Final Examination
2018/2019 Academic Session

June 2019

JIM411 – Introductory Analysis
(Pengenalan Analisis)

Duration : 3 hours
(*Masa: 3 jam*)

Please check that this examination paper consists of **TEN (10)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEPULUH (10)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini].*

Instructions : Answer **ALL** questions.

*[**Arahan** : Jawab **SEMUA** soalan].*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].

1. (a). Find the infimum and supremum of the set

$$E = \left\{ x \in \mathbb{R} : x = 1 - \frac{(-1)^n}{n} \text{ for } n \in \mathbb{N} \right\}.$$

(10 marks)

(b). Prove by mathematical induction that the following formula is true for all $n \in \mathbb{N}$.

$$\sum_{k=1}^{n} \frac{a-1}{a^k} = 1 - \frac{1}{a^n}, \quad a \neq 0$$

(25 marks)

(c). Prove that the sequence $\left\{ \frac{2n}{n+2} \right\}$ is Cauchy.

(30 marks)

(d). Use formal definition of the limit of a function to prove that the following limit exists.

$$\lim_{x \to -1} \frac{x^2 - x - 2}{x+1} = -3$$

(35 marks)

2. (a). Use definition to prove that $f(x) = kx + c$ where $k, c \in \mathbb{R}, k \neq 0$, is uniformly continuous on $\mathbb{R}$.

(25 marks)

(b). Suppose that $f, g : [a,b] \to \mathbb{R}$. Decide which of the following statement is true and which is false. Prove the true one and provide counterexample for the false one.

(i). If f and g are increasing on $[a,b]$, then fg is increasing on $[a,b]$.

(ii). If f and g are differentiable on $[a,b]$ and $|f'(x)| \leq 1 \leq |g'(x)|$ for all $x \in [a,b]$, then $|f(x) - f(a)| \leq |g(x) - g(a)|$ for all $x \in [a,b]$.

(40 marks)

…3/-

(c). Let $f(x) = \cos x$ and $n \in \mathbb{N}$.

(i). Find the Taylor polynomial of order four generated by f centered at $\frac{\pi}{3}$.

(ii). Hence, find the error bound for $\left| x - \frac{\pi}{3} \right| < \frac{\pi}{6}$.

(35 marks)

3. (a). Prove that the Dirichlet function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable on $[0,1]$.

(30 marks)

(b). Let $f(x) = x,\ x \in [0, a]$. $P_n := \{x_0, x_1, \ldots, x_n\}$ is a partition of the interval $[0, a]$ and $t_i = \frac{1}{2}(x_{i-1} + x_i),\ i = 1, 2, \ldots, n$.

(i). Show that the Riemann sums of $f(x)$ on P_n is

$$\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \frac{1}{2}a^2 .$$

(ii). Use the definition and results in (i). to conclude that

$$\int_0^a x\,dx = \frac{1}{2}a^2.$$

(50 marks)

(c). Suppose that f is integrable on $[0.5, 1]$ and $\int_{0.5}^{1} x^k f(x)\,dx = 3 + k^2$ for $k = 0, 1, 2.$ Prove that the following statement is correct.

$$\int_0^{\sqrt{3}/2} \frac{x^3}{\sqrt{1-x^2}} f\left(\sqrt{1-x^2}\right) dx = -4$$

(20 marks)

4. (a). If $\sum_{k=1}^{\infty} a_k$ converges conditionally, prove that $\sum_{k=1}^{\infty} k^p a_k$ diverges for all $p > 1$.

(25 marks)

(b). Find set A such that

$$f_n(x) = \begin{cases} \dfrac{\sin 2nx}{2nx} , & 0 < x \le 1 \\ 1 , & x = 0 \end{cases}$$

converge pointwise on A. Hence, obtain the limit function.

(25 marks)

(c). Prove that $\sum_{k=1}^{\infty} \frac{(-1)^k e^{-kx}}{k^2}$ converges uniformly on $[0, \infty)$.

(30 marks)

(d). Given the following theorem:

"Let $S(x) = \sum_{k=0}^{\infty} a_k x^k$ be a power series centered at 0.

