UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2004/2005

October 2004

MAT 517 – COMPUTATIONAL LINEAR ALGEBRA AND FUNCTION APPROXIMATION
[ALJABAR LINEAR PENGKOMPUTERAN DAN PENGHAMPIRAN FUNGSI]

Duration : 3 hours

[Masa : 3 jam]

Please check that this examination paper consists of **EIGHT [8]** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **LAPAN [8]** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Answer **all FOUR [4]** questions.

Jawab **semua EMPAT [4]** soalan.

1. The Jacobi iterative method for finding an approximate solution to an $n \times n$ linear system of the form $\mathbf{Ax} = \mathbf{b}$ generates a sequence of approximation to $\mathbf{x}$, namely $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$, by employing the following recursive equation to obtain the ith component of $\mathbf{x}^{(k)}$,

$$x_i^{(k)} = \frac{\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-a_{ij} x_j^{(k-1)} \right) + b_i}{a_{ii}}, \qquad \text{for } i = 1, \ldots, n. \qquad (1.1)$$

where $\mathbf{A} = [a_{ij}]$, $\mathbf{x}^{(k)} = [x_i^{(k)}]$ and $\mathbf{b} = [b_i]$, $1 \le i, j \le n$, $k \ge 1$.

(a) Show that the Jacobi iteration can be written in the form

$$\mathbf{x}^{(k)} = \mathbf{T}_J \mathbf{x}^{(k-1)} + \mathbf{c}_J,$$

where $\mathbf{T}_J = \mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})$ and $\mathbf{c}_J = \mathbf{D}^{-1}\mathbf{b}$ such that $\mathbf{D}$ is a diagonal matrix whose diagonal entries are those of $\mathbf{A}$, $-\mathbf{L}$ is the strictly lower triangular part of $\mathbf{A}$ and $-\mathbf{U}$ is the strictly upper triangular part of $\mathbf{A}$.

(b) Modify equation (1.1) to give the Gauss-Seidel iterative technique and show that the Gauss-Seidel technique can be written in the form

$$\mathbf{x}^{(k)} = \mathbf{T}_G \mathbf{x}^{(k-1)} + \mathbf{c}_G$$

where $\mathbf{T}_G = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{U}$ and $\mathbf{c}_G = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{b}$, with matrices $\mathbf{D}$, $\mathbf{L}$ and $\mathbf{U}$ are the same as given in part a).

(c) Given that $\mathbf{A} = \begin{pmatrix} 1 & 1/2 \\ -1 & 1 \end{pmatrix}$, find the spectral radius of $\mathbf{T}_J$ and $\mathbf{T}_G$ and, without solving the system $\mathbf{Ax} = \mathbf{b}$, confirm that both Jacobi and Gauss-Seidel iterative technique will converge to the unique solution of the system $\mathbf{Ax} = \mathbf{b}$, for any choice of $\mathbf{b}$.

[100 marks]

1. *Kaedah lelaran Jacobi untuk mencari penghampiran kepada penyelesaian sistem linear $n \times n$ dalam bentuk $\mathbf{Ax} = \mathbf{b}$, membentuk satu jujukan penghampiran kepada $\mathbf{x}$, iaitu $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$, melalui penggunaan persamaan rekursif berikut untuk mendapatkan komponen ke- i vektor $\mathbf{x}^{(k)}$,*

$$x_i^{(k)} = \frac{\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-a_{ij} x_j^{(k-1)} \right) + b_i}{a_{ii}}, \qquad \textit{untuk } i = 1, \ldots, n.$$

(1.1)

di mana $\mathbf{A} = [a_{ij}]$, $\mathbf{x}^{(k)} = [x_i^{(k)}]$ *dan* $\mathbf{b} = [b_i]$, $1 \le i, j \le n$, $k \ge 1$.

