Final Examination
2018/2019 Academic Session

June 2019

JIM310 – Introductory Numerical Methods
(Pengantar Kaedah Berangka)

Duration: 3 hours
(Masa: 3 jam)

Please check that this examination paper consists of **FOURTEEN (14)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **EMPAT BELAS (14)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini].*

Instructions : Answer **ALL** questions.

*[**Arahan** : Jawab **SEMUA** soalan].*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].

1. (a). Assume a cylinder with height, $h = 10$ cm and radius, $r = 6$ cm. An error in the measurement of h or r leads to an error in calculating the volume of the cylinder $V = \pi r^2 h$. Let $\Delta h = 0.05$ and $\Delta r = 0.02$. Estimate the resulting error in V.

(20 marks)

(b). Show that the equation $(x-5)^2 - \ln x = 0$ has at least one solution in the interval $[\pi, 5]$.

(10 marks)

(c). Given a nonlinear equation

$$f(x) = 2\sin\left(\sqrt{x}\right) - x$$

(i). Use the theorem to show that $g(x) = 2\sin\left(\sqrt{x}\right)$ has a unique fixed point on the initial value $x_0 = 0.7$.

(ii). Hence, use fixed-point iteration method to locate the root of $f(x)$ and iterate until a tolerance of 0.5% is achieved.

(50 marks)

(d). Given

$$f(x) = \sin x + \cos\left(1 + x^2\right) - 1$$

where x is in radians. Use four iterations of the secant method with initial guesses of $x_0 = 1.5$ and $x_1 = 2.25$ to locate the root of $f(x)$.

(20 marks)

2. (a). Given that $f(0.2) = 0.93562$, $f(0.5) = 0.82671$, $f(0.9) = 0.91466$.

(i). Construct a Lagrange interpolating polynomial of degree two.

(ii). Hence, approximate $f(0.6)$.

(25 marks)

(b). Given the following data:

x	1.0	1.1	1.3	1.5	1.9	2.0
y	1.46	1.69	2.28	2.45	2.51	2.81

(i). Use linear regression to fit these data.

(ii). Compute the standard error of the estimate and the correlation coefficient.

(iii). Is the regression in (i). a good fit for the above data? Explain your answer.

(45 marks)

(c). The efficiency, w of a gear pump is found to vary with the volumetric flow rate of the fluid, v and the pressure of the fluid, p as follows:

v	42	53	58	36	46	57	49	65
p	3000	3000	3000	2000	2000	2000	1000	1000
w	83	86	88	83	86	88	83	86

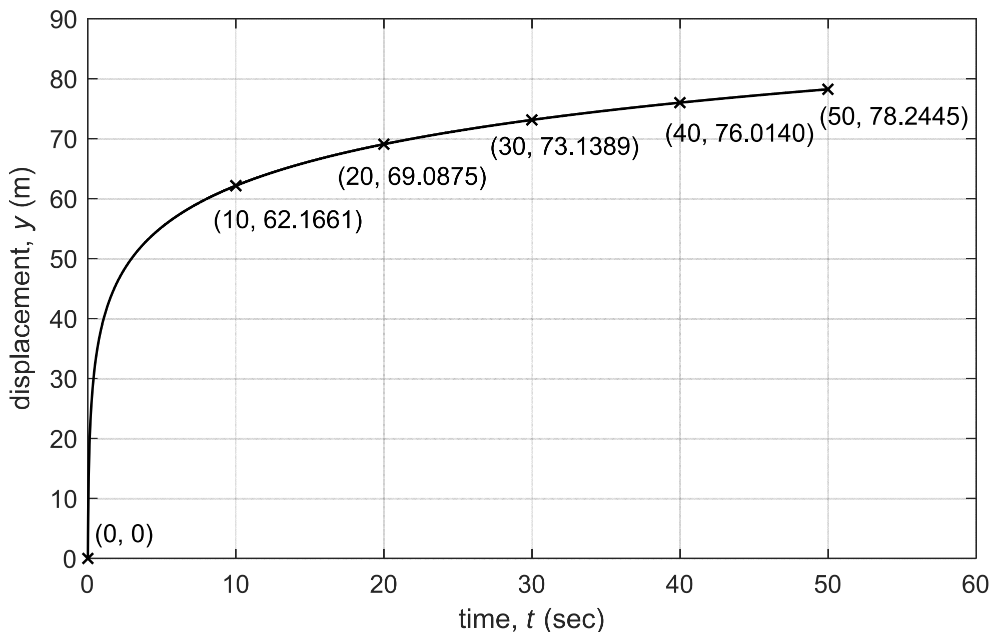
Obtain a linear relationship between the variables w, v and p.

