Final Examination
2018/2019 Academic Session

June 2019

JIM414 – Statistical Inference
(Pentaabiran Statistik)

Duration: 3 hour
(Masa: 3 jam)

Please check that this examination paper consists of **ELEVEN (11)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEBELAS (11)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini].*

Instructions : Answer **ALL** questions.

*[**Arahan** : Jawab **SEMUA** soalan].*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].

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1. (a). An engineer wants to estimate the mean yield from a chemical process that resulted in measurements X_1, X_2, X_3 from 3 runs of an experiment. Given that $E[X_i] = \mu$, Var$(X_i) = \sigma^2$, and Cov$(X_i, X_j) = 0$, $i = 1,2,3$ and $j = 1,2,3$. Consider the following estimators of μ:

$$T_1 = \frac{X_1 + 2X_2 + X_3}{4},$$

$$T_2 = \frac{3X_1 + X_2 + X_3}{5},$$

$$T_3 = \frac{2X_1 + X_2 + 2X_3}{5}.$$

(i). Determine whether these estimators are unbiased.

(ii). Calculate the variances of these three estimators and verify that they are all larger than the variance of $\bar{X} = \dfrac{X_1 + X_2 + X_3}{3}$.

(iii). What knowledge of X_1, X_2, X_3 is needed to prove that $\bar{X}$ is the uniformly minimum variance unbiased estimator (UMVUE)?

(30 marks)

(b). Given that $f(x;\theta) = \theta e^{-\theta x}$, $x > 0$. Find the maximum likelihood estimator of $E(X)$.

(20 marks)

(c). Let $X_1, X_2, \ldots X_n$ be a random sample from a population with distribution $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $\theta > 0$. Let $T(X_1, X_2, \ldots, X_n) = \frac{1}{n}\sum_{i=1}^{n} \ln X_i$.

(i). What is estimated by $T(X_1, X_2, \ldots, X_n)$?

(ii). Obtain the Fisher information of the parameter in (i).

(iii). Did Var$\left(T(X_1, X_2, \ldots, X_n)\right)$ attain the Cramer-Rao lower bound?

(iv). Is $T(X_1, X_2, \ldots, X_n)$ an UMVUE for the parameter in (i)?

(50 marks)

2. (a). An efficiency expert wants to determine the mean time to drill three holes in a metal vice. What is the sample size needed to obtain a difference between sample and population means within 15 seconds? Assume that σ = 40 seconds.

(20 marks)

(b). $X_1, X_2, \ldots X_n$ is a random sample from a population distributed as Normal(μ_1, σ_1^2) and $Y_1, Y_2, \ldots Y_m$ is a random sample from a population distributed as Normal(μ_2, σ_2^2). Determine the 100(1 – α)% confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$.

(30 marks)

(c). Let 25 measurements of resistance to breakage of a specific alloy result in $\bar{x} = \frac{1}{25}\sum_{i=1}^{25} x_i = 11.1$ and $s = \sqrt{\frac{1}{24}\sum_{i=1}^{25}(x_i - \bar{x})^2} = 3.4$. Suppose this sample is taken from a population with Normal(μ, σ^2) distribution. Obtain the 90% confidence interval for $\mu + \sigma$.

(50 marks)

3. (a). (i). Given $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$ as a statistic for sample variance. Let the sample size be $n = 2$. Find c so that $S^2 = c\left(X_1 - X_2\right)^2$.

(ii). $X_1, X_2, \ldots X_{10}$ and $Y_1, Y_2, \ldots Y_{10}$ are two independent random samples taken from Normal$(0,1)$ and Normal$(1,4)$ distributions, repectively. Let $\overline{X} = \frac{1}{10}\sum_{i=1}^{10} X_i$ and $\overline{Y} = \frac{1}{10}\sum_{j=1}^{10} Y_j$ be the mean samples for the two random samples. Find $P\left(\overline{X} > \overline{Y}\right)$.

(50 marks)

(b). $Y_1 < Y_2$ are order statistics taken from a sample of size 2 from the Uniform$(0,1)$ distribution. Find $E\left(Y_1\right)$.

(20 marks)

(c). Given

$$f_n\left(x\right) = \begin{cases} 1, & x = 2 + \frac{1}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

Get the limiting distribution of X_n.

