Final Examination
2018/2019 Academic Session

June 2019

JIM413 – Differential Equations II
(Persamaan Pembezaan II)

Duration : 3 hours
(*Masa: 3 jam*)

Please check that this examination paper consists of **ELEVEN (11)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEBELAS (11)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini].*

Instructions : Answer **ALL** questions.

*[**Arahan** : Jawab **SEMUA** soalan].*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].

1. Given the differential equation

$$x^2 y'' + xy' + \left(x^2 - 1\right) y = 0$$

   (a). Show that $x = 0$ is a regular singular point of the above differential equation.

   (10 marks)

   (b). Obtain the indicial equation and the exponents.

   (10 marks)

   (c). By considering the positive exponent, use the method of Frobenius to show that one of the solution is

$$y_1 = a_0 x - \frac{1}{8} a_0 x^3 + \frac{1}{192} a_0 x^5 + \ldots$$

   (40 marks)

   (d). The initial values are given as

$$y\left(0\right) = 0,\ y'\left(0\right) = \frac{1}{2}.$$

   Obtain the particular solution for y_1.

   (5 marks)

   (e). Hence, use the method of reduction of order to obtain the second solution which has the logarithmic term.

   (35 marks)

2. (a). State two properties of Sturm-Liouville Problem.

   (10 marks)

   (b). Given a second-order differential equation

$$y'' + 2xy' + (x + \lambda) y = 0$$

   Show that the above differential equation can be transformed into the Sturm-Liouville problem by multiplying with e^{x^2}.

   (10 marks)

(c). Given that the Sturm-Liouville problem

$$y'' + \lambda y = 0$$

is continuous for $0 < x < 1$ and has the boundary conditions

$$y'(0) = 0, \qquad hy(1) + y'(1) = 0 \qquad (h > 0)$$

with positive eigenvalues.

(i). Verify that the eigenvalues are $\lambda_n = \beta_n^2$ and the associated eigenfunctions are $y_n(x) = \cos \beta_n x$ where $n \geq 1$ and β_n is the nth positive root of $\tan x = \dfrac{h}{x}$.

(ii). Hence, represent the function $f(x) = x$ as a series of eigenfunctions in the form

$$f(x) = \sum_{n=1}^{\infty} c_n y_n$$

where y_n is as given in (i).

(80 marks)

3. (a). Given the autonomous differential equation

$$\frac{dx}{dt} = 7x - x^2 - 10$$

(i). Find all the critical points and write the equilibrium solutions.

(ii). Sketch the phase diagram and the solution curves.

(iii). State the stability of the critical points and describe the long term behavior of the solutions.

(40 marks)

(b). The eigenvalue-eigenvector method can be used to investigate the critical point $(0,0)$ of a linear system

$$x'(t) = ax + by$$
$$y'(t) = cx + dy$$

with constant-coefficient matrix $\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The behavior of the critical point depends on the two nonzero eigenvalues λ_1 and λ_2 of A, which can be classified into five cases:

(i). Real and unequal eigenvalues with the same signs.
(ii). Real and unequal eigenvalues with opposite signs.
(iii). Real and equal eigenvalues, with two linearly independant eigenvectors.
(iv). Complex conjugate eigenvalues, $\lambda = p \pm qi$.
(v). Pure imaginary eigenvalues, $\lambda = \pm qi$.

For each case, sketch the example of phase portrait; determine the type of critical point (nodal sink or source, spiral sink or source, saddle point and center) and its stability (stable, asymptotically stable or unstable).

(60 marks)

4. Suppose that the populations $x(t)$ and $y(t)$ satisfy the competition system

$$\frac{dx}{dt} = 10x - \frac{1}{2}x^2 - xy$$
$$\frac{dy}{dt} = 16y - \frac{1}{2}y^2 - xy$$

(a). Find all the critical points.

(25 marks)

(b). Use the Jacobian matrix to linearize the above system. Consequently, obtain the eigenvalues for each critical point.

(45 marks)

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(c). Determine the type and the stability of the critical points.

(15 marks)

(d). From the results in (c)., sketch the phase plane portraits for the above system. Give comment on the interaction or competition between the two populations.

(15 marks)

5. (a). Discuss two types of errors, namely the round-off error and the truncation error that occurred in numerical solution of ordinary differential equation. Give one suggestion for each type of error in order to improve the accuracy of the results.

(10 marks)

(b). A population $y(t)$ that satisfied the logistic equation is modelled by

$$\frac{dy}{dt} = 0.0004y^2 - 0.06y$$

If the initial population is 100 at time $t = 0$, use the fourth-order Runge-Kutta method with step size $h = 1$ year to approximate the number of population after 2 years.