(25 marks)

(d). Both the methods of interpolation and least square regression can be used to find the polynomial function from a given set of data. Give one (1) main difference between these two methods.

(5 marks)

3. (a). The displacement of a simulated car at six different time is represented in the graph below:



(i). Find the velocity $\left.\frac{dy}{dt}\right|_{t=10\text{ sec}}$ using centered-difference and forward-difference formula of $O(h^2)$. In your opinion, which answer is more reliable? Explain your answer.

(ii). Find the acceleration $\frac{d^2y}{dt^2}$ at time $t = 30$ sec using centered-difference formula of $O(h^4)$.

(30 marks)

(b). Given the function $f(x)$ at the following values:

x	0	0.05	0.10	0.15	0.20	0.25	0.30
$f(x)$	0	0.00263	0.01105	0.02614	0.04886	0.08025	0.12149

(i). Evaluate $\int_0^{0.3} f(x)\,dx$ using composite trapezoidal rule with step size $h = 0.1$.

(ii). Repeat question (i). by using step size $h = 0.05$.

(iii). Use the results in (i). and (ii). to find an improved approximation using Richardson extrapolation.

(iv). The exact value is given by $\int_0^{0.3} f(x)\,dx = 0.01129$. Compare the results obtained in (i)., (ii). and (iii). by finding the relative error.

(v). Give two (2) comments regarding the comparison in (iv).

(45 marks)

(c). Approximate integral

$$\int_{0.1}^{1.0} (\sin x)^{-x}\,dx.$$

using Gaussian quadrature with $n = 2$.

(25 marks)

4. (a). Explain briefly the procedure how to solve a system of linear equation $A\mathbf{x} = \mathbf{b}$ using $A = LU$ factorization method.

(20 marks)

(b). A matrix A has the factorization LU with L and U as given below:

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 5 & 0 \\ 3 & \frac{7}{2} & \frac{13}{5} \end{bmatrix}; \quad U = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Solve the linear system $A\mathbf{x} = (4, 6, 15)^T$ using LU factorization.

(30 marks)

(c). Solve the following linear system using Gauss-Seidel iteration, taking $\mathbf{x}^{(0)} = (0, 0, 1)^T$ and tolerance $\varepsilon = 10^{-2}$.

$$\begin{aligned} -5x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + 6x_2 - 3x_3 &= 2 \\ 2x_1 + x_2 + 7x_3 &= 32 \end{aligned}$$

(50 marks)

5. (a). Use Newton's method with initial values $\mathbf{x}^{(0)} = (2, 4)^t$ to find $\mathbf{x}^{(2)}$ for the following nonlinear system:

$$\begin{aligned} x_1^2 + x_2^2 - 8x_1 - 4x_2 + 11 &= 0 \\ x_1^2 + x_2^2 - 20x_1 + 75 &= 0 \end{aligned}$$

(50 marks)

(b). Given a matrix

$$A = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 4 & 5 \\ 10 & 5 & 7 \end{bmatrix}$$

(i). Use the Geršgorin Circle Theorem to determine the bounds for the eigenvalues of the matrix A.

(ii). Assume the initial eigenvector is $\mathbf{x}^{(0)} = (1,1,1)^t$. Use the power method to approximate the largest eigenvalue and its corresponding eigenvector of the matrix A. Iterate until the eigenvalue is correct to two decimal places.

(50 marks)

1. (a). Andaikan suatu silinder dengan tinggi, $h = 10$ sm dan jejari, $r = 6$ sm. Ralat yang wujud dalam pengukuran h atau r akan menghasilkan ralat dalam pengiraan isipadu silinder $V = \pi r^2 h$. Katakan $\Delta h = 0.05$ dan $\Delta r = 0.02$. Anggarkan ralat yang terhasil pada V .

(20 markah)

(b). Tunjukkan bahawa persamaan $(x-5)^2 - \ln x = 0$ mempunyai sekurang-kurangnya satu punca di dalam selang $[\pi, 5]$.

(10 markah)

(c). Diberi satu persamaan tak linear

$$f(x) = 2\sin\left(\sqrt{x}\right) - x$$

(i). Guna teorem untuk menunjukkan bahawa $g(x) = 2\sin\left(\sqrt{x}\right)$ mempunyai titik tetap unik pada nilai awal $x_0 = 0.7$.

(ii). Seterusnya, guna kaedah lelaran titik tetap untuk menentukan punca bagi $f(x)$ dan lelarkan sehingga toleransi 0.5% tercapai.

