Final Examination
2018/2019 Academic Session

June 2019

JIM213 – Differential Equations I
(Persamaan Pembezaan I)

Duration : 3 hours
(*Masa: 3 jam*)

Please check that this examination paper consists of **NINE (9)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEMBILAN (9)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini].*

Instructions : Answer **ALL** questions.

*[**Arahan** : Jawab **SEMUA** soalan].*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].

1. (a). Find the solution for the following differential equation.

$$xy' + 2y = x^3, \quad y(1) = 2.$$

(40 marks)

(b). Show that $y = \dfrac{1}{Cx^2 - x^3}$ where C is a constant, is the general solution for the Bernoulli equation

$$\frac{dy}{dx} + \frac{2}{x}y = x^2 y^2$$

(60 marks)

2. (a). Verify that the integrating factor of

$$x^2 y' = 3y + x^4$$

is $\mu(x) = \sqrt[x]{e^3}$.

(20 marks)

(b). Show that the following equation

$$\left(e^x \sin y - 2y \sin x\right) dx = -\left(2\cos x + e^x \cos y\right) dy$$

is an exact equation. Hence, find the general solution.

(30 marks)

(c). Find the explicit solution for the following initial-value problem.

$$\frac{dy}{dx} - xe^{2y} = e^{2y} \sin x, \quad y(0) = 0.$$

(50 marks)

3. (a). The population of a town $P(t)$ is modelled by the following differential equation

$$\frac{dP}{dt} = k\,P$$

at time t. The initial population of 5000 increases by 20% in 5 years. What will the population in 12 years?

(30 marks)

(b). A 250-volt electromotive force is applied to an RC series circuit where the resistance is 1000 ohms and the capacitance is 5×10^{-6} farad. Find the charge $q(t)$ on capacitor if the initial charge is $q(0)=0$. Hence, find the current $I(t)$.

[Hint: Kirchhoff's Second Law, $RI+\frac{1}{C}q=E(t)$ and $I=\frac{dq}{dt}$].

(30 marks)

(c). Given LRC series circuit

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t)$$

with L = 1 henry, R = 20 ohms, C = 0.005 farad and $E(t)$ = 300 volt.

(i). Show that the general solution of the charge on the capacitor is $q(t)=e^{-10t}\left[c_1\cos(10t)+c_2\sin(10t)\right]+1.5$

(ii). Hence, find particular solution if $q(0)=1$ and $I(0)=0$.

(iii). What is the charge on the capacitor when $t\to\infty$?

(40 marks)

4. (a). Solve the initial value problem of Cauchy-Euler differential equation

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+3y=3x^4,\quad y(1)=6,\quad y'(1)=7.$$

(30 marks)

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(b). Use the Laplace transform to solve the following initial-value problem

$$\frac{dy}{dt} - 2y = e^{-t}, \quad y(0) = 0.$$

(30 marks)

(c). Using Laplace transform, prove that the particular solution of

$$y'' + 4y' + 4y = te^{-2t} \quad , \ y(0) = 1 \, , \, y'(0) = 1$$

is

$$y(t) = \frac{1}{6}t^3 e^{-2t} + e^{-2t} + 3te^{-2t}.$$

(40 marks)

5. Given a system of homogenous linear differential equations

$$\frac{dx}{dt} = x + y + 2z,$$
$$\frac{dy}{dt} = x + 2y + z,$$
$$\frac{dz}{dt} = 2x + y + z.$$

(a). Write the system of equations in the form

$$\frac{dX}{dt} = AX,$$

where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and identify the matrix A.

(15 marks)

(b). Show that the characteristic equation is

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

(25 marks)

(c). Find all eigen values and the corresponding eigenvectors.

(50 marks)

(d). Hence, write down the general solution of the given system.

(10 marks)

1. (a). Cari penyelesaian bagi persamaaan pembezaan yang berikut

$$xy' + 2y = x^3, \quad y(1) = 2.$$

(40 markah)

(b). Tunjukkan bahawa $y = \dfrac{1}{Cx^2 - x^3}$ dengan C adalah pemalar, ialah penyelesaian umum bagi persamaan Bernoulli

$$\frac{dy}{dx} + \frac{2}{x}y = x^2y^2$$

(60 markah)

2. (a). Tentusahkan faktor kamiran bagi

$$x^2y' = 3y + x^4$$

adalah $\mu(x) = \sqrt[x]{e^3}$.

