Final Examination
2018/2019 Academic Session

June 2019

JIM417 – Partial Differential Equations
(Persamaan Pembezaan Separa)

Duration : 3 hours
(*Masa: 3 jam*)

Please check that this examination paper consists of **TEN (10)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEPULUH (10)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini].*

Instructions : Answer **ALL** questions.

*[**Arahan** : Jawab **SEMUA** soalan].*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].

1. (a). Consider a given piecewise continuous function

$$f(t)=\begin{cases} t^2+3t, & 0\le t<7, \\ t^2+3t+te^{-t}+5, & t\ge 7 \end{cases}$$

Rewrite the following function in terms of unit step function. Then, find its Laplace transform.

(50 marks)

(b). Solve the following initial value problem by using Laplace transform

$$y''-y'-2y=\sin t,$$

subject to the conditions

$$y(0)=1,\ y'(0)=0$$

(50 marks)

2. Consider a given function of period of 2π ,

$f(x)=x^2$, over the interval $-3\pi<x<3\pi$.

(a). Sketch the graph of $f(x)$ for $-3\pi<x<3\pi$.

(15 marks)

(b). Show that the Fourier series for $f(x)$ in the interval of $-\pi<x<\pi$ is

$$\frac{\pi^2}{3}-4\left[\cos x-\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x-...\right]$$

(55 marks)

(c). By giving an appropriate value of x , show that

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...$$

(30 marks)

3. Given a partial differential equation

$$x\frac{\partial y}{\partial x}+\frac{\partial y}{\partial t}+ay=bx^2, \quad x>0, \quad t>0, \quad a \text{ and } b \text{ are constants}$$

with initial and boundary conditions

$$y(0,t)=0 \quad \text{and} \quad y(x,0)=0.$$

(a). By using Laplace transform, show that

$$Y(x,s)=\frac{bx^2}{s(s+a+2)}.$$

(70 marks)

(b). Find the inverse Laplace transform of (a).

(30 marks)

4. (a). Classify each of the following partial differential equations as hyperbolic, elliptic, or parabolic:

(i). $u_{xx}+u_{xy}-2u_{yy}-3u_x-6u_y=9(2x-y)$

(ii). $x^2u_{xx}-2xy\,u_{xy}+y^2u_{yy}=e^x$

(20 marks)

(b). Given the Tricomy equation

$$u_{xx}+x\,u_{yy}=0.$$

Show that Tricomy equation is an elliptic partial differential equation. Find its canonical form.

(40 marks)

(c). Show that the one-dimensional heat equation

$$\alpha\,u_{xx}=u_t.$$

is parabolic, choose the appropriate characteristic variables, and write the equation in equivalent canonical form.

(40 marks)

…4/-

5. Find the solution to the wave problem

$$c^2 u_{xx} = u_{tt}, \text{ for constant } c > 0,\ 0 < x < 1 \text{ and } t \geq 0$$

$$u_x(0,t) = 0,\ u(1,t) = 0,\ t > 0$$

$$u(x,0) = \cos\left(\frac{3\pi}{2}x\right) - 3\cos\left(\frac{7\pi}{2}x\right), \quad u_t(x,0) = 0, \quad 0 < x < 1$$

(100 marks)

1. (a). Pertimbangkan fungsi cebis demi cebis selanjar

$$f(t)=\begin{cases} t^2+3t, & 0\le t<7, \\ \\ t^2+3t+te^{-t}+5, & t\ge 7 \end{cases}$$

Tulis semula fungsi tersebut sebagai fungsi langkah unit. Seterusnya, cari jelmaan Laplace bagi fungsi tersebut.

(50 markah)

(b). Selesaikan masalah nilai awal dengan menggunakan jelmaan Laplace

$$y''-y'-2y=\sin t,$$

tertakluk pada syarat

$$y(0)=1,\ y'(0)=0$$

(50 markah)

2. Pertimbangkan fungsi yang diberikan dengan kala 2π ,

$f(x)=x^2$, pada selang $-3\pi<x<3\pi$.

(a). Lakarkan graf $f(x)$ untuk $-3\pi<x<3\pi$.

