



UNIVERSITI SAINS MALAYSIA



Final Examination  
2017/2018 Academic Session

May/June 2018

**JIM319 – Vector Calculus**  
**[Kalkulus Vektor]**

Duration : 3 hours  
[Masa: 3 jam]

Please ensure that this examination paper contains **ELEVEN** printed pages before you begin the examination.

Answer **ALL** questions.

Read the instructions carefully before answering.

Each question is worth 100 marks.

In the event of any discrepancies, the English version shall be used.

*Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEBELAS** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.*

*Jawab **SEMUA** soalan.*

*Baca arahan dengan teliti sebelum anda menjawab soalan.*

*Setiap soalan diperuntukkan 100 markah.*

*Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunakan.*

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1. (a). Define the scalar and vector product of two vectors  $\underline{a}$  and  $\underline{b}$  in term of  $|\underline{a}|$ ,  $|\underline{b}|$  and the angle  $\theta$  between the vectors, explaining carefully any sign conventions used.

Given two vectors,

$$\underline{a} = \underline{i} + \alpha \underline{k} + 2 \underline{k}, \quad \underline{b} = \alpha \underline{i} + 4 \underline{j} + 4 \underline{k}.$$

Determine the values of parameter  $\alpha$  such that the vectors  $\underline{a}$  and  $\underline{b}$  are

- (i). parallel,  
(ii). orthogonal.

(30 marks)

- (b). Three vectors  $\underline{a}$ ,  $\underline{b}$  and  $\underline{c}$  are such that  $\underline{a} \neq 0$  and  $\underline{a} \times \underline{b} = 2 \underline{a} \times \underline{c}$ .

Show that  $\underline{b} - 2\underline{c} = \lambda \underline{a}$ , where  $\lambda$  is a scalar.

Given that  $|\underline{a}| = |\underline{c}| = 1$ ,  $|\underline{b}| = 4$  and the angle between  $\underline{b}$  and  $\underline{c}$  is  $\arccos \frac{1}{4}$ . Show that  $\lambda = +4$  or  $\lambda = -4$ .

For each of these cases, find the cosine of the angle between  $\underline{a}$  and  $\underline{c}$ .

(45 marks)

- (c). Show that

$$(\underline{a} + \underline{b} - \underline{c}) \times (\underline{a} - \underline{b} + \underline{c}) = \beta \underline{a} \times (\underline{b} - \underline{c}),$$

where  $\beta$  is an integer to be found.

(25 marks)

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2. The planes  $\Pi_1$  and  $\Pi_2$  have equations

$$x + y - 3z = 6 \text{ and } 2x - y + z = 4$$

respectively, with respect to Cartesian axes Oxyz.

- (a). Find the perpendicular distance of the plane  $\Pi_1$  from the origin.  
(10 marks)
- (b). Given a point  $P$  with position vector  $\underline{i} - 2\underline{k} - 3\underline{k}$ . Find the distance of point  $P$  from the plane  $\Pi_2$ .  
(15 marks)
- (c). Find a normal to each plane.  
(10 marks)
- (d). Find the angle in degrees between the normals to the plane  $\Pi_1$  and  $\Pi_2$ .  
(15 marks)
- (e). Show that the vector equation of the line  $L$  of intersection of the plane  $\Pi_1$  and  $\Pi_2$  can be written in the form
- $$\underline{r} = -9\underline{j} - 5\underline{k} + \lambda(2\underline{i} + 7\underline{j} + 3\underline{k})$$
- (20 marks)
- (f). Find the coordinates of the point  $A$  which correspond to  $\lambda = 1$ .  
(10 marks)
- (g). Find the Cartesian equation of the plane which is perpendicular to  $L$  and passes through the point  $A$ .  
(20 marks)

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3. (a). Compute the rate of change of the scalar field

$$\phi = x^2 + 3xy + xyz^2$$

at the point  $P(1,1,-1)$  in the direction towards the point  $Q(2,2,-3)$ .

Starting at  $P$ , in what direction does  $\phi$  increase the most quickly?

Give the direction as a unit vector.

(40 marks)

- (b). Find an equation in the form  $Ax + By + Cz = D$  for the tangent plane to the surface

$$x^2 - e^{xy} + z^2 = 1$$

at the point  $(1,0,1)$ .

Determine the parametric equation for the line perpendicular to this surface through the point  $(1,0,1)$ .

(60 marks)

4. (a). Given the scalar field  $\phi$  and vector field  $\underline{F}$ , where

$$\phi(x, y, z) = x^2y - xz^3 \text{ and } \underline{F}(x, y, z) = -z^2 \underline{j} + yz\underline{k}.$$

- (i). Verify the identities

$$\text{curl grad } \phi = \underline{0} \text{ and } \text{div curl } \underline{F} = 0.$$

- (ii). Does there exist a vector  $\underline{G}$  such that  $\nabla \times \underline{G} = \underline{F}$ ? Explain why or why not.

- (iii). Is  $\underline{F}$  a gradient vector field? Justify your answer.