(20 markah)

(b). Tunjukkan bahawa persamaan berikut

$$\left(e^x \sin y - 2y\sin x\right)dx = -\left(2\,\text{kos}\,x + e^x\,\text{kos}\,y\right)dy$$

adalah persamaan tepat. Seterusnya, cari penyelesaian umum.

(30 markah)

(c). Cari penyelesaian tak tersirat bagi masalah nilai awal berikut

$$\frac{dy}{dx} - xe^{2y} = e^{2y}\sin x, \quad y(0) = 0.$$

(50 markah)

3. (a). Populasi sebuah bandar $P(t)$ boleh dimodelkan oleh persamaan pembezaan seperti berikut

$$\frac{dP}{dt} = k\,P$$

pada masa t. Nilai awal populasi adalah 5000 dan meningkat sebanyak 20% dalam masa 5 tahun. Berapakah nilai populasi dalam masa 12 tahun?

(30 markah)

(b). Satu daya elektromotif 250-volt digunakan pada litar siri RC di mana rintangan adalah 1000 ohms dan kapasitinya adalah 5×10^{-6} farad. Cari caj $q(t)$ pada kapasitor jika caj awal adalah $q(0)=0.$ Seterusnya, cari arus $I(t)$.

[Petunjuk: Hukum Kedua Kirchhoff, $RI+\frac{1}{C}q=E(t)$ dan $I=\frac{dq}{dt}$].

(30 markah)

(c). Diberi litar siri LRC

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t)$$

dengan L = 1 henry, R = 20 ohms, C = 0.005 farad dan $E(t)$ = 300 volt.

(i). Tunjukkan bahawa penyelesaian umum bagi caj pada kapasitor adalah $q(t)=e^{-10t}\left[c_1\,\text{kos}(10t)+c_2\sin(10t)\right]+1.5$

(ii). Seterusnya, cari penyelesaian khusus jika $q(0)=1$ dan $I(0)=0$.

(iii). Apakah nilai caj pada kapasitor apabila $t\to\infty$?

(40 markah)

4. (a). Selesaikan masalah nilai awal bagi persamaan pembezaan Cauchy-Euler

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+3y=3x^4,\quad y(1)=6,\quad y'(1)=7.$$

(30 markah)

(b). Guna jelmaan Laplace untuk menyelesaikan masalah nilai awal berikut

$$\frac{dy}{dt} - 2y = e^{-t}, \quad y(0) = 0.$$

(30 markah)

(c). Dengan menggunakan jelmaan Laplace, buktikan penyelesaian khusus bagi

$$y'' + 4y' + 4y = te^{-2t} \quad , \ y(0) = 1 \, , \, y'(0) = 1$$

adalah

$$y(t) = \frac{1}{6}t^3 e^{-2t} + e^{-2t} + 3te^{-2t}.$$

(40 markah)

5. Diberi satu sistem persamaan pembezaan linear homogen

$$\frac{dx}{dt} = x + y + 2z,$$
$$\frac{dy}{dt} = x + 2y + z,$$
$$\frac{dz}{dt} = 2x + y + z.$$

(a). Tulis sistem persamaan dalam bentuk

$$\frac{dX}{dt} = AX,$$

dengan $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ dan kenalpasti matriks A.

(15 markah)

(b). Tunjukkan bahawa persamaan cirian adalah

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

(25 markah)

(c). Cari semua nilai eigen dan vektor eigen yang sepadan.

(50 markah)

(d). Seterusnya, tuliskan penyelesaian am bagi sistem yang diberi.

(10 markah)

Table 1/Jadual 1
Elementary Laplace Transforms

$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2 + a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2 + a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2 - a^2}, \quad s > \lvert a \rvert$
8. $\cosh at$	$\frac{s}{s^2 - a^2}, \quad s > \lvert a \rvert$
9. $e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}, \quad s > a$
10. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$
11. $t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$

13. $u_c(t)\, f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$f(s-c)$
15. $f'(t)$	$sF(s)-f(0)$
16. $f''(t)$	$s^2F(s)-sf(0)-f'(0)$

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