...3/-

(a) *Tunjukkan bahawa lelaran Jacobi boleh ditulis dalam bentuk*

$$\mathbf{x}^{(k)} = \mathbf{T}_J\mathbf{x}^{(k-1)} + \mathbf{c}_J,$$

di mana $\mathbf{T}_J = \mathbf{D}^{-1}(\mathbf{L}+\mathbf{U})$ *dan* $\mathbf{c}_J = \mathbf{D}^{-1}\mathbf{b}$, $\mathbf{D}$ *ialah matriks pepenjuru yang mempunyai pemasukan pepenjuru yang sama dengan pepenjuru* $\mathbf{A}$, $-\mathbf{L}$ *ialah bahagian segitiga bawah tegas* $\mathbf{A}$ *dan* $-\mathbf{U}$ *ialah bahagian segitiga atas tegas* $\mathbf{A}$.

(b) *Ubahsuai persamaan (1.1) untuk memberikan teknik lelaran Gauss-Seidel dan tunjukkan bahawa teknik Gauss-Seidel boleh ditulis dalam bentuk*

$$\mathbf{x}^{(k)} = \mathbf{T}_G\mathbf{x}^{(k-1)} + \mathbf{c}_G$$

di mana $\mathbf{T}_G = (\mathbf{D}-\mathbf{L})^{-1}\mathbf{U}$ *dan* $\mathbf{c}_G = (\mathbf{D}-\mathbf{L})^{-1}\mathbf{b}$. *Matriks* $\mathbf{D}$, $\mathbf{L}$ *dan* $\mathbf{U}$ *sama seperti yang diberikan dalam bahagian a).*

(c) *Diberikan* $\mathbf{A} = \begin{pmatrix} 1 & 1/2 \\ -1 & 1 \end{pmatrix}$, *cari jejari spektrum* $\mathbf{T}_J$ *dan* $\mathbf{T}_G$, *dan, tanpa menyelesaikan sistem* $\mathbf{Ax} = \mathbf{b}$, *sahkan bahawa kedua-dua teknik lelaran Jacobi dan Gauss-Seidel akan menumpu kepada penyelesaian unik system tersebut untuk sebarang pilihan* $\mathbf{b}$.

[100 markah]

2. Consider the regression model of the form $\mathbf{b} = \mathbf{Ax}$ for some $m \times n$ data matrix $\mathbf{A}$, $m \times 1$ observation vector $\mathbf{b}$ and $n \times 1$ vector of independent variables $\mathbf{x}$.

(a) Show that

$$\|\mathbf{Ax}-\mathbf{b}\|_2^2 = \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} - \mathbf{x}^T\mathbf{A}^T\mathbf{b} - \mathbf{b}^T\mathbf{Ax} + \mathbf{b}^T\mathbf{b}.$$

(Note that for any vector $\mathbf{y}$, $\|\mathbf{y}\|_2^2 = \mathbf{y}^T\mathbf{y}$).

(b) Show that both $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ are square and symmetric.

(c) Find the value $\mathbf{x}_{LS}$ that minimizes squared error norm $\|\mathbf{Ax}-\mathbf{b}\|_2^2$ by considering the partial derivative of $\|\mathbf{Ax}-\mathbf{b}\|_2^2$ with respect to $\mathbf{x}$ and show that

$$\mathbf{x}_{LS} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}.$$

(*HINT*: Solve for $\mathbf{x}$ such that $\frac{\partial}{\partial \mathbf{x}}\|\mathbf{Ax}-\mathbf{b}\|_2^2 = \mathbf{0}$)

(d) Let $\mathbf{A} = \mathbf{U\Sigma V}^T$ be the singular value decomposition of matrix $\mathbf{A}$ so that $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices and Σ is a diagonal matrix whose diagonal elements are the singular values of $\mathbf{A}$. By considering matrices $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ and their properties, describe how $\mathbf{U}, \mathbf{V}$ and Σ can be obtained. Hence show that

$$\mathbf{x}_{LS} = \mathbf{V\Sigma}^{-1}\mathbf{U}^T\mathbf{b}$$

[100 marks]

2. *Pertimbangkan model regresi dalam bentuk* $\mathbf{b} = \mathbf{Ax}$ *untuk suatu matriks data* $m \times n$ $\mathbf{A}$, *vektor pemerhatian* $m \times 1$ $\mathbf{b}$ *dan vektor pembolehubah bebas* $n \times 1$ $\mathbf{x}$.

