Final Examination
2017/2018 Academic Session

May/June 2018

JIM317 – Differential Equations II
[Persamaan Pembezaan II]

Duration : 3 hours
[*Masa: 3 jam*]

Please ensure that this examination paper contains **SIX** printed pages before you begin the examination.

Answer **ALL** questions.

Read the instructions carefully before answering.

Each question is worth 100 marks.

In the event of any discrepancies, the English version shall be used.

Sila pastikan bahawa kertas peperiksaan ini mengandungi ***ENAM*** *muka surat yang bercetak sebelum anda memulakan peperiksaan ini.*

Jawab ***SEMUA*** *soalan.*

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah.

Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai.

1. Consider the boundary value problem

$$x^2 y'' + xy' + 3y + \lambda y = 0\,, \quad 1 < x < 2\,.$$

(a). Rewrite the problem as a Sturm-Liouville problem.

(20 marks)

(b). Is the problem regular? Explain.

(20 marks)

(c). Find all eigenvalues and eigenfunctions with the boundary conditions $y(1) = 0,\ y(2) = 0$.

(40 marks)

(d). Write down the orthogonal expansion of $f(x) = \ln x,\ 1 < x < 2$, in terms of the eigenfunctions, and specify the formula for its coefficients.

(20 marks)

2. (a). Find all solutions $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ with $a_0 \neq 0$ of the Bessel equation

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0\,, \quad x > 0\,,$$

where α is any real nonnegative constant, using the Frobenius method centered at $x_0 = 0$.

(50 marks)

(b). Find the solution y near the regular singular point $x_0 = 0$ of the equation

$$x^2 y'' - x(x+3)y' + (x+3)y = 0\,.$$

(50 marks)

3. (a). Find all the regular points of the equation

$$xy'' + y' + x^2 y = 0 .$$

(20 marks)

(b). Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - xy = 0 .$$

(80 marks)

4. (a). Consider the initial value problem

$$x' = \sin(t^2 + x),$$
$$x(0) = 0$$

(i). Use Euler's method with a step size of $h = 1/4$ in order to obtain an approximation of $x(2)$.

(ii). Use the improved Euler method with a step size of $h = 1/2$ in order to obtain an approximation of $x(2)$.

(iii). Use the Runge–Kutta method with a step size of $h = 1$ in order to obtain an approximation of $x(2)$.

(60 marks)

(b). Consider the system of equations:

$$y_1' = 3y_1 - 2y_2$$
$$y_2' = -y_1 + 4y_2$$

(i). Transform the system of equations to an equivalent second-order system.

(ii). Solve the auxiliary equation to determine the constant rates.

(40 marks)

1. Pertimbangkan masalah nilai sempadan

$$x^2 y'' + xy' + 3y + \lambda y = 0\,, \quad 1 < x < 2\,.$$

(a). Tuliskan semula masalah tersebut dalam bentuk masalah Sturm-Liouville.

(20 markah)

(b). Adakah masalah tersebut biasa? Jelaskan.

(20 markah)

(c). Cari semua nilai eigen dan fungsi eigen dengan syarat-syarat sempadan $y(1) = 0,\ y(2) = 0$.

(40 markah)

(d). Tuliskan pengembangan ortogon kepada $f(x) = \ln x,\ 1 < x < 2$, dari segi fungsi eigen, dan nyatakan rumus untuk pekali tersebut.

(20 markah)

2. (a). Cari semua penyelesaian $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ dengan $a_0 \neq 0$ kepada persamaan Bessel

$$x^2 y'' + xy' + (x^2 - \alpha^2) y = 0\,, \quad x > 0\,,$$

di mana α adalah sebarang pemalar bukan negatif, menggunakan kaedah Frobenius berpusat di $x_0 = 0$.

(50 markah)

(b). Cari penyelesaian y berhampiran titik tunggal biasa $x_0 = 0$ kepada persamaan

$$x^2 y'' - x(x+3) y' + (x+3) y = 0\,.$$

(50 markah)

3. (a). Cari semua titik biasa kepada persamaan

$$xy'' + y' + x^2 y = 0 .$$

(20 markah)

(b). Cari tiga istilah pertama pengembangan siri kuasa di sekitar titik $x_0 = 2$ daripada setiap penyelesaian asas kepada persamaan pembezaan

$$y'' - xy = 0 .$$

(80 markah)

4. (a). Pertimbangkan masalah nilai awal

$$x' = \sin(t^2 + x),$$
$$x(0) = 0.$$

(i). Gunakan kaedah Euler dengan saiz langkah $h = 1/4$ untuk mendapatkan anggaran $x(2)$.

(ii). Gunakan kaedah Euler diperbaiki dengan saiz langkah $h = 1/2$ untuk mendapatkan anggaran $x(2)$.

(iii). Gunakan kaedah Runge–Kutta dengan saiz langkah $h = 1$ untuk mendapatkan anggaran $x(2)$.

(60 markah)

(b). Pertimbangkan sistem persamaan:

$$y_1' = 3y_1 - 2y_2$$
$$y_2' = -y_1 + 4y_2.$$

(i). Ubah sistem persamaan itu kepada sistem tahap dua yang sama.

(ii). Selesaikan persamaan bantu untuk menentukan kadar pemalar.

(40 markah)

Appendix

Trigonometry identities

$\sin^2 x + \cos^2 x = 1$

$\sin 2x = 2\sin x \cos x$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \end{aligned}$$

Power series representation of elementary functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Sturm-Liouville problem

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx} \right] - q(x)y + \lambda r(x)y = 0 \qquad (a < x < b)$$

$$\alpha_1 y(a) - \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0$$

Eigenfunction expansions

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

where

$$c_n = \frac{\int_a^b f(x) y_n(x) r(x)\, dx}{\int_a^b [y_n(x)]^2 r(x)\, dx}$$

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