(30 marks)

4. (a). Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a sample of size 4 taken from a population distributed as Uniform$(0,\theta)$. The hypothesis $H_0 : \theta = 1$ is rejected and $H_1 : \theta > 1$ accepted when $Y_4 \geq c$.

(i). Find c such that the significance level of this test is 0.05.

(ii). Determine the power function of this test.

(50 marks)

(b). $X_1, X_2, \ldots X_n$ is a random sample from a population with a distribution characterized by the density function $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$. Find the critical region of size-α to test $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

(20 marks)

(c). State the test that uses the minimum complete and sufficient statistic. How was the statistic used?

(30 marks)

5. (a). Verify 1. (c). (iv). using Lehmann-Scheffe' Theorem.

(25 marks)

(b). Refer 2. (c). Obtain the 90% confidence region for (μ, σ).

(25 marks)

(c). What is the difference between a limiting distribution and an asymptotic distribution? Explain this difference with an example.

(25 marks)

(d). $X_1, X_2, \ldots X_n$ is a random sample from a population with Normal$(\theta, 1)$ distribution. We want to versus $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. The maximum likelihood estimator of $\widehat{\theta}$ is given by $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the likelihood ratio test.

(25 marks)

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1. (a). Seorang jurutera ingin menganggar min hasil daripada suatu proses kimia yang memberi sukatan X_1, X_2, X_3 daripada 3 larian suatu eksperimen. Diberikan $E[X_i] = \mu$, Var$(X_i) = \sigma^2$, dan Kov$(X_i, X_j) = 0$, $i = 1,2,3$ dan $j = 1,2,3$. Pertimbangkan penganggar-penganggar bagi μ berikut:

$$T_1 = \frac{X_1 + 2X_2 + X_3}{4},$$

$$T_2 = \frac{3X_1 + X_2 + X_3}{5},$$

$$T_3 = \frac{2X_1 + X_2 + 2X_3}{5}.$$

(i). Tentukan sama ada penganggar-penganggar ini saksama.

(ii). Hitungkan varians ketiga-tiga penganggar ini dan sahkan bahawa kesemuanya adalah lebih besar daripada varians $\bar{X} = \frac{X_1 + X_2 + X_3}{3}$.

(iii). Pengetahuan apakah yang diperlukan terhadap X_1, X_2, X_3 supaya kita dapat membuktikan bahawa $\bar{X}$ adalah penganggar saksama bervarians minimum secara seragam (PSVMS)?

(30 markah)

(b). Diberikan $f(x;\theta) = \theta e^{-\theta x}$, $x > 0$. Cari penganggar kebolehjadian maksimum bagi $E(X)$.

(20 markah)

(c). Andaikan $X_1, X_2, \ldots X_n$ sebagai suatu sampel rawak daripada populasi bertaburan $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $\theta > 0$. Andaikan $T(X_1, X_2, \ldots, X_n) = \frac{1}{n}\sum_{i=1}^{n} \ln X_i$.

(i). Apakah yang dianggar oleh $T(X_1, X_2, \ldots, X_n)$?

(ii). Dapatkan maklumat Fisher bagi parameter dalam (i).

(iii). Adakah Var$(T(X_1, X_2, \ldots, X_n))$ mencapai batas bawah Cramer-Rao?

(iv). Adakah $T(X_1, X_2, \ldots, X_n)$ suatu PSVMS bagi parameter dalam (i)?

(50 markah)

2. (a). Seorang pakar kecekapan ingin menentukan purata masa yang diperlukan untuk menggerudi tiga lubang di dalam suatu pencengkam logam. Berapakah saiz sampel yang perlu dicerap supaya perbezaan min sampel ini dengan min populasi di dalam lingkungan 15 saat? Anggapkan σ = 40 saat.

(20 markah)

(b). $X_1, X_2, \ldots X_n$ adalah suatu sampel rawak daripada populasi bertaburan Normal(μ_1, σ_1^2) dan $Y_1, Y_2, \ldots Y_m$ adalah suatu sampel rawak daripada populasi bertaburan Normal(μ_2, σ_2^2). Tentukan selang keyakinan 100(1 – α)% bagi $\frac{\sigma_1^2}{\sigma_2^2}$.

