Final Examinaton
2018/2019 Academic Session

June 2019

JIM312 – Probability Theory
[Teori Kebarangkalian]

Duration: 3 hours
(Masa: 3 jam)

Please check that this examination paper consists of **TWELVE (12)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **DUA BELAS (12)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini].*

Instructions : Answer **ALL** questions.

*[**Arahan** : Jawab **SEMUA** soalan].*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].

…2/-

1. (a). Given the following probability distribution.

X	0	1	2	3
$f(x)$	0.2	0.4	0.3	0.1

Find

(i). Var(X).

(ii). $E(Y)$ when $Y=(2X-8)^2$.

(iii). Median of X .

(iv). Validate the Chebyshev Theorem.

(50 marks)

(b). Use the probability mass function and not the moment generating function of the Poisson distribution to

(i). show $E(X)=\lambda$,

(ii). find $E\left[X(X-1)\right]$,

(iii). deduce Var$(X)=\lambda$ using (i). and (ii).

(30 marks)

(c). Determine the quartiles of the standard normal distribution.

(20 marks)

2. (a). Given a table representing the joint probability mass function of a bivariate random variable (X,Y).

y	x = 0	1	2
0	$\frac{6}{28}$	$\frac{3}{28}$	$\frac{6}{28}$
1	$\frac{9}{28}$	$\frac{1}{28}$	0
2	$\frac{3}{28}$	0	0

(i). Find $P(X+Y>1)$.

(ii). Prove or disprove that X and Y are not independent.

(40 marks)

(b). Consider the joint probability density function

$$f(x,y) = \begin{cases} \frac{x}{4}(1+3y^2), & 0<x<2, 0<y<1 \\ 0, & \text{otherwise.} \end{cases}$$

(i). Obtain the joint cumulative distribution function $F(x,y)$.

(ii). Find $P\left(0<X<1, \frac{1}{4}<Y<\frac{1}{2}\right)$ using (i). only.

(iii). Find $P\left(\frac{1}{4}<Y<\frac{1}{2} \mid X=\frac{1}{3}\right)$.

(60 marks)

3. (a). Suppose there is a toll gate on a busy highway. Assume an average of 6 cars pass through this toll gate within a minute. What is the exact probability of 8 cars through this toll gate between 11:00 am and 11:01 am?

(20 marks)

(b). X_i is a random variable with Binomial$\left(n_i, \frac{1}{2}\right)$ distribution, $i = 1, 2$. X_1 and X_2 are independent. Find the distribution $Y = X_1 - X_2$.

(30 marks)

(c). $X_1, X_2, \ldots, X_k$ are independent Gamma random variables with parameters $n = \alpha_i$ and $\lambda = 1,\ i = 1, 2, \ldots, k$. Identify the distribution $Y = \sum_{i=1}^{k} X_i$.

(50 marks)

4. (a). Let $X_1, X_2, \ldots, X_5$ be a random sample from a population with Normal$\left(0, \sigma^2\right)$ distribution. Given $U = X_1 - X_2$ and $V = X_3^2 + X_4^2 + X_5^2$.

(i). Modify V such that the distribution of V is chi-square.

(ii). Let $T = \frac{cU}{\sqrt{V}}$. Evaluate c such that T is distributed as t.

(iii). What is the degrees of freedom of this t distribution?

(60 marks)

(b). Let T be distributed as t with 14 degrees of freedom. Evaluate b such that $P\left(|T| < b\right) = 0.90$.

(20 marks)

(c). How is the χ^2 distribution related to the Gamma distribution?

(20 marks)

5. (a). Suppose that the performances of a student in Mathematics and English subjects are independent. Given the probability that a student passes Mathematics is 0.85, the probability of failing in English is 0.12 and the probability of failing in both subjects is 0.05. Obtain the probability that a student fails in Mathematics given that he passes in English.

(25 marks)

(b). A store supplying house building materials bought stone chips for use in the construction of decorative concrete blocks. The supply of stone chip comes in mixed sizes. Once sorted there are three types of sizes labeled with grades A, B and C as follows:

GRADE	SIZE, x
C	$x > 160$
A	$150 \leq x \leq 160$
B	$115 \leq x < 150$
C	$x < 115$

The distribution of the stone chips size is Normal(130, 196). After deducting all costs, the store earns a net profit per tonne for each grade as follows: A \$50, B \$25 and C –\$5 (loss). Get the expected net profit from a shipment of a tonne of stone chips.

(25 marks)

(c). Let (X,Y) be a bivariate random variable. Give an example where X and Y are independent.