(50 marks)

(c). Consider a linear system

$$x' = 2x + 3y, \quad x(0) = 1$$
$$y' = 2x + y, \quad y(0) = -1$$

(i). Approximate the values of $x(0.2)$ and $y(0.2)$ by the improved Euler method with a single step of size $h = 0.2$.

(ii). The exact solutions are given by $x(t) = e^{-t}$ and $y(t) = -e^{-t}$. Compare the approximate values in (i). with the exact values by calculating the percent relative error.

(40 marks)

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1. Diberi persamaan pembezaan

$$x^2 y'' + xy' + \left(x^2 - 1\right) y = 0$$

(a). Tunjukkan bahawa $x = 0$ ialah titik singular sekata bagi persamaan pembezaan di atas.

(10 markah)

(b). Dapatkan persamaan indeksan serta eksponen.

(10 markah)

(c). Pertimbangkan eksponen bernilai positif, guna kaedah Frobenius untuk tunjukkan bahawa salah satu penyelesaian ialah

$$y_1 = a_0 x - \frac{1}{8} a_0 x^3 + \frac{1}{192} a_0 x^5 + \ldots .$$

(40 markah)

(d). Diberi nilai awal

$$y\left(0\right) = 0, \; y'\left(0\right) = \frac{1}{2}$$

Dapatkan penyelesaian khusus bagi y_1.

(5 markah)

(e). Seterusnya, guna kaedah penurunan peringkat untuk memperolehi penyelesaian kedua yang mempunyai terma logaritma.

(35 markah)

2. (a). Nyatakan dua sifat bagi masalah Sturm-Liouville.

(10 markah)

(b). Diberi persamaan pembezaan berperingkat dua

$$y'' + 2xy' + (x + \lambda) y = 0$$

Tunjukkan bahawa persamaan pembezaan di atas dapat dijelmakan kepada masalah Sturm-Liouville dengan mendarabkannya dengan e^{x^2}.

(10 markah)

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(c). Diberi bahawa masalah Sturm-Liouville

$$y'' + \lambda y = 0$$

adalah selanjar bagi $0 < x < 1$ dan mempunyai syarat sempadan

$$y'(0) = 0, \quad hy(1) + y'(1) = 0 \qquad (h > 0)$$

dengan nilai eigen positif.

(i). Buktikan bahawa nilai eigen ialah $\lambda_n = \beta_n^2$ dan fungsi eigen sepadan ialah $y_n(x) = \cos \beta_n x$ di mana $n \geq 1$ dan β_n ialah nth punca positif bagi $\tan x = \dfrac{h}{x}$.

(ii). Seterusnya, persembahkan fungsi $f(x) = x$ sebagai suatu siri fungsi eigen dalam bentuk

$$f(x) = \sum_{n=1}^{\infty} c_n y_n$$

di mana y_n adalah diberi seperti di (i).

(80 markah)

3. (a). Diberi persamaan pembezaan autonomous

$$\frac{dx}{dt} = 7x - x^2 - 10$$

(i). Dapatkan semua titik genting dan tuliskan penyelesaian keseimbangannya.

(ii). Lakarkan gambarajah fasa dan lengkung penyelesaian.

(iii). Tentukan kestabilan titik genting dan huraikan perilaku jangka panjang penyelesaian.

(40 markah)

(b). Kaedah nilai eigen-fungsi eigen boleh digunakan untuk mengkaji titik genting (0,0) bagi suatu sistem linear

$$x'(t) = ax + by$$
$$y'(t) = cx + dy$$

dengan matriks pekali-pemalar $\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Sifat titik genting bergantung kepada kedua-dua nilai eigen tak sifar λ_1 dan λ_2 bagi A, di mana ia boleh diklasifikasikan kepada lima kes:

(i). Nilai eigen nyata dan berlainan dengan tanda yang sama.

(ii). Nilai eigen nyata dan berlainan dengan tanda yang berlainan.

(iii). Nilai eigen nyata dan sama, dengan dua fungsi eigen tak bersandar secara linear.

(iv). Nilai eigen konjugat kompleks, $\lambda = p \pm qi$.

(v). Nilai eigen khayalan tulen, $\lambda = \pm qi$.

Untuk setiap kes, lakarkan contoh potret fasa; tentukan jenis titik genting (nod sink atau nod sumber, lingkaran sink atau lingkaran sumber, titik pelana dan pusat) dan kestabilannya (stabil, stabil secara asimptot atau tidak stabil).

