Final Examination
2017/2018 Academic Session

May/June 2018

JIM315 – Introduction to Analysis
[Pengantar Analisis]

Duration : 3 hours
[*Masa: 3 jam*]

Please ensure that this examination paper contains **EIGHT** printed pages before you begin the examination.

Answer **ALL** questions. You may answer either in Bahasa Malaysia or in English.

Read the instructions carefully before answering.

Each question is worth 100 marks.

In the event of any discrepancies, the English version shall be used.

*Sila pastikan bahawa kertas peperiksaan ini mengandungi **LAPAN** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.*

*Jawab **SEMUA** soalan. Anda dibenarkan menjawab sama ada dalam Bahasa Malaysia atau Bahasa Inggeris.*

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah.

Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai.

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1. (a). Prove that there is no rational number w such that $w^2 = 2$.

(30 marks)

(b). (i). Are the set of rationals $\mathbb{Q}$ and set of reals $\mathbb{R}$ countable?

(ii). Let $\mathbb{N}$ be the set of natural numbers and A be the set of even natural numbers. Show that $\mathbb{N}$ ~ A (i.e. they have the same cardinality).

(30 marks)

(c). (i). State the Axiom of Completeness.

(ii). A sequence $\{x_n\}$ is defined as $x_1 = 1, x_{n+1} = \frac{1}{2}x_n + 1, n \in \mathbb{N}$.

Show that 2 is an upper bound for $\{x_n\}$.

(40 marks)

2. (a). (i). A sequence $\{x_n\}$ is defined by $x_n = \frac{1}{\sqrt{n}}$. Using the $\in$ – argument, show that $\lim_{n\to\infty} x_n = 0$.

(ii). Given a series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, show that the sum is less that 2, by considering the partial sums and appealing to the Monotone Convergence Theorem.

(40 marks)

(b). (i). State the Bolzano-Weierstrass Theorem.

(ii). Consider the sequence {1, -1, 1, -1, 1, -1, 1, ….}. By taking appropriate subsequences, show that the sequence diverges.

(30 marks)

(c). Give an example of each of the following:

(i). A Cauchy sequence that is not monotonic.

(ii). A monotonic sequence that is not Cauchy.

(iii). A Cauchy sequence with a divergent subsequence.

(30 marks)

3. (a). (i). Show that $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist.

(ii). Consider the function

$$f(x)=\begin{cases} x\sin\left(\frac{1}{2}\right) & x\neq 0 \\ 0 & x=0 \end{cases}$$

Using the definition of continuity, show that $f(x)$ is continuous at the origin.

(40 marks)

(b). Prove the following theorem (Intermediate Value Theorem):

"If f : [a, b] $\to$ □ is continuous and L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a,b)$ where $f(c) = L$."

(30 marks)

(c). Prove that if $f : A \to$ □ is differentiable at a point $c \in A$, then f is continuous at c as well.

(30 marks)

4. (a). Show that if a function f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

(30marks)

(b). Verify that the Taylor series for $\sin(x)$ is given by

$$\sin(x) \,□\, x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

(30 marks)

(c). Let $S_5(x)$ denotes the first three non-zero terms of the Taylor series for $\sin(x)$ over the interval [-2, 2].

(i). Use the Lagrange Remainder Theorem to find the remainder $E_5(x)$.

(ii). Show that $|E_5(x)| \le 0.09$.

(40marks)

5. (a). Consider $f(x) = 2x+1$ over the interval $[1,3]$. Let P be partitions consisting of $\left\{1, \frac{3}{2}, 2, 3\right\}$.

(i). Evaluate the lower sum $L(f,P)$ and the upper sum $U(f,P)$

(ii). What happen to the value $U(f,P) - L(f,P)$ when we add $\frac{5}{2}$ to the partition?

(30 marks)

(b). Let $H(x) = \int_1^x \frac{1}{t}\,dt, \quad x > 0$.

(i). What is $H(1)$? Find $H'(x)$.

(ii). Show that H is strictly increasing.

(30 marks)

(c). Let $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$

Show that *f* is integrable on [0, 1] and compute $\int_0^1 f$.

(40 marks)

1. (a). Buktikan bahawa tiada nombor rasional w supaya $w^2 = 2$.

(30 markah)

(b). (i). Adakah set-set □ dan □ boleh bilang?

(ii). Biarkan A sebagai set nombor-nombor asli genap. Tunjukkan bahawa □ ~ A (iaitu kedua-duanya mempunyai kardinalitas yang sama).

(30 markah)

(c). (i). Nyatakan Aksion Kelengkapan.

