Final Examination
2017/2018 Academic Session

May/June 2018

**JIM312 – Probability Theory**
***[Teori Kebarangkalian]***

Duration : 3 hours
*[Masa: 3 jam]*

Please ensure that this examination paper contains **EIGHT** printed pages before you begin the examination.

Answer **ALL** questions. You may answer either in Bahasa Malaysia or in English.

Read the instructions carefully before answering.

Each question is worth 100 marks.

In the event of any discrepancies, the English version shall be used.

*Sila pastikan bahawa kertas peperiksaan ini mengandungi **LAPAN** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.*

*Jawab **SEMUA** soalan. Anda dibenarkan menjawab sama ada dalam Bahasa Malaysia atau Bahasa Inggeris.*

*Baca arahan dengan teliti sebelum anda menjawab soalan.*

*Setiap soalan diperuntukkan 100 markah.*

*Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai.*

1. An urn contains two red, three orange and five blue balls. Two balls are randomly selected without replacement.

   (a). List the elements of the sample space of this experiment.

   (20 marks)

   (b). Find the probability that two blue balls are selected.

   (20 marks)

   (c). Find the probability that the second ball selected is orange given that the first ball selected is red.

   (20 marks)

   (d). Let $X$ represents the occasion that one red and one blue ball are selected. Construct the probability mass function of $X$.

   (20 marks)

   (e). What is $P(X=0)$?

   (20 marks)

2. Let $Y$ have the probability density function

$$f_Y(y)=\begin{cases} y, & 0\le y<1 \\ k-y, & 1\le y\le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

   Find

   (a). *k*.

   (25 marks)

   (b). $F_Y(y)$.

   (25 marks)

   (c). $P(Y>1.5)$.

   (25 marks)

   (d). $f_X(x)$, when $X=|Y-1|$.

   (25 marks)

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3. (a). Suppose $f_{X,Y}(x,y)=\frac{2}{3}(2x+y), 0\le x\le 1, 0\le y\le 1.$ Find the correlation between $X$ and $Y$.

(50 marks)

(b). Given two discrete random variables $X$ and $Y$ with joint probability mass function

$$P_{X,Y}(x,y)=k|x-y| \text{ for } x=1,2,3 \text{ and } y=1,2,3.$$

(i). Find $k$.

(ii). Evaluate $E(Y|X=x)$ for all values of $x$.

(50 marks)

4. (a). The moment generating function of $X$ is $m_X(t)=\exp\left\{\mu t+\frac{\sigma^2 t^2}{2}\right\}$.

Prove or disprove that $Y=aX+b$ has the same mean and variance of $X$.

(20 marks)

(b). Let $X_1,\ldots,X_n$ be independent Bernoulli($p$) random variables. Then prove that $Y=X_1+\ldots+X_n$ has the binomial($n$, $p$) distribution.

(20 marks)

(c). Suppose $X$ ~ gamma(4, 2). $X$ can also be chi-squared with $r$ degrees of freedom. What is the value of $r$?

(20 marks)

(d). How long should we expect to have to wait to get two 6s in a sequence of dice tosses?

(20 marks)

(e). Evaluate $\int_{-1.24}^{1.24} e^{-\frac{z^2}{2}}dz$ .

(20 marks)

5. (a). If $X_1 \sim \chi^2_m$ and $X_2 \sim \chi^2_n$ are two independent random variables, and $Y = X_1 + X_2$, then what is the distribution of $Y$?

(25 marks)

(b). State the Chebyshev Inequality. Demonstrate the inequality on Poisson(1) random variable.

(25 marks)

(c). State the Bayes Theorem. What happens to the theorem when the random variables involved are independent? Demonstrate with an example.

(25 marks)

(d). State the Central Limit Theorem. Demonstrate the Central Limit Theorem on $X_1, \ldots, X_{30}$, a random sample from the negative binomial(10,2) distribution by finding $P\left(\bar{X} \le 11\right)$ where $\bar{X} = \frac{1}{30}\sum_{i=1}^{30} X_i$.

(25 marks)

1. Sebuah balang mengandungi dua biji bola merah, tiga biji bola jingga dan lima biji bola biru. Dua biji bola dipilih secara rawak tanpa pengembalian.

   (a). Senaraikan unsur-unsur ruang sampel ujikaji ini.

   (20 markah)

   (b). Cari kebarangkalian dua biji bola biru dipilih.

   (20 markah)

   (c). Cari kebarangkalian bola yang kedua dipilih berwarna jingga diberikan bola yang pertama berwarna merah.

   (20 markah)

   (d). Andaikan $X$ mewakili peristiwa suatu bola merah dan suatu bola biru dipilih. Binakan fungsi jisim kebarangkalian bagi $X$.

   (20 markah)

   (e). Apakah $P(X=0)$?

   (20 markah)

2. Biar $Y$ mempunyai fungsi ketumpatan kebarangkalian

$$f_Y(y)=\begin{cases} y, & 0\le y<1 \\ k-y, & 1\le y\le 2 \\ 0, & \text{di tempat lain.} \end{cases}$$

   Cari

   (a). *k*.

   (25 markah)

   (b). $F_Y(y)$.

   (25 markah)

   (c). $P(Y>1.5)$.

   (25 markah)

   (d). $f_X(x)$, apabila $X=|Y-1|$.

   (25 markah)

3. (a). Andaikan $f_{X,Y}(x,y) = \frac{2}{3}(2x+y), 0 \le x \le 1, 0 \le y \le 1$. Cari korelasi di antara $X$ dan $Y$.