(25 marks)

(d). Prove the following statement. If W is distributed as t with n degrees of freedom, then W^2 is distributed as $F_{1,n}$.

(25 marks)

1. (a). Diberikan taburan kebarangkalian

X	0	1	2	3
$f(x)$	0.2	0.4	0.3	0.1

Cari

(i). Var(X).

(ii). $E(Y)$ sekiranya $Y=(2X-8)^2$.

(iii). Median bagi X.

(iv). Sahkan Teorem Chebyshev.

(50 markah)

(b). Gunakan fungsi jisim kebarangkalian dan tanpa menggunakan fungsi penjana momen taburan Poisson untuk

(i). tunjukkan $E(X)=\lambda$,

(ii). carikan $E[X(X-1)]$,

(iii). deduksikan Var$(X)=\lambda$ dengan menggunakan (i). dan (ii).

(30 markah)

(c). Tentukan kuartil-kuartil taburan normal piawai.

(20 markah)

2. (a). Diberikan jadual fungsi jisim kebarangkalian tercantum bagi pembolehubah rawak bivariat (X,Y).

y \ x	0	1	2
0	$\frac{6}{28}$	$\frac{3}{28}$	$\frac{6}{28}$
1	$\frac{9}{28}$	$\frac{1}{28}$	0
2	$\frac{3}{28}$	0	0

(i). Cari $P(X+Y>1)$.

(ii). Buktikan atau sangkalkan X dan Y tak bersandar.

(40 markah)

(b). Pertimbangkan fungsi ketumpatan tercantum

$$f(x,y)=\begin{cases}\frac{x}{4}\left(1+3y^2\right), & 0<x<2, 0<y<1\\ 0, & \text{di tempat lain.}\end{cases}$$

(i). Dapatkan fungsi taburan longgokan $F(x,y)$.

(ii). Cari $P\left(0<X<1, \frac{1}{4}<Y<\frac{1}{2}\right)$ dengan menggunakan (i). sahaja.

(iii). Cari $P\left(\frac{1}{4}<Y<\frac{1}{2} \mid X=\frac{1}{3}\right)$.

(60 markah)

3. (a). Katakan terdapat satu pintu tol pada sebuah lebuhraya yang sibuk. Andaikan purata 6 buah kereta melalui pintu tol ini dalam jangkamasa seminit. Berapakah kebarangkalian tepat 8 buah kereta melalui pintu tol ini di antara jam 11:00 pagi dan 11:01 pagi?

(20 markah)

(b). X_i adalah pembolehubah rawak bertaburan Binomial$\left(n_i, \frac{1}{2}\right)$, $i = 1, 2$. X_1 dan X_2 tak bersandar. Dapatkan taburan $Y = X_1 - X_2$.

(30 markah)

(c). $X_1, X_2, \ldots, X_k$ adalah pembolehubah-pembolehubah gamma tak bersandar yang mempunyai parameter $n = \alpha_i$ dan $\lambda = 1,\ i = 1, 2, \ldots, k$. Camkan taburan $Y = \sum_{i=1}^{k} X_i$.

(50 markah)

4. (a). Andaikan $X_1, X_2, \ldots, X_5$ adalah sampel rawak daripada populasi bertaburan Normal$\left(0, \sigma^2\right)$. Diberikan $U = X_1 - X_2$ dan $V = X_3^2 + X_4^2 + X_5^2$.

(i). Ubahsuai V supaya taburan V adalah khi-kuasa dua.

(ii). Andaikan $T = \frac{cU}{\sqrt{V}}$. Dapatkan nilai c supaya T tertabur secara t.

(iii). Berapakah darjah kebebasan taburan t ini?

(60 markah)

(b). Andaikan T mempunyai taburan t dengan 14 darjah kebebasan. Cari nilai b supaya $P\left(|T| < b\right) = 0.90$.

(20 markah)

(c). Bagaimanakah taburan χ^2 berkait dengan taburan Gamma?

(20 markah)

…10/-

5. (a). Andaikan prestasi seseorang pelajar di dalam matapelajaran Matematik dan Bahasa Inggeris adalah tidak bersandar. Diberikan kebarangkalian seorang pelajar lulus Matematik ialah 0.85, kebarangkalian gagal di dalam Bahasa Inggeris ialah 0.12 dan kebarangkalian gagal di dalam kedua-dua matapelajaran ialah 0.05. Dapatkan kebarangkalian seseorang pelajar itu gagal di dalam Matematik diberikan dia lulus di dalam Bahasa Inggeris.