(30 markah)

(c). Andaikan 25 ukuran ketahanan aloi tertentu kepada kepatahan menghasilkan $\bar{x} = \frac{1}{25}\sum_{i=1}^{25} x_i = 11.1$ dan $s = \sqrt{\frac{1}{24}\sum_{i=1}^{25}(x_i - \bar{x})^2} = 3.4$. Andaikan sampel ini dicerap daripada populasi bertaburan Normal(μ, σ^2). Dapatkan selang keyakinan 90% bagi $\mu + \sigma$.

(50 markah)

3. (a). (i). Diberikan $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$ sebagai statistik bagi varians sampel. Andaikan saiz sampel $n = 2$. Cari c supaya $S^2 = c\left(X_1 - X_2\right)^2$.

(ii). $X_1, X_2, \ldots X_{10}$ dan $Y_1, Y_2, \ldots Y_{10}$ adalah dua sampel rawak tak bersandar, masing-masing dicerap daripada taburan Normal$(0,1)$ dan Normal$(1,4)$. Andaikan $\bar{X} = \frac{1}{10}\sum_{i=1}^{10} X_i$ dan $\bar{Y} = \frac{1}{10}\sum_{j=1}^{10} Y_j$ adalah min-min sampel bagi kedua-dua sampel rawak tersebut. Dapatkan $P\left(\bar{X} > \bar{Y}\right)$.

(50 markah)

(b). $Y_1 < Y_2$ adalah statistik tertib daripada sampel bersaiz 2 yang dicerap daripada taburan Seragam$(0,1)$. Dapatkan $E\left(Y_1\right)$.

(20 markah)

(c). Diberikan

$$f_n\left(x\right) = \begin{cases} 1, & x = 2 + \frac{1}{n}, \\ 0, & \text{di tempat lain.} \end{cases}$$

Dapatkan taburan penghad bagi X_n.

(30 markah)

4. (a). Andaikan $Y_1 < Y_2 < Y_3 < Y_4$ adalah statistik tertib suatu sampel bersaiz 4 yang dicerap daripada populasi yang bertaburan Seragam$(0,\theta)$. Hipotesis $H_0 : \theta = 1$ ditolak dan $H_1 : \theta > 1$ diterima apabila $Y_4 \geq c$.

(i). Cari nilai c supaya aras keertian ujian ini ialah 0.05.

(ii). Tentukan fungsi kuasa ujian ini.

(50 markah)

(b). $X_1, X_2, \ldots X_n$ adalah sampel rawak daripada populasi bertaburan $f(x;\theta) = \theta x^{\theta-1},\ 0 < x < 1$. Dapatkan rantau genting paling berkuasa saiz-α untuk menguji $H_0 : \theta = 1$ menentang $H_1 : \theta = 2$.

(20 markah)

(c). Nyatakan ujian yang menggunakan statistik cukup minimum lengkap. Bagaimanakah statistik tersebut digunakan?

(30 markah)

5. (a). Sahkan 1. (c). (iv). dengan menggunakan Teorem Lehmann-Scheffe'.

(25 markah)

(b). Pertimbangkan semula 2. (c). Dapatkan rantau keyakinan 90% bagi (μ, σ).

(25 markah)

(c). Apakah perbezaan di antara taburan penghad dengan taburan asimptot? Jelaskan perbezaan ini dengan menggunakan satu contoh.

(25 markah)

(d). $X_1, X_2, \ldots X_n$ adalah sampel rawak daripada populasi bertaburan Normal$(\theta, 1)$. Kita ingin menguji $H_0 : \theta = 0$ menentang $H_1 : \theta > 0$. Diberikan penganggar kebolehjadian maksimum bagi $\hat{\theta}$ ialah $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Dapatkan statistik ujian nisbah kebolehjadian.

(25 markah)

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Formulas

1. $\lim_{n\to\infty} P\left(\left|X_n - c\right| < \varepsilon\right) = 1$, for any $\varepsilon > 0$.