(50 markah)

(b). Diberikan dua pembolehubah rawak diskrit $X$ dan $Y$ yang mempunyai fungsi jisim tercantum

$$P_{X,Y}(x,y) = k|x-y| \text{ for } x = 1,2,3 \text{ dan } y = 1,2,3.$$

(i). Cari $k$.

(ii). Nilaikan $E(Y|X=x)$ bagi semua nilai $x$.

(50 markah)

4. (a). Fungsi penjana momen bagi $X$ ialah $m_X(t) = \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$.

Buktikan atau sangkalkan $Y = aX + b$ mempunyai min dan varians yang sama dengan $X$.

(20 markah)

(b). Andaikan $X_1, \ldots, X_n$ sebagai pembolehubah-pembolehubah rawak Bernoulli($p$) yang tak bersandar. Buktikan bahawa $Y = X_1 + \ldots + X_n$ tertabur secara binomial($n$, $p$).

(20 markah)

(c). Andaikan $X$ ~ gamma(4, 2). $X$ pun boleh tertabur secara khi-kuasa dua dengan darjah kebebasan $r$. Apakah nilai $r$?

(20 markah)

(d). Berapa lama kah kita harus menunggu untuk mendapatkan keputusan 6 muncul dua kali di dalam turutan lemparan dadu?

(20 markah)

(e). Nilaikan $\int_{-1.24}^{1.24} e^{-\frac{z^2}{2}} dz$ .

(20 markah)

5. (a). Jika $X_1 \sim \chi^2_m$ dan $X_2 \sim \chi^2_n$ adalah dua pembolehubah rawak tak bersandar, dan $Y = X_1 + X_2$, maka apakah taburan $Y$?

(25 markah)

(b). Nyatakan Ketaksamaan Chebyshev. Tunjukkan ketaksamaan ini pada pembolehubah rawak Poisson(1).

(25 markah)

(c). Nyatakan Teorem Bayes. Apakah yang berlaku kepada teorem ini apabila pembolehubah-pembolehubah tak bersandar dilibatkan? Tunjukkan dengan suatu contoh.

(25 markah)

(d). Nyatakan Teorem Had Memusat. Tunjukkan Teorem Had Memusat ke atas suatu sampel rawak $X_1, \ldots, X_{30}$ daripada taburan binomial negatif(10, 2) dengan mencari $P\left(\overline{X} \leq 11\right)$ apabila $\overline{X} = \frac{1}{30}\sum_{i=1}^{30} X_i$.

(25 markah)

**Formulas**

1. $\rho(X,Y)=\dfrac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y}$, $\text{Cov}(X,Y)=E(XY)-E(X)E(Y)$, $\sigma_X=\sqrt{E(X^2)-(E(X))^2}$

2. $f(x)=p^x(1-p)^{1-x}$, $x=0,1$, $0<p<1$. $E(X)=p$, $\text{Var}(X)=p(1-p)$. $m(t)=1-p+pe^t$.

3. $f(x)=\binom{n}{x}p^x(1-p)^{n-x}$, $x=0,1,2,\ldots,n$, $0<p<1$. $E(X)=np$, $\text{Var}(X)=np(1-p)$.

   $m(t)=(1-p+pe^t)^n$.

4. $f(x)=p(1-p)^x$, $x=0,1,2,\ldots$, $0<p<1$. $E(X)=\dfrac{1-p}{p}$, $\text{Var}(X)=\dfrac{1-p}{p^2}$.

   $m(t)=\dfrac{p}{1-(1-p)e^t}$.

5. $f(x)=\binom{r+x-1}{x}p^r(1-p)^x$, $x=0,1,2,\ldots$, $0<p<1$. $E(X)=\dfrac{r(1-p)}{p}$, $\text{Var}(X)=\dfrac{r(1-p)}{p^2}$.

   $m(t)=\left(\dfrac{p}{1-(1-p)e^t}\right)^r$, $t<-\log(1-p)$.

6. $f(x)=\dfrac{e^{-\lambda}\lambda^x}{x!}$, $x=0,1,2,\ldots$, $\lambda\ge 0$. $E(X)=\text{Var}(X)=\lambda$. $m(t)=e^{\lambda(e^t-1)}$.

7. $f(x)=\dfrac{1}{\sigma\sqrt{2\pi}}\exp\left[-\dfrac{(x-\mu)^2}{2\sigma^2}\right]$, $\sigma>0$. $E(X)=\mu$, $\text{Var}(X)=\sigma^2$. $m(t)=\exp\left[\mu t+\tfrac{1}{2}\sigma^2 t\right]$.

8. $f(x)=\dfrac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)}x^{\frac{k}{2}-1}e^{-\frac{x}{2}}$, $x\ge 0$. $E(X)=k$, $\text{Var}(X)=2k$. $m_X(t)=\left(\dfrac{1}{1-2t}\right)^{\frac{k}{2}}$, $t<\dfrac{1}{2}$.

9. $f(x)=\dfrac{1}{\beta}e^{-x/\beta}$, $x\ge 0,\beta>0$. $E(X)=\beta$, $\text{Var}(X)=\beta^2$. $m(t)=\dfrac{1}{1-\beta t}$, $t<\dfrac{1}{\beta}$.

10. $f(x)=\dfrac{1}{\Gamma(\alpha)\beta}x^{\alpha-1}e^{-x/\beta}$, $x\ge 0,\alpha>0,\beta>0$. $E(X)=\alpha\beta$, $\text{Var}(X)=\alpha\beta^2$.

    $m(t)=\left(\dfrac{1}{1-\beta t}\right)^{\alpha}$, $t<\dfrac{1}{\beta}$.

11. $P(|X-\mu|\ge t\alpha)\le\dfrac{1}{t^2}$.

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