(50 markah)

(d). Diberi

$$f(x) = \sin x + \text{kos}\left(1 + x^2\right) - 1$$

di mana x adalah dalam radian. Guna kaedah sekan dengan empat lelaran dan nilai awal $x_0 = 1.5$ dan $x_1 = 2.25$ untuk menentukan punca bagi $f(x)$.

(20 markah)

2. (a). Diberi bahawa $f(0.2) = 0.93562$, $f(0.5) = 0.82671$, $f(0.9) = 0.91466$.

(i). Bina satu polinomial interpolasi Lagrange darjah dua.

(ii). Seterusnya, anggarkan $f(0.6)$.

(25 markah)

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(b). Diberi data berikut:

x	1.0	1.1	1.3	1.5	1.9	2.0
y	1.46	1.69	2.28	2.45	2.51	2.81

(i). Guna regresi linear untuk menyuaikan data tersebut.

(ii). Kira anggaran ralat piawai dan pekali korelasi.

(iii). Adakah regresi dalam (i). penyuai yang baik bagi data di atas? Jelaskan jawapan anda.

(45 markah)

(c). Kecekapan, w bagi suatu pam gear didapati berubah terhadap kadar aliran isipadu bendalir, v and tekanan bendalir, p seperti berikut:

v	42	53	58	36	46	57	49	65
p	3000	3000	3000	2000	2000	2000	1000	1000
w	83	86	88	83	86	88	83	86

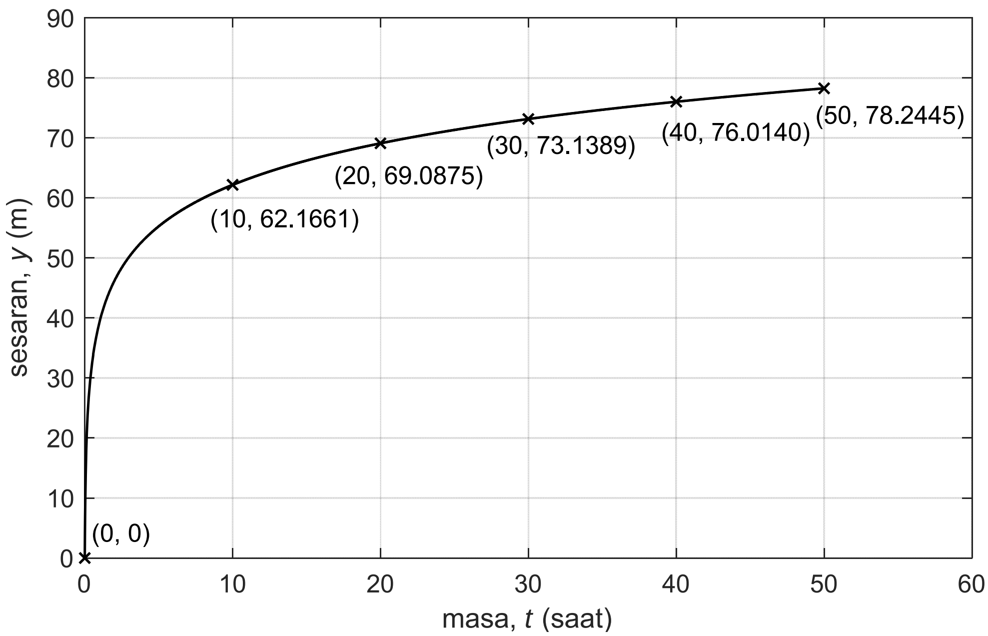
Dapatkan satu hubungan linear antara pembolehubah w, v dan p .

(25 markah)

(d). Kedua-dua kaedah interpolasi dan regresi kuasa dua terkecil boleh digunakan untuk mendapatkan fungsi polinomial daripada set data yang diberi. Nyatakan satu (1) perbezaan utama antara kedua-dua kaedah tersebut.

(5 markah)

3. (a). Sesaran bagi suatu kereta simulasi pada enam masa berbeza diwakilkan dalam rajah di bawah:



(i). Cari halaju $\left.\frac{dy}{dt}\right|_{t=10\text{ saat}}$ dengan formula beza-tengah dan beza-depan peringkat $O(h^2)$. Pada pandangan anda, jawapan yang mana lebih dipercayai? Jelaskan jawapan anda.

(ii). Cari pecutan $\frac{d^2y}{dt^2}$ pada masa $t = 30$ saat dengan menggunakan formula beza-tengah peringkat $O(h^4)$.