2. $\prod_{i=1}^{n} f\left(x_i;\theta\right) = k_1\left[u_1\left(x_1, x_2, \ldots, x_n\right);\theta\right]k_2\left(x_1, x_2, \ldots, x_n\right)$

3. $I\left(\theta\right) = E\left[\left(\frac{\partial \log f\left(X;\theta\right)}{\partial\theta}\right)^2\right] = -E\left(\frac{\partial^2 \log f\left(X;\theta\right)}{\partial\theta^2}\right)$

4. $Var\left(Y\right) \geq \frac{\left[k'\left(\theta\right)\right]^2}{nI\left(\theta\right)}$

5. Let $Y_1 < Y_2 < \ldots < Y_n$. $g(y_1, y_2, \ldots, y_n) = n!f(y_1)f(y_2)\ldots f(y_n)$, $y_1 < y_2 < \ldots < y_n$.

6. Let $Y_1 < Y_2 < \ldots < Y_n$. $g_k\left(y_k\right) = \frac{n!}{(k-1)!(n-k)!}\left[F\left(y_k\right)\right]^{k-1}\left[1 - F\left(y_k\right)\right]^{n-k} f\left(y_k\right)$

7. Let $Y_1 < Y_2 < \ldots < Y_n$.

$$g_{ij}\left(y_i, y_j\right) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\left[F\left(y_i\right)\right]^{i-1}\left[F\left(y_j\right) - F\left(y_i\right)\right]^{(j-i-1)}$$
$$\times\left[1 - F\left(y_j\right)\right]^{n-j} f\left(y_i\right) f\left(y_j\right), y_i < y_j.$$

8. $f(x) = p^x(1-p)^{1-x}$, $x = 0, 1$, $0 < p < 1$. $E(X) = p$, $\text{Var}(X) = p(1-p)$. $m(t) = 1 - p + pe^t$.

9. $f\left(x\right) = \binom{n}{x} p^x\left(1-p\right)^{n-x}$, $x = 0, 1, 2, \ldots, n$, $0 < p < 1$. $E(X) = np$, $\text{Var}(X) = np(1-p)$.

$m(t) = (1 - p + pe^t)^n$.

10. $f(x) = p\,(1-p)^x$, $x = 0, 1, 2, \ldots$, $0 < p < 1$. $E\left(X\right) = \frac{1-p}{p}$, $\text{Var}\left(X\right) = \frac{1-p}{p^2}$.

$m\left(t\right) = \frac{p}{1-\left(1-p\right)e^t}$.

11. $f\left(x\right) = \binom{r+x-1}{x} p^r\left(1-p\right)^x$, $x = 0, 1, 2, \ldots$, $0 < p < 1$. $E\left(X\right) = \frac{r\left(1-p\right)}{p}$,

$\text{Var}\left(X\right) = \frac{r\left(1-p\right)}{p^2}$. $m\left(t\right) = \left(\frac{p}{1-\left(1-p\right)e^t}\right)^r$, $t < -\log(1-p)$.

12. $f\left(x\right) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots, \lambda \geq 0$. $E\left(X\right) = \text{Var}(X) = \lambda$. $m(t) = e^{\lambda\left(e^t - 1\right)}$.

13. $f(x) = \frac{1}{b-a}, a < x < b.$ $E(X) = \frac{a+b}{2},$ $\text{Var}(X) = \frac{(b-a)^2}{12}.$ $m(t) = \frac{e^{bt} - e^{at}}{(b-a)t}.$

14. $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \sigma > 0.$ $E(X) = \mu,$ $\text{Var}(X) = \sigma^2.$ $m(t) = \exp\left[\mu t + \frac{1}{2}\sigma^2 t\right].$

15. $f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1},\ 0 \le x \le 1,\ \alpha > 0,\ \beta > 0.$ $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$

16. $f(x) = \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}, x \ge 0.$ $E(X) = k,$ $\text{Var}(X) = 2k.$ $m_X(t) = \left(\frac{1}{1-2t}\right)^{\frac{k}{2}}, t < \frac{1}{2}.$

17. $f(x) = \frac{1}{\beta} e^{-x/\beta}, x \ge 0, \beta > 0.$ $E(X) = \beta,$ $\text{Var}(X) = \beta^2.$ $m(t) = \frac{1}{1-\beta t}, t < \frac{1}{\beta}.$

18. $f(x) = \frac{1}{\Gamma(\alpha)\beta} x^{\alpha-1} e^{-x/\beta}, x \ge 0, \alpha > 0, \beta > 0.$ $E(X) = \alpha\beta,$ $\text{Var}(X) = \alpha\beta^2.$

$m(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha}, t < \frac{1}{\beta}.$

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