(a) *Tunjukkan bahawa*

$$\|\mathbf{Ax} - \mathbf{b}\|_2^2 = \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} - \mathbf{x}^T\mathbf{A}^T\mathbf{b} - \mathbf{b}^T\mathbf{Ax} + \mathbf{b}^T\mathbf{b}.$$

(Perhatikan bahawa untuk sebarang vektor $\mathbf{y}$, $\|\mathbf{y}\|_2^2 = \mathbf{y}^T\mathbf{y}$*).*

(b) *Tunjukkan bahawa kedua-dua* $\mathbf{AA}^T$ *dan* $\mathbf{A}^T\mathbf{A}$ *adalah bersegiempatsama dan bersimetri.*

(c) *Cari nilai* $\mathbf{x}_{LS}$ *yang meminimumkan norma ralat kuasa dua* $\|\mathbf{Ax} - \mathbf{b}\|_2^2$ *dengan mempertimbangkan beza separa* $\|\mathbf{Ax} - \mathbf{b}\|_2^2$ *terhadap* $\mathbf{x}$ *dan tunjukkan bahawa*

$$\mathbf{x}_{LS} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}.$$

(PETUNJUK: Selesaikan persamaan $\frac{\partial}{\partial \mathbf{x}}\|\mathbf{Ax} - \mathbf{b}\|_2^2 = 0$ *untuk* $\mathbf{x}$*)*

(d) *Biar* $\mathbf{A} = \mathbf{U\Sigma V}^T$ *menjadi penghuraian nilai singular matriks* $\mathbf{A}$ *yang mana matriks* $\mathbf{U}$ *dan* $\mathbf{V}$ *adalah berortogon dan* Σ *ialah matriks pepenjuru dengan pemasukkan pepenjurunya adalah nilai-nilai singular matriks* $\mathbf{A}$. *Dengan mempertimbangkan matriks* $\mathbf{AA}^T$, $\mathbf{A}^T\mathbf{A}$ *dan ciri-ciri matriks tersebut, huraikan bagaimana* $\mathbf{U}, \mathbf{V}$ *dan* Σ *boleh diperolehi. Seterusnya tunjukkan bahawa*

$$\mathbf{x}_{LS} = \mathbf{V\Sigma}^{-1}\mathbf{U}^T\mathbf{b}$$

[100 markah]

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3. (a) The nth degree Lagrange interpolating polynomial for some function f is defined as

$$P(x) = f(x_0)L_{n,0}(x) + \ldots + f(x_n)L_{n,n}(x)$$

where, for each $k = 0,1,\ldots,n$,

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

and $x_0, x_1, \ldots x_n$ are $k+1$ distinct numbers such that

$$f(x_k) = P(x_k) \text{ for each } k = 0,1,\ldots,n.$$

(i) Let $P_3(x)$ be the Lagrange interpolating polynomial for the data $(0,0)$, $(0.5, y)$, $(1,3)$ and $(2,2)$. Find y if the coefficient of x^3 in $P_3(x)$ is 5.

(ii) For the function $f(x) = \sqrt{1+x}$ let $x_0 = 0$, $x_1 = 0.6$ and $x_2 = 0.9$. Construct the Lagrange interpolating polynomial of degree 2 to approximate $f(0.45)$ and find the actual error.

(b) The Hermite polynomial of degree at most $2n+1$ for some function $f \in C^1[a,b]$ is given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_{n,j}(x),$$

where

$$H_{n,j}(x) = [1 - 2(x-x_j)L'_{n,j}(x_j)]L^2{}_{n,j}(x)$$

and

$$\hat{H}_{n,j}(x) = (x-x_j)L^2{}_{n,j}(x),$$

with distinct numbers $x_0, x_1, \ldots, x_n \in [a,b]$. $L_{n,j}(x)$ denotes the jth Lagrange coefficient of degree n as defined in part a).

(i) Show that the Hermite polynomial with definition given above agrees with f at $x_0, x_1, \ldots, x_n$ (i.e. show that $f(x_k) = H_{2n+1}(x_k)$ for each $k = 0,1,\ldots,n$).

(ii) Show that the Hermite polynomial with definition given above agrees with f' at $x_0, x_1, \ldots, x_n$ (i.e. show that $f'(x_k) = H'_{2n+1}(x_k)$ for each $k = 0,1,\ldots,n$).