(25 markah)

(b). Sebuah kedai pembekal bahan-bahan membina rumah membeli cip batu untuk digunakan di dalam pembentukan blok-blok konkrit hiasan. Bekalan cip-cip batu sampai di dalam saiz yang bercampur. Setelah diasingkan terdapat tiga jenis saiz yang dilabelkan gred A, B dan C seperti berikut:

GRADE	SAIZ, x
C	$x > 160$
A	$150 \leq x \leq 160$
B	$115 \leq x < 150$
C	$x < 115$

Taburan saiz cip-cip batu itu adalah Normal(135, 196). Setelah ditolak kesemua kos, kedai tersebut memperolehi keuntungan bersih per tan bagi setiap gred seperti berikut: A \$50, B \$25 dan C –\$5 (kerugian). Dapatkan jangkaan keuntungan bersih daripada hantaran cip-cip batu seberat satu tan.

(25 markah)

(c). Andaikan (X,Y) adalah pembolehubah rawak bivariat. Berikan satu contoh di mana X dan Y tak bersandar.

(25 markah)

(d). Buktikan pernyataan berikut. Jika W mempunyai taburan t dengan darjah kebebasan n, maka W^2 tertabur secara $F_{1,n}$.

(25 markah)

Formulas

1. $\rho(X,Y)=\dfrac{\operatorname{Cov}(X,Y)}{\sigma_X\sigma_Y}$, $\operatorname{Cov}(X,Y)=E(XY)-E(X)E(Y)$, $\sigma_X=\sqrt{E(X^2)-(E(X))^2}$
2. $f(x)=p^x(1-p)^{1-x}$, $x=0,1$, $0<p<1$. $E(X)=p$, $\operatorname{Var}(X)=p(1-p)$. $m(t)=1-p+pe^t$.
3. $f(x)=\binom{n}{x}p^x(1-p)^{n-x}$, $x=0,1,2,\ldots,n$, $0<p<1$. $E(X)=np$, $\operatorname{Var}(X)=np(1-p)$.

   $m(t)=(1-p+pe^t)^n$.
4. $f(x)=p(1-p)^x$, $x=0,1,2,\ldots$, $0<p<1$. $E(X)=\dfrac{1-p}{p}$, $\operatorname{Var}(X)=\dfrac{1-p}{p^2}$.

   $m(t)=\dfrac{p}{1-(1-p)e^t}$.
5. $f(x)=\binom{r+x-1}{x}p^r(1-p)^x$, $x=0,1,2,\ldots$, $0<p<1$. $E(X)=\dfrac{r(1-p)}{p}$,

   $\operatorname{Var}(X)=\dfrac{r(1-p)}{p^2}$. $m(t)=\left(\dfrac{p}{1-(1-p)e^t}\right)^r$, $t<-\log(1-p)$.
6. $f(x)=\dfrac{e^{-\lambda}\lambda^x}{x!}$, $x=0,1,2,\ldots,\lambda\ge 0$. $E(X)=\operatorname{Var}(X)=\lambda$. $m(t)=e^{\lambda(e^t-1)}$.
7. $f(x)=\dfrac{1}{\sigma\sqrt{2\pi}}\exp\left[-\dfrac{(x-\mu)^2}{2\sigma^2}\right]$, $\sigma>0$. $E(X)=\mu$, $\operatorname{Var}(X)=\lambda^2$.

   $m(t)=\exp\left[\mu t+\tfrac{1}{2}\sigma^2 t\right]$.
8. $f(x)=\dfrac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)}x^{\frac{k}{2}-1}e^{-\frac{x}{2}}$, $x\ge 0$. $E(X)=k$, $\operatorname{Var}(X)=2k$. $m_X(t)=\left(\dfrac{1}{1-2t}\right)^{\frac{k}{2}}$, $t<\dfrac{1}{2}$.
9. $f(x)=\dfrac{1}{\beta}e^{-x/\beta}$, $x\ge 0,\beta>0$. $E(X)=\beta$, $\operatorname{Var}(X)=\beta^2$. $m(t)=\dfrac{1}{1-\beta t}$, $t<\dfrac{1}{\beta}$.
10. $f(x)=\dfrac{1}{\Gamma(\alpha)\beta}x^{\alpha-1}e^{-x/\beta}$, $x\ge 0,\alpha>0,\beta>0$. $E(X)=\alpha\beta$, $\operatorname{Var}(X)=\alpha\beta^2$.

    $m(t)=\left(\dfrac{1}{1-\beta t}\right)^{\alpha}$, $t<\dfrac{1}{\beta}$.
11. $P(|X-\mu|\ge t\alpha)\le\dfrac{1}{t^2}$.

- oooOooo -