(40 marks)

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- (b). A vector field is given by

$$\underline{F}(x, y, z) = 6xyz\underline{i} + 3x^2z\underline{j} + 3x^2y\underline{k}$$

- (i). Show that  $\underline{F}$  is a conservative field.
- (ii). Find a potential function  $\phi$  such that  $\underline{F} = \nabla\phi$ .
- (iii). Use the potential function to evaluate the line integral

$$\int_C \underline{F} \cdot d\underline{r},$$

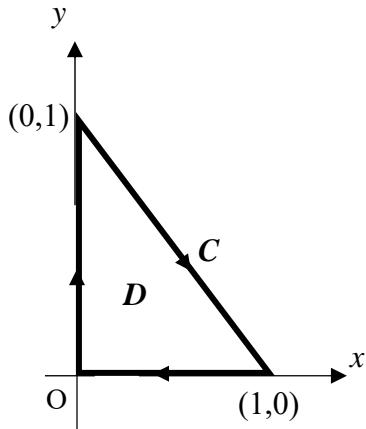
where  $C$  is a curve with parametrization

$$\underline{r}(t) = t\underline{i} + \sin t \underline{j} + t \sin t \underline{k}, \quad 0 \leq t \leq \pi.$$

(60 marks)

5. (a). State Green's theorem.

For the curve  $C$  in  $R^2$  as shown below



and the vector field  $\underline{F}(x, y, z) = \ln \sin(x) \underline{i} + (\cos(\sin y) + x) \underline{j}$ ,

evaluate

$$\int_C \underline{F} \cdot d\underline{r}$$

by using the Green's theorem.

(40 marks)

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- (b). State Stokes' theorem for a differentiable vector field  $\underline{F}$  over a surface  $S$  bounded by a closed curve  $C$ .

Let  $S$  be the surface in  $R^3$  parametrised by

$$\underline{r}(u, v) = (2 - 2v^2)\underline{i} + (v \cos u)\underline{j} + (v \sin u)\underline{k},$$

where

$$0 \leq u \leq 2\pi \quad \text{and} \quad 0 \leq v \leq 1.$$

For the vector field

$$\underline{F}(x, y, z) = -z\underline{j} + y\underline{k},$$

- (i). directly evaluate the surface integral

$$\iint_S (\nabla \times \underline{F}) \cdot d\underline{S}$$

where  $\underline{n}$  is unit normal vector field that points in the positive  $x$ -direction,

- (ii). using the Stokes' theorem, check your answer in (i).

(60 marks)

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1. (a). Takrifkan hasildarab skalar dan hasildarab vektor bagi dua vektor  $\underline{a}$  dan  $\underline{b}$  dalam sebutan  $|\underline{a}|$ ,  $|\underline{b}|$  dan sudut  $\theta$  di antara vektor berkenaan, dengan menerangkan secara jelas sebarang tanda yang digunakan.

Diberi dua vektor,

$$\underline{a} = \underline{i} + \alpha \underline{k} + 2 \underline{k}, \quad \underline{b} = \alpha \underline{i} + 4 \underline{j} + 4 \underline{k}.$$

Tentukan nilai parameter  $\alpha$  supaya vektor  $\underline{a}$  dan  $\underline{b}$  adalah

- (i). selari,  
(ii). serenjang.

(30 markah)

- (b). Tiga vektor  $\underline{a}$ ,  $\underline{b}$  dan  $\underline{c}$  sedemikian rupa  $\underline{a} \neq 0$  dan  $\underline{a} \times \underline{b} = 2 \underline{a} \times \underline{c}$ .

Tunjukkan bahawa  $\underline{b} - 2\underline{c} = \lambda \underline{a}$ , di mana  $\lambda$  adalah skalar.

Diberi  $|\underline{a}| = |\underline{c}| = 1$ ,  $|\underline{b}| = 4$  dan sudut di antara  $\underline{b}$  dan  $\underline{c}$  adalah  $\arccos \frac{1}{4}$ . Tunjukkan bahawa  $\lambda = +4$  atau  $\lambda = -4$ .

Untuk setiap kes ini, cari kosinus sudut di antara  $\underline{a}$  dan  $\underline{c}$ .

(45 markah)

- (c). Tunjukkan bahawa

$$(\underline{a} + \underline{b} - \underline{c}) \times (\underline{a} - \underline{b} + \underline{c}) = \beta \underline{a} \times (\underline{b} - \underline{c}),$$

di mana  $\beta$  adalah suatu integer yang perlu dicari.

(25 markah)

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2. Satah  $\Pi_1$  dan  $\Pi_2$  masing-masing mempunyai persamaan

$$x + y - 3z = 6 \text{ dan } 2x - y + z = 4$$

terhadap paksi Kartesian Oxyz.

- (a). Cari jarak serenjang satah  $\Pi_1$  dari asalan.

(10 markah)

- (b). Diberi titik  $P$  dengan vektor kedudukan  $\underline{i} - 2\underline{k} - 3\underline{k}$ . Cari jarak titik  $P$  dari satah  $\Pi_2$ .

(15 markah)

- (c). Cari normal kepada setiap satah.

(10 markah)

- (d). Cari sudut dalam darjah di antara normal kepada satah  $\Pi_1$  dan  $\Pi_2$ .