(60 markah)

4. Katakan populasi $x(t)$ dan $y(t)$ masing-masing memenuhi sistem persaingan

$$\frac{dx}{dt} = 10x - \frac{1}{2}x^2 - xy$$
$$\frac{dy}{dt} = 16y - \frac{1}{2}y^2 - xy$$

(a). Cari semua titik genting.

(25 markah)

(b). Gunakan matriks Jacobian untuk melinearkan sistem di atas. Seterusnya, dapatkan nilai eigen untuk setiap titik genting.

(45 markah)

(c). Tentukan jenis dan kestabilan titik-titik genting tersebut.

(15 markah)

(d). Daripada keputusan (c)., lakarkan fasa potret satah untuk sistem di atas. Berikan pandangan anda terhadap interaksi atau persaingan antara kedua-dua populasi tersebut.

(15 markah)

5. (a). Bincangkan dua jenis ralat, iaitu ralat pembundaran dan ralat pangkasan yang wujud dalam penyelesaian berangka bagi persamaan pembezaan biasa. Bagi setiap jenis ralat, berikan satu cadangan untuk memperbaiki kejituan penyelesaian.

(10 markah)

(b). Suatu populasi $y(t)$ yang memenuhi persamaan logistik dapat dimodelkan dengan

$$\frac{dy}{dt} = 0.0004y^2 - 0.06y$$

Jika populasi awal ialah 100 pada masa $t = 0$, guna kaedah Runge-Kutta berperingkat empat dengan saiz langkah $h = 1$ tahun untuk mendapat anggaran bilangan populasi selepas 2 tahun.

(50 markah)

(c). Pertimbangkan sistem linear

$$x' = 2x + 3y, \quad x(0) = 1$$
$$y' = 2x + y, \quad y(0) = -1$$

(i). Dapatkan nilai penghampiran bagi $x(0.2)$ dan $y(0.2)$ menggunakan kaedah ubahsuai Euler dengan saiz langkah tunggal $h = 0.2$.

(ii). Diberi penyelesaian tepat ialah $x(t) = e^{-t}$ dan $y(t) = -e^{-t}$. Dengan mengirakan peratus ralat relatif, bandingkan nilai penghampiran dalam (i). dengan nilai tepat.

(40 marks)

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Formula

Trigonometric identities

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\sin 2x = 2\sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$

$\cos 2x = 2\cos^2 x - 1$

Power series representation of elementary functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Reduction of oder

$$y_2 = y_1 \int \frac{\exp\left(-\int P(x)dx\right)}{y_1^2} dx$$

Sturm-Liouville problem

A Sturm-Liouville problem is an endpoint value problem of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x)y = 0 \qquad (a < x < b)$$

$$\alpha_1 y(a) - \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0$$

with neither α_1 and α_2 both zeros nor β_1 and β_2 both zeros.

Eigenfunction expansions

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) \quad \text{for} \quad a < x < b$$

where
$$c_n = \frac{\int_a^b f(x) y_n(x) r(x)\, dx}{\int_a^b [y_n(x)]^2 r(x)\, dx}$$

Linearized system

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \mathbf{J} \begin{bmatrix} u \\ v \end{bmatrix}$$

where $\mathbf{J}$ is the Jacobian matrix defined as

$$\mathbf{J}(x_0, y_0) = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}$$

evaluated at the point (x_0, y_0)

Euler method

$$x_{n+1} = x_n + h \cdot f(t_n, x_n, y_n)$$

$$y_{n+1} = y_n + h \cdot g(t_n, x_n, y_n)$$

Improved Euler method

Predictor: $u_{n+1} = x_n + h \cdot f(t_n, x_n, y_n)$

$v_{n+1} = y_n + h \cdot g(t_n, x_n, y_n)$

Corrector: $x_{n+1} = x_n + \frac{h}{2}\left[f(t_n, x_n, y_n) + f(t_{n+1}, u_{n+1}, v_{n+1}) \right]$

$y_{n+1} = y_n + \frac{h}{2}\left[g(t_n, x_n, y_n) + g(t_{n+1}, u_{n+1}, v_{n+1}) \right]$

where $\frac{dx}{dt} = f(t, x, y)$ and $\frac{dy}{dt} = g(t, x, y)$

Fourth-order Runge Kutta method

$$y_{n+1} = y_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4 \right];$$

$$k_1 = h \cdot f(t_n, y_n)$$

$$k_2 = h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{1}{2}k_1 \right)$$

$$k_3 = h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{1}{2}k_2 \right)$$

$$k_4 = h \cdot f\left(t_{n+1}, y_n + k_3 \right)$$

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