Final Examination
2017/2018 Academic Session

May/June 2018

JIM417 – Partial Differential Equations
[Persamaan Pembezaan Separa]

Duration : 3 hours
[*Masa: 3 jam*]

Please ensure that this examination paper contains **EIGHT** printed pages before you begin the examination.

Answer **ALL** questions. You may answer either in Bahasa Malaysia or in English.

Read the instructions carefully before answering.

Each question is worth 100 marks.

In the event of any discrepancies, the English version shall be used.

*Sila pastikan bahawa kertas peperiksaan ini mengandungi **LAPAN** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.*

*Jawab **SEMUA** soalan. Anda dibenarkan menjawab sama ada dalam Bahasa Malaysia atau Bahasa Inggeris.*

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah

Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai.

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1. (a). The Heavisde function is defined by

$$U(t-c)=\begin{cases}0, & t<c,\\ 1, & t\geq c\end{cases}$$

By using the definition of Laplace Transform, prove that

$$L\{U(t-c)f(t-c)\}=e^{-cs}F(s)$$

where

$$L\{f(t)\}=F(s).$$

(40 marks)

(b). Using Laplace Transform, solve the initial value problem,

$$u_{tt}+\omega^2 u=0, \quad \omega>0, \quad t>0$$

subject to the conditions

$$u(0)=0,$$
$$u_t(0)=\omega$$

(60 marks)

2. Consider a given function

$$f(x)=\begin{cases}0, & -\pi<x\leq-\frac{\pi}{2}\\ 1, & -\frac{\pi}{2}<x\leq\frac{\pi}{2}\\ 0, & \frac{\pi}{2}<x\leq\pi\end{cases}$$

(a). State the precise numerical value of $f(x)$ for each x in the interval $-\pi\leq x\leq\pi.$

(20 marks)

(b). Compute the Fourier coefficients a_j, b_n for $f(x)$.

(50 marks)

(c). Using the fact that $\int_0^x f(t)dt=x$ for $-\frac{\pi}{2}<x<\frac{\pi}{2}$, integrate the Fourier series for $f(x)$ to obtain the expansion

$$\frac{\pi x}{4}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin(2k-1)x, \qquad -\frac{\pi}{2}<x<\frac{\pi}{2}.$$

(30 marks)

3. Given a partial differential equation

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}+u=0, \quad x>0, \quad t>0,$$

with boundary and initial conditions

$$u(0,t)=0 \quad t>0, \text{ and } u(x,0)=\sin(x), \quad x>0.$$

(a). By using Laplace transform, show that

$$U(x,s)=\frac{(s+1)\sin(x)-\cos(x)+\mathrm{e}^{-(s+1)x}}{s^2+2s+2}.$$

(60 marks)

(b). Find the inverse Laplace transform of (a).

(40 marks)

4. (a). Classify each of the following partial differential equations as hyperbolic, elliptic, or parabolic:

(i). $u_{xx}+2u_{xy}+u_{yy}+u_x+u_y=0$

(ii). $u_{xx}+2u_{xy}+2u_{yy}+u_x+u_y=\sin(xy)$

(iii). $2u_{xx}-4u_{xy}-6u_{yy}+u_x=0$

(30 marks)

(b). Find the canonical form of the following hyperbolic partial differential equations. Be sure to show the change of coordinates that reduces the partial differential equations to canonical form

$$u_{xx}+6u_{xy}-16u_{yy}=0.$$

(70 marks)

5. Find the solution to the heat conduction problem

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \qquad 0\le x\le \pi, t>0 \\ u(0,t) &= 0 \\ u_x(\pi,t) &= 0 \\ u(x,0) &= 3\sin\left(\frac{5x}{2}\right)=f(x). \end{aligned}$$

(100 marks)

1. (a). Fungsi Heavisde function ditakrifkan oleh

$$U(t-c)=\begin{cases}0, & t<c,\\ 1, & t\geq c\end{cases}$$

Dengan menggunakan takrifan Jelmaan Laplace, buktikan bahawa

$$L\{U(t-c)f(t-c)\}=e^{-cs}F(s)$$

di mana

$$L\{f(t)\}=F(s).$$

(40 markah)

(b). Dengan menggunakan jelmaan Laplace, selesaikan masalah nilai awal-sempadan

$$u_{tt}+\omega^2 u=0, \quad \omega>0, \quad t>0$$

Tertakluk kepada syarat

$$u(0)=0,$$
$$u_t(0)=\omega.$$

(60 markah)

2. Pertimbangkan fungsi yang diberikan

$$f(x)=\begin{cases}0, & -\pi<x\leq-\frac{\pi}{2}\\ 1, & -\frac{\pi}{2}<x\leq\frac{\pi}{2}\\ 0, & \frac{\pi}{2}<x\leq\pi\end{cases}$$

(a). Nyatakan nilai berangka tepat $f(x)$ bagi setiap x dalam selang $-\pi\leq x\leq\pi$.