If $R = 1 \Big/ \left(\limsup_{k \to \infty} |a_k|^{1/k} \right)$, then R is the radius of convergence of S."

Find the radius of convergence of $\sum_{k=0}^{\infty} a_k^2 x^k$.

(20 marks)

5. (a). If $\|2\mathbf{x} - \mathbf{y}\| < 2$ and $\|\mathbf{y}\| < 1$, prove that $\left| \|\mathbf{x} - \mathbf{y}\|^2 - \mathbf{x} \cdot \mathbf{x} \right| < 2$.

(20 marks)

(b). If set V is open and set E is closed, show that $V \setminus E$ is open and $E \setminus V$ is closed.

(20 marks)

(c). Given a set $E = \left\{(x, y) : y \geq x^2, 0 \leq y < 4\right\}$.

(i). Sketch the set E.

(ii). Determine whether the set E in $\mathbb{R}$ is open, closed or neither.

(iii). Find the interior, closure and boundary of the set E.

(30 marks)

(d). For any $A \subseteq \mathbb{R}$, the closure of A is denoted as $\overline{A}$ and is defined as $\overline{A} = A \cup A'$. Suppose that $A \subseteq B \subseteq \mathbb{R}$. Prove that $\overline{A} \subseteq \overline{B}$.

(30 marks)

1. (a). Cari infimum dan supremum bagi set

$$E = \left\{ x \in \mathbb{R} : x = 1 - \frac{(-1)^n}{n} \text{ untuk } n \in \mathbb{N} \right\}.$$

(10 markah)

(b). Buktikan secara aruhan matematik bahawa formula berikut adalah benar bagi semua $n \in \mathbb{N}$.

$$\sum_{k=1}^{n} \frac{a-1}{a^k} = 1 - \frac{1}{a^n}, \quad a \neq 0$$

(25 markah)

(c). Buktikan jujukan $\left\{ \frac{2n}{n+2} \right\}$ adalah Cauchy.

(30 markah)

(d). Guna definisi formal bagi had suatu fungsi untuk membuktikan had fungsi berikut wujud.

$$\operatorname*{had}_{x \to -1} \frac{x^2 - x - 2}{x+1} = -3$$

(35 markah)

2. (a). Guna definisi untuk membuktikan $f(x) = kx + c$ di mana $k, c \in \mathbb{R}$, $k \neq 0$, adalah menumpu secara seragam pada $\mathbb{R}$.

(25 markah)

(b). Katakan $f, g : [a,b] \to \mathbb{R}$. Tentukan pernyataan berikut adalah betul atau salah. Buktikan pernyataan yang betul dan beri contoh penyangkal bagi pernyataan yang salah.

(i). Jika f dan g menokok pada $[a,b]$, maka fg menokok pada $[a,b]$.

(ii). Jika f dan g terbezakan pada $[a,b]$ dan $|f'(x)| \leq 1 \leq |g'(x)|$ untuk semua $x \in [a,b]$, maka $|f(x) - f(a)| \leq |g(x) - g(a)|$ untuk semua $x \in [a,b]$.

(40 markah)

…7/-

(c). Diberi $f(x) = \operatorname{kos} x$ dan $n \in \mathbb{N}$.

(i). Cari polynomial Taylor berperingkat empat bagi f di sekitar $\frac{\pi}{3}$.

(ii). Seterusnya, cari ralat batas untuk $\left| x - \frac{\pi}{3} \right| < \frac{\pi}{6}$.

(35 markah)

3. (a). Tunjukkan bahawa fungsi Dirichlet

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

tak terkamirkan secara Riemann pada $[0,1]$.

(30 markah)

(b). Diberi $f(x) = x, \; x \in [0,a]$. $P_n := \{x_0, x_1, \dots, x_n\}$ merupakan petak bagi selang $[0,a]$ dan $t_i = \frac{1}{2}\left(x_{i-1} + x_i\right), \; i = 1, 2, \dots, n$.