(iii) Show that $H_{2n+1}(x)$ is the unique polynomial of least degree agreeing with f and f' at $x_0, x_1, \ldots, x_n$. (*HINT*: Assume that $P(x)$ is another such polynomial then consider $D = H_{2n+1} - P$ and D' at $x_0, x_1, \ldots, x_n$).

[100 marks]

3. (a) *Polinomial interpolasi Lagrange berdarjah n untuk sesuatu fungsi ditakrifkan sebagai*

$$P(x) = f(x_0)L_{n,0}(x) + \ldots + f(x_n)L_{n,n}(x)$$

di mana, untuk setiap $k = 0,1,\ldots,n$,

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

dan $x_0, x_1, \ldots x_n$ *adalah* $k+1$ *nombor yang berbeza supaya*

$$f(x_k) = P(x_k) \text{ untuk setiap } k = 0,1,\ldots,n.$$

(i) *Biar* $P_3(x)$ *mewakili polinomial interpolasi Lagrange untuk data* $(0,0)$, $(0.5,y)$, $(1,3)$ *dan* $(2,2)$. *Cari* y *jika pekali* x^3 *dalam* $P_3(x)$ *ialah 5.*

(ii) *Untuk fungsi* $f(x) = \sqrt{1+x}$ *biar* $x_0 = 0$, $x_1 = 0.6$ *dan* $x_2 = 0.9$. *Bina polinomial interpolasi Lagrange berdarjah 2 untuk mendapatkan penghampiran kepada* $f(0.45)$ *dan cari ralat sebenar penghampiran tersebut.*

(b) *Polinomial Hermite berdarjah paling tinggi* $2n+1$ *untuk sesuatu fungsi* $f \in C^1[a,b]$ *diberikan oleh*

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_{n,j}(x),$$

di mana

$$H_{n,j}(x) = [1 - 2(x-x_j)L'_{n,j}(x_j)]L^2{}_{n,j}(x)$$

dan

$$\hat{H}_{n,j}(x) = (x-x_j)L^2{}_{n,j}(x),$$

dengan nombor-nombor berbeza $x_0, x_1, \ldots, x_n \in [a,b]$. $L_{n,j}(x)$ *melambangkan pekali Lagrange ke j berdarjah n dan ditakrifkan seperti dalam bahagian a).*

(i) *Tunjukkan bahawa polinomial Hermite dengan takrifan di atas menyamai nilai f pada* $x_0, x_1, \ldots, x_n$ *(dalam perkataan lain tunjukkan bahawa* $f(x_k) = H_{2n+1}(x_k)$ *untuk setiap* $k = 0,1,\ldots,n$*).*

(ii) *Tunjukkan bahawa polinomial Hermite dengan takrifan di atas menyamai nilai f' pada* $x_0, x_1, \ldots, x_n$ *(dalam perkatan lain tunjukkan bahawa* $f'(x_k) = H'_{2n+1}(x_k)$ *untuk setiap* $k = 0,1,\ldots,n$*).*

(iii) *Tunjukkan bahawa* $H_{2n+1}(x)$ *ialah polinomial unik berdarjah paling rendah yang menyamai nilai* f *dan* f' *pada* $x_0, x_1, \ldots, x_n$. *(PETUNJUK: Andaikan* $P(x)$ *ialah polynomial lain yang menepati ciri-ciri tersebut, kemudian pertimbangkan* $D = H_{2n+1} - P$ *dan* D' *pada* $x_0, x_1, \ldots, x_n$*).*

[100 markah]

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4. The Chebyshev polynomials $\{T_n(x)\}$ are defined as follows. For $x \in [-1,1]$,

$$T_n(x) = \cos(n\cos^{-1} x), \qquad \text{for each } n \geq 0.$$

(a) Show that
(i) $T_0(x) = 1$;
(ii) $T_1(x) = x$;

(b) For $n \geq 1$, show that

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x).$$

(*HINT*: Use the substitution $\theta = \cos^{-1} x$ and trigonometric identities
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin \sin b$$
on $T_{n+1}(\theta)$ and $T_{n-1}(\theta)$).

(c) Show that the Chebyshev polynomials are orthogonal on $(-