(15 markah)

(b). Tunjukkan bahawa siri Fourier bagi $f(x)$ pada selang $-\pi<x<\pi$ adalah

$$\frac{\pi^2}{3}-4\left[\operatorname{kos} x-\frac{1}{2^2}\operatorname{kos}2x+\frac{1}{3^2}\operatorname{kos}3x-...\right]$$

(55 markah)

(c). Dengan memberikan nilai x yang sesuai, tunjukkan bahawa

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...$$

(30 markah)

3. Diberi persamaan pembezaan separa

$$x\frac{\partial y}{\partial x}+\frac{\partial y}{\partial t}+ay=bx^2, \quad x>0, \quad t>0, \quad a \text{ dan } b \text{ pemalar}$$

dengan syarat awal dan syarat sempadan

$$y(0,t)=0 \text{ and } y(x,0)=0.$$

(a). Dengan menggunakan jelmaan Laplace, tunjukkan bahawa

$$Y(x,s)=\frac{bx^2}{s(s+a+2)}.$$

(70 markah)

(b). Cari jelmaan Laplace songsang bagi (a).

(30 markah)

4. (a). Klasifikasi setiap persamaan pembezaan separa berikut sebagai hiperbolik, eliptik, atau parabola:

(i). $u_{xx}+u_{xy}-2u_{yy}-3u_x-6u_y=9(2x-y)$

(ii). $x^2u_{xx}-2xy\,u_{xy}+y^2u_{yy}=e^x$

(20 markah)

(b). Diberi persamaan Tricomy

$$u_{xx}+x\,u_{yy}=0.$$

Tunjukkan bahawa persamaan Tricomy ialah sejenis persamaan pembezaan separa eliptik. Cari bentuk berkanun bagi persamaan tersebut.

(40 markah)

(c). Tunjukkan bahawa persamaan haba satu dimensi

$$\alpha\, u_{xx}=u_t.$$

ialah parabolik, pilih pembolehubah cirian yang sesuai, dan tulis persamaan tersebut dalam bentuk berkanun.

(40 markah)

5. Cari penyelesaian bagi masalah gelombang

$$c^2 u_{xx} = u_{tt}, \text{ untuk pemalar } c > 0,\ 0 < x < 1 \text{ dan } t \geq 0$$

$$u_x(0,t) = 0,\ u(1,t) = 0,\ t > 0$$

$$u(x,0) = \operatorname{kos}\left(\frac{3\pi}{2}x\right) - 3\operatorname{kos}\left(\frac{7\pi}{2}x\right), \quad u_t(x,0) = 0, \quad 0 < x < 1$$

(100 markah)

Formula

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}\,.$$

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

with

$$a_o = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\ n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx,\ n = 1, 2, 3, \ldots$$

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

with

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\ n = 1, 2, 3, \ldots$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

with

$$b_n = \frac{2}{L}\int_0^L f(x)\left(\frac{n\pi x}{L}\right)dx,\ n = 1,\ 2,\ 3,\ ...$$

$$f(x) = \frac{1}{2}\sum_{-\infty}^{\infty} c_n e^{inx}$$

with

$$c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx,\ n = 0,\ \pm 1, \pm 2,\ ...$$

$\dfrac{d^2 y}{dx^2} - \alpha^2 y = 0$ has solution

$$y = Ae^{\alpha x} + Be^{-\alpha x} \text{ or } C \cosh\ \alpha x + D \sinh\ \alpha x.$$

$\dfrac{d^2 y}{dx^2} + \alpha^2 y = 0$ has solution

$$y = A \cos \alpha x + B \sin \alpha x.$$

$r^2 \dfrac{d^2 R}{dr^2} + r\dfrac{dR}{dr} - n^2 R = 0$ has solution

$$R_n = C_n r^n + \frac{D_n}{r^n}$$

$r\dfrac{d^2 R}{dr^2} + r\dfrac{dR}{dr} = 0$ has solution

$$R = A + B\,\ell n\, r.$$

$$\text{L}\ \left[e^{\alpha t} f(t)\right] = F(s-\alpha).$$

$$\text{L}\ \left\{H(t-a)\right\} = \frac{e^{-as}}{s}$$

$$\text{L}\ \left\{f(t-a)H(t-a)\right\} = e^{-as}F(s)$$

$$\text{L}\ \left[f^n(t)\right] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\mathrm{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n}\left[F(s)\right]$$

$$\mathrm{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$$

$$\mathrm{L}^{-1}\left\{F(s)G(s)\right\} = \int_0^t f(u)\,g(t-u)\,du = f * g$$

Laplace Transforms

$f(t)$	$\mathrm{L}\left\{f(t)\right\} = F(s)$
1	$\frac{1}{s}$
$t^n,\ n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$t\cos bt$	$\frac{s^2-a^2}{\left(s^2+b^2\right)^2}$
$t\sin bt$	$\frac{2bs}{\left(s^2+b^2\right)^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$

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