(20 markah)

(b). Kirakan pekali Fourier a_j , b_n kepada $f(x)$.

(50 markah)

(c). Menggunakan fakta $\int_0^x f(t)dt=x$ kepada $-\frac{\pi}{2}<x<\frac{\pi}{2}$, kamirkan siri Fourier untuk $f(x)$ untuk dapakan pengembangan.

$$\frac{\pi x}{4}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin(2k-1)x, \qquad -\frac{\pi}{2}<x<\frac{\pi}{2}.$$

(30 markah)

3. Diberi persamaan pembezaan separa

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}+u=0, \quad x>0, \quad t>0,$$

dengan syarat sempadan dan syarat awal

$$u(0,t)=0 \quad t>0, \text{ dan } u(x,0)=\sin(x), \quad x>0.$$

(a). Dengan menggunakan transformasi Laplace, tunjukkan bahawa

$$U(x,s)=\frac{(s+1)\sin(x)-\cos(x)+\mathrm{e}^{-(s+1)x}}{s^2+2s+2}.$$

(60 markah)

(b). Cari transformasi Laplace songsang bagi (a).

(40 markah)

4. (a). Klasifikasi setiap persamaan pembezaan separa berikut sebagai hiperbolik, eliptik, atau parabola:

(i). $u_{xx}+2u_{xy}+u_{yy}+u_x+u_y=0$

(ii). $u_{xx}+2u_{xy}+2u_{yy}+u_x+u_y=\sin(xy)$

(iii). $2u_{xx}-4u_{xy}-6u_{yy}+u_x=0$

(30 markah)

(b). Cari bentuk berkanun persamaan pembezaan separa hiperbola yang berikut. Pastikan untuk menunjukkan perubahan koordinat yang mengurangkan persamaan pembezaan separa kepada bentuk berkanun

$$u_{xx}+6u_{xy}-16u_{yy}=0.$$

(70 markah)

5. Cari penyelesaian kepada masalah konduksi haba

$$u_t=\alpha^2 u_{xx}, \qquad 0\le x\le \pi, t>0$$
$$u(0,t)=0$$
$$u_x(\pi,t)=0$$
$$u(x,0)=3\sin\left(\frac{5x}{2}\right)=f(x).$$

(100 markah)

Formulae

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy} .$$

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

with

$$a_o = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\ n = 1, 2, 3, ...$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx,\ n = 1, 2, 3, ...$$

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

with

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\ n = 1, 2, 3, ...$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

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with

$$b_n = \frac{2}{L}\int_0^L f(x)\left(\frac{n\pi x}{L}\right)dx, \ \ n = 1,\ 2,\ 3,\ ...$$

$$f(x) = \frac{1}{2}\sum_{-\infty}^{\infty} c_n e^{inx}$$

with

$$c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ e^{-inx}\ dx, \ \ n = 0,\ \pm 1, \pm 2,\ ...$$

$\dfrac{d^2y}{dx^2} - \alpha^2 y = 0$ has solution

$$y = Ae^{\alpha x} + Be^{-\alpha x} \text{ or } C \cosh\ \alpha x + D \sinh\ \alpha x.$$

$\dfrac{d^2y}{dx^2} + \alpha^2 y = 0$ has solution

$$y = A \cos \alpha x + B \sin \alpha x.$$

$r^2\dfrac{d^2R}{dr^2} + r\dfrac{dR}{dr} - n^2R = 0$ has solution

$$R_n = C_n r^n + \frac{D_n}{r^n}$$

$r\dfrac{d^2R}{dr^2} + r\dfrac{dR}{dr} = 0$ has solution

$$R = A + B\,\ell n\ r.$$

$$\mathrm{L}\left[e^{\alpha t} f(t)\right] = F(s-\alpha).$$

$$\mathrm{L}\ \{H(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathrm{L}\ \{f(t-a)H(t-a)\} = e^{-as}F(s)$$

$$\mathrm{L}\ \left[f^n(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

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$$\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n}\left[F(s)\right]$$

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$$

$$\mathcal{L}^{-1}\left\{F(s)G(s)\right\} = \int_0^t f(u)\,g(t-u)\,du = f * g$$

Laplace Transforms

$f(t)$	$\mathcal{L}\left\{f(t)\right\} = F(s)$
1	$\frac{1}{s}$
t^n, $n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$t \cos bt$	$\frac{s^2-a^2}{\left(s^2+b^2\right)^2}$
$t \sin bt$	$\frac{2bs}{\left(s^2+b^2\right)^2}$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$

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