(ii). Satu jujukan $\{x_n\}$ ditakrif sebagai $x_1 = 1,\ x_{n+1} = \frac{1}{2}x_n + 1,\ n \in$ □ . Tunjukkan bahawa $\{x_n\}$ di batas dari atas dengan 2.

(40 markah)

2. (a). (i). Satu jujukan $\{x_n\}$ ditakrif sebagai $x_n = \frac{1}{\sqrt{n}}$. Menggunakan hujah-$\in$, tunjukkan bahawa $\lim_{n\to\infty} x_n = 0$.

(ii). Diberi satu siri $\sum_{n=1}^{\infty} \frac{1}{n^2}$, tunjukkan bahawa hasil tambahnya adalah kurang daripada 2, dengan mempertimbangkan hasil tambah separa dan merujuk kepada Teorem Penumpuan Ekanada.

(40 markah)

(b). (i). Nyatakan Teorem Bolzano-Weierstrass.

(ii). Pertimbangkan jujukan {1, -1, 1, -1, 1, -1, 1, ….}. Dengan mengambil subjujukan yang patut, tunjukkan bahawa jujukan mencapah.

(30 markah)

(c). Beri satu contoh untuk setiap yang berikut:

(i). Satu jujukan Cauchy yang tidak ekanada.

(ii). Satu jujukan ekanada yang tidak Cauchy.

(iii) . Satu jujukan tak terbatas yang mengandungi satu subjujukan Cauchy.

(30 markah)

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3. (a). (i). Tunjukkan bahawa $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ tidak wujud.

(ii). Pertimbangkan fungsi

$$f(x) = \begin{cases} x \sin\left(\frac{1}{2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Menggunakan takrif keselanjaran, tunjukkan bahawa $f(x)$ selanjar pada asalan.

(40 markah)

(b). Buktikan teorem berikut (Teorem Nilai Pertengahan):

"Jika f : $[a, b] \to \mathbb{R}$ selanjar dan L satu nombor nyata memenuhi $f(a) < L < f(b)$ atau $f(a) > L > f(b)$, maka wujud satu titik $c \in (a,b)$ dengan $f(c) = L$."

(30 markah)

(c). Buktikan bahawa jika $f : A \to \mathbb{R}$ terbezakan pada satu titik $c \in A$, maka f selanjar pada c.

(30 markah)

4. (a). Tunjukkan bahawa jika satu fungsi f terbezakan atas satu selang dengan $f'(x) \neq 1$, maka f mempunyai sebanyaknya satu titik tetap.

(30 markah)

(b). Sahkan bahawa siri Taylor untuk $\sin(x)$ diberi oleh

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

(30 markah)

(c). Biar $S_5(x)$ menandakan sebutan tiga ungkapan pertama siri Taylor bagi $\sin(x)$ atas selang [-2, 2].

(i). Gunakan Teorem Baki Lagrange untuk mendapatkan $E_5(x)$.

(ii). Tunjukkan bahawa $|E_5(x)| \leq 0.09$.

(40 markah)

5. (a). Pertimbangkan $f(x) = 2x+1$ atas selang $[1,3]$. Biar P sebagai petakan-petakan $\left\{1, \frac{3}{2}, 2, 3\right\}$.

   (i). Nilaikan hasil tambah bawah $L(f,P)$ dan hasil tambah atas $U(f,P)$.

   (ii). Apa akan jadi kepada nilai $U(f,P) - L(f,P)$ apabila kita tambahkan $\frac{5}{2}$ ke petakan?

(30 markah)

(b). Biar $H(x) = \int_1^x \frac{1}{t}\,dt, \quad x > 0$.

   (i). Dapatkan $H(1)$? Carikan $H'(x)$.

   (ii). Tunjukkan bahawa H menokok tegas.

(30 markah)

(c). Biar $f(x) = \begin{cases} 1 & \text{jika } x = \frac{1}{n} \text{ untuk suatu } n \in \mathbb{N} \\ 0 & \text{sebaliknya} \end{cases}$

Tunjukkan bahawa f terkamirkan pada [0, 1] dan hitungkan $\int_0^1 f$.

(40 markah)

APPENDIX

The Taylor's Formula

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)$$

The Taylor polynomial of order n generated by f centred at x_0

$$P_n^{f,\,x_0}(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Upper Riemann sum

$$U(f,P) := \sum_{j=1}^{n} M_j(f)\,\Delta x_j$$

where $M_j(f) := \sup f([x_{j-1}, x_j])$ and $\Delta x_j := x_j - x_{j-1}$

Lower Riemann sum

$$L(f,P) := \sum_{j=1}^{n} m_j(f)\,\Delta x_j$$

where $m_j(f) := \inf f([x_{j-1}, x_j])$ and $\Delta x_j := x_j - x_{j-1}$

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