}^m_+$ *such that* $x_i = 0$.

Theorem 2.2. *(Cresson and Szafrańska, 2017) Let* f *be locally Lipschitz. Assume that* f *satisfies condition of Theorem 2.1 then the fractional differential equation satisfies the positivity property.*

The following lemmas and corollary are needed to study the boundedness of the solutions of the fractional-order eco-epidemiological models.

Lemma 2.2. *(Choi et al., 2014) (Standard comparison theorem of fractional-order) Suppose that* $u \in C_p(\mathbb{R}^+, \mathbb{R})$ *satisfies*

$$^cD^q u(t) \leq \lambda u(t) + d, \quad u(0) = u_0,$$

where λ, $d \in \mathbb{R}$. *Then one has*

$$u(t) \leq u(0) E_q\left(\lambda (t)^q\right) + d(t)^q E_{q,q+1}(\lambda (t)^q).$$

Lemma 2.3. *(Choi et al., 2014) Let* $q \in (0,1)$ *and* $\lambda < 0$. *Then,* $E_q(\lambda t^q)$ *tend monotonically to zero as* $t \to \infty$.