(30 markah)

(b). Diberi fungsi $f(x)$ pada nilai berikut:

x	0	0.05	0.10	0.15	0.20	0.25	0.30
$f(x)$	0	0.00263	0.01105	0.02614	0.04886	0.08025	0.12149

(i). Nilaikan $\int_0^{0.3} f(x)\,dx$ dengan menggunakan petua gubahan trapezium dengan saiz langkah $h = 0.1$.

(ii). Ulang soalan (i). dengan menggunakan saiz langkah $h = 0.05$.

(iii). Guna jawapan daripada (i). dan (ii). untuk mencari penghampiran yang lebih baik dengan menggunakan kaedah ekstrapolasi Richardson.

(iv). Diberi penyelesaian tepat ialah $\int_0^{0.3} f(x)\,dx = 0.01129$. Bandingkan jawapan yang diperoleh daripada (i)., (ii). dan (iii). dengan mencari ralat relatif.

(v). Beri dua (2) komen tentang perbandingan di (iv).

(45 markah)

(c). Anggarkan kamiran

$$\int_{0.1}^{1.0} (\sin x)^{-x}\,dx.$$

menggunakan kuadratur Gaussian dengan $n = 2$.

(25 markah)

4. (a). Jelaskan dengan ringkas tatacara bagaimana menyelesaikan sistem persamaan linear $A\mathbf{x} = \mathbf{b}$ menggunakan kaedah pemfaktoran $A = LU$.

(20 markah)

(b). Matriks A terfaktorkan secara LU, dengan L dan U seperi berikut:

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 5 & 0 \\ 3 & \frac{7}{2} & \frac{13}{5} \end{bmatrix}; \quad U = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Selesaikan sistem persamaan linear $A\mathbf{x} = (4, 6, 15)^T$ menggunakan pemfaktoran LU.

(30 markah)

(c). Selesaikan sistem linear berikut dengan menggunakan kaedah lelaran Gauss-Seidel, ambil $\mathbf{x}^{(0)} = (0, 0, 1)^T$ dan toleransi $\varepsilon = 10^{-2}$.

$$\begin{aligned} -5x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + 6x_2 - 3x_3 &= 2 \\ 2x_1 + x_2 + 7x_3 &= 32 \end{aligned}$$

(50 markah)

5. (a). Dengan nilai awal $\mathbf{x}^{(0)} = (2, 4)^t$, guna kaedah Newton untuk mencari $\mathbf{x}^{(2)}$ bagi sistem tak linear berikut:

$$\begin{aligned} x_1^2 + x_2^2 - 8x_1 - 4x_2 + 11 &= 0 \\ x_1^2 + x_2^2 - 20x_1 + 75 &= 0 \end{aligned}$$

(50 markah)

(b). Diberi satu matriks

$$A = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 4 & 5 \\ 10 & 5 & 7 \end{bmatrix}.$$

(i). Gunakan Teorem Bulatan Geršgorin untuk menentukan selang nilai eigen bagi matriks A.

(ii). Andaikan vektor eigen awal ialah $\mathbf{x}^{(0)} = (1,1,1)^t$. Guna kaedah kuasa untuk menganggarkan nilai eigen terbesar dan vektor eigen yang sepadan bagi matriks A. Lelarkan sehingga nilai eigen adalah betul kepada dua tempat perpuluhan.

(50 markah)

List of Formula

1. $$P_n(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$$

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)} = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}$$

2. $$P_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \ldots + f[x_0,x_1,x_2,\ldots,x_n](x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

3. $$y = a + bx$$

$$b = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a = \overline{y} - b\overline{x}$$

4. $$y = a + bx + cx^2$$

$$(n)a + \left(\sum x_i\right)b + \left(\sum x_i^2\right)c = \sum y_i$$

$$\left(\sum x_i\right)a + \left(\sum x_i^2\right)b + \left(\sum x_i^3\right)c = \sum x_i y_i$$

$$\left(\sum x_i^2\right)a + \left(\sum x_i^3\right)b + \left(\sum x_i^4\right)c = \sum x_i^2 y_i$$

5. $$y = a + bx_1 + cx_2$$

$$(n)a + \left(\sum x_{1i}\right)b + \left(\sum x_{2i}\right)c = \sum y_i$$

$$\left(\sum x_{1i}\right)a + \left(\sum x_{1i}^2\right)b + \left(\sum x_{1i}x_{2i}\right)c = \sum x_{1i} y_i$$

$$\left(\sum x_{2i}\right)a + \left(\sum x_{1i}x_{2i}\right)b + \left(\sum x_{2i}^2\right)c = \sum x_{2i} y_i$$