(i). Tunjukkan bahawa hasil tambah Riemann bagi $f(x)$ pada P_n ialah

$$\sum_{i=1}^{n} f(t_i)\left(x_i - x_{i-1}\right) = \frac{1}{2}a^2 .$$

(ii). Gunakan definisi dan keputusan dari (i). untuk tunjukkan bahawa

$$\int_0^a x\,dx = \frac{1}{2}a^2 .$$

(50 markah)

(c). Katakan f terkamirkan pada $[0.5,1]$ dan $\int_{0.5}^{1} x^k f(x)\,dx = 3 + k^2$ untuk $k = 0,1,2$. Tunjukkan pernyataan berikut adalah betul.

$$\int_0^{\sqrt{3}/2} \frac{x^3}{\sqrt{1-x^2}} f\left(\sqrt{1-x^2}\right) dx = -4$$

(20 markah)

4. (a). Jika $\sum_{k=1}^{\infty} a_k$ menumpu secara bersyarat, tunjukkan bahawa $\sum_{k=1}^{\infty} k^p a_k$ mencapah bagi semua $p > 1$.

(25 markah)

(b). Cari set A supaya

$$f_n(x) = \begin{cases} \dfrac{\sin 2nx}{2nx}, & 0 < x \le 1 \\ 1, & x = 0 \end{cases}$$

menumpu secara titik demi titik pada A. Kemudian, dapatkan fungsi had.

(25 markah)

(c). Buktikan $\sum_{k=1}^{\infty} \frac{(-1)^k e^{-kx}}{k^2}$ menumpu secara seragam pada $[0, \infty)$.

(30 markah)

(d). Diberi teorem berikut:

"Biar $S(x) = \sum_{k=0}^{\infty} a_k x^k$ suatu siri kuasa berpusat pada 0.

Jika $R = 1 \Big/ \left(\operatorname{had\,sup}_{k\to\infty} |a_k|^{1/k} \right)$, maka R ialah jejari penumpuan bagi S".

Cari jejari penumpuan bagi $\sum_{k=0}^{\infty} a_k^2 x^k$.

(20 markah)

5. (a). Jika $\|2\mathbf{x} - \mathbf{y}\| < 2$ dan $\|\mathbf{y}\| < 1$, buktikan bahawa $\left| \|\mathbf{x} - \mathbf{y}\|^2 - \mathbf{x} \cdot \mathbf{x} \right| < 2$.

(20 marks)

(b). Jika set V adalah terbuka dan set E adalah tertutup, buktikan bahawa $V \setminus E$ adalah terbuka dan $E \setminus V$ adalah tertutup.

(20 markah)

…9/-

(c). Diberi suatu set $E = \{(x, y) : y \geq x^2, 0 \leq y < 4\}$.

(i). Lakarkan set E.

(ii). Tentukan sama ada set E dalam $\mathbb{R}$ adalah terbuka, tertutup atau bukan kedua-keduanya.

(iii). Cari pedalaman, tutupan dan sempadan bagi set E.

(30 markah)

(d). Untuk sebarang $A \subseteq \mathbb{R}$, tutupan set A dilambangkan dengan $\overline{A}$ dan ditakrifkan sebagai $\overline{A} = A \cup A'$. Katakan $A \subseteq B \subseteq \mathbb{R}$. Tunjukkan bahawa $\overline{A} \subseteq \overline{B}$.

(30 markah)

Formula

The Taylor's Formula

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

The Taylor polynomial of order n generated by f centred at x_0

$$P_n^{f,\,x_0}(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

The error bound of a Taylor Polynomial

$$\left| f(x) - P_n^{f,x_0} \right| \le \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \right|$$

for some c between x and x_0

Upper Riemann sum

$$U(f,P) := \sum_{j=1}^{n} M_j(f)\, \Delta x_j$$

where $M_j(f) := \sup f([x_{j-1}, x_j])$ and $\Delta x_j := x_j - x_{j-1}$

Lower Riemann sum

$$L(f,P) := \sum_{j=1}^{n} m_j(f)\, \Delta x_j$$

where $m_j(f) := \inf f([x_{j-1}, x_j])$ and $\Delta x_j := x_j - x_{j-1}$

- oooOooo -