(15 markah)

- (e). Tunjukkan bahawa persamaan vektor garis  $L$  bagi persilangan satah  $\Pi_1$  dan  $\Pi_2$  boleh ditulis dalam bentuk

$$\underline{r} = -9\underline{j} - 5\underline{k} + \lambda(2\underline{i} + 7\underline{j} + 3\underline{k})$$

(20 markah)

- (f). Cari koordinat titik  $A$  yang bersepadan dengan  $\lambda = 1$ .

(10 markah)

- (g). Cari persamaan Kartesian bagi satah yang berserenjang kepada  $L$  dan melalui titik  $A$ .

(20 markah)

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3. (a). Kira kadar perubahan medan skalar

$$\phi = x^2 + 3xy + xyz^2$$

di titik  $P(1,1,-1)$  dalam arah kepada titik  $Q(2,2,-3)$ .

Bermula di  $P$ , dalam arah manakah  $\phi$  meningkat dengan pantasnya? Beri arah tersebut dalam vektor unit.

(40 markah)

- (b). Cari persamaan dalam bentuk  $Ax + By + Cz = D$  untuk satah tangent kepada permukaan

$$x^2 - e^{xy} + z^2 = 1$$

di titik  $(1,0,1)$ .

Tentukan persamaan parametrik bagi garis serenjang kepada permukaan tersebut yang melalui titik  $(1,0,1)$ .

(60 markah)

4. (a). Diberi medan skalar  $\phi$  dan medan vektor  $\underline{F}$ , dengan

$$\phi(x, y, z) = x^2y - xz^3 \text{ dan } \underline{F}(x, y, z) = -z^2\underline{j} + yz\underline{k}.$$

- (i). Tentusahkan identiti

$$\operatorname{curl} \operatorname{grad} \phi = \underline{0} \text{ dan } \operatorname{div} \operatorname{curl} \underline{F} = 0.$$

Adakah wujud suatu vektor  $\underline{G}$  supaya  $\nabla \times \underline{G} = \underline{F}$ ?

Terangkan kenapa.

- (ii). Adakah  $\underline{F}$  suatu medan vektor kecerunan? Beri justifikasi kepada jawapan anda.

(40 markah)

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- (b). Suatu medan vektor diberi oleh

$$\underline{F}(x, y, z) = 6xyz\underline{i} + 3x^2z\underline{j} + 3x^2y\underline{k}$$

- (i). Tunjukkan bahawa  $\underline{F}$  adalah medan abadi.
- (ii). Cari fungsi keupayaan  $\phi$  supaya  $\underline{F} = \nabla\phi$ .
- (iii). Gunakan fungsi keupayaan untuk menilai kamiran garis

$$\int_C \underline{F} \cdot d\underline{r},$$

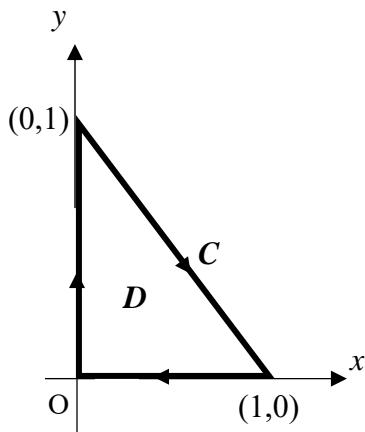
di mana  $C$  adalah suatu lengkung dengan persamaan parameter

$$\underline{r}(t) = t\underline{i} + \sin t \underline{j} + t \sin t \underline{k}, \quad 0 \leq t \leq \pi.$$

(60 markah)

5. (a). Nyatakan Teorem Green.

Untuk lengkung  $C$  dalam  $R^2$  seperti yang ditunjukkan di bawah



dan medan vektor  $\underline{F}(x, y, z) = \ln \sin(x)\underline{i} + (\cos(\sin y) + x)\underline{j}$ , nilaiakan

$$\int_C \underline{F} \cdot d\underline{r}$$

dengan menggunakan teorem Green.

(40 markah)

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- (b). Nyatakan teorem Stokes untuk suatu medan vektor terbeza  $\underline{F}$  ke atas permukaan  $S$  yang dibatasi oleh lengkung tertutup  $C$ .

Katakan  $S$  adalah permukaan dalam  $R^3$  yang diparameterkan oleh

$$\underline{r}(u, v) = (2 - 2v^2)\underline{i} + (v \cos u)\underline{j} + (v \sin u)\underline{k},$$

dengan

$$0 \leq u \leq 2\pi \quad \text{dan} \quad 0 \leq v \leq 1.$$

Untuk medan vektor

$$\underline{F}(x, y, z) = -z\underline{j} + y\underline{k},$$

- (i). nilaiakan secara terus kamiran permukaan

$$\iint_S (\nabla \times \underline{F}) \cdot d\underline{S}$$

di mana  $\underline{n}$  adalah vektor unit normal yang mengarah dalam positif  $x$ ,

- (ii). dengan menggunakan teorem Stokes, semak jawapan anda dalam (i).

(60 markah)

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