6. $$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - a - bx_i\right)^2$$

$$S_t = \sum_{i=1}^{n} \left(y_i - \overline{y}_i\right)^2$$

$$S_{y/x} = \sqrt{\frac{S_r}{n-2}}$$

$$r = \sqrt{\frac{S_t - S_r}{S_t}}$$

$$r = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{\sqrt{n\sum x_i^2 - \left(\sum x_i\right)^2}\sqrt{n\sum y_i^2 - \left(\sum y_i\right)^2}}$$

7. $$f'(x_0) \approx \frac{1}{h}\left[f(x_0+h) - f(x_0) \right]$$

$$f'(x_0) \approx \frac{1}{h}\left[f(x_0) - f(x_0-h) \right]$$

$$f'(x_0) \approx \frac{1}{2h}\left[f(x_0+h) - f(x_0-h) \right]$$

$$f'(x_0) \approx \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h) \right]$$

$$f'(x_0) \approx \frac{1}{2h}\left[f(x_0-2h) - 4f(x_0-h) + 3f(x_0) \right]$$

$$f'(x_0) \approx \frac{1}{12h}\left[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h) \right]$$

8. $$f''(x_0) \approx \frac{1}{h^2}\left[f(x_0) - 2f(x_0+h) + f(x_0+2h) \right]$$

$$f''(x_0) \approx \frac{1}{h^2}\left[f(x_0) - 2f(x_0-h) + f(x_0-2h) \right]$$

$$f''(x_0) \approx \frac{1}{h^2}\left[f(x_0-h) - 2f(x_0) + f(x_0+h) \right]$$

$$f''(x_0) \approx \frac{1}{h^2}\left[2f(x_0) - 5f(x_0+h) + 4f(x_0+2h) - f(x_0+3h) \right]$$

$$f''(x_0) \approx \frac{1}{h^2}\left[2f(x_0) - 5f(x_0-h) + 4f(x_0-2h) - f(x_0-3h) \right]$$

$$f''(x_0) \approx \frac{1}{12h^2}\left[-f(x_0-2h) + 16f(x_0-h) - 30f(x_0) + 16f(x_0+h) - f(x_0+2h) \right]$$

9. $$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)}{4^{j-1} - 1} \quad \text{for } j = 2,3,\ldots$$

10. $$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1) \right]$$

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2h f(x_0)$$

$$\int_{x_{-1}}^{x_2} f(x)\,dx = \frac{3h}{2}\left[f(x_0) + f(x_1) \right]$$

$$\int_{x_{-1}}^{x_3} f(x)\,dx = \frac{4h}{3}\left[2f(x_0) - f(x_1) + 2f(x_2) \right]$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right]$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right]$$

11. $$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2\sum_{j=1}^{n-1} f(x_j) + f(x_n)\right]$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + f(x_n)\right]$$

12. $$h_k = \frac{(b-a)}{2^{k-1}} \quad \text{for } k = 1,2,3,\ldots$$

$$R_{1,1} = \frac{h_1}{2}\left[f(a) + f(b)\right]$$

$$R_{k,1} = \frac{1}{2}\left[R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f(a + (2i-1)h_k)\right] \quad \text{for } k = 2,3,\ldots$$

$$R_{k,j} = R_{k,j-1} + \frac{1}{4^{j-1}-1}\left(R_{k,j-1} - R_{k-1,j-1}\right) \quad \text{for } k = j, j+1,\ldots$$

13. $$x = \frac{1}{2}\left[(b-a)t + (a+b)\right]$$

$$\int_{-1}^{1} f(x)\;\; dx \approx f\left(0.5773503\right) + f\left(-0.5773503\right)$$

$$\int_{-1}^{1} f(x)\;\; dx \approx 0.5555556 f\left(0.7745967\right) + 0.8888889 f\left(0\right) + 0.5555556 f(-0.7745967)$$

14. $$R_i = \left\{ z \in C \,\middle|\, |z - a_{ii}| \le \sum_{\substack{j=1 \\ j\ne 1}}^{n} |a_{ij}| \right\}$$

15. $$\mathrm{x}^{(k+1)} = \frac{1}{m_{k+1}} A\mathrm{x}^{(k)}$$

16. $$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + \mathbf{y}^{(k-1)} \text{ for } k \ge 1,\; \mathbf{y}^{(k-1)} = -J\left(\mathbf{x}^{(k-1)}\right)^{-1}\mathbf{F}\left(\mathbf{x}^{(k-1)}\right)$$

$$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}) & \frac{\partial f_n}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\mathbf{x}) \end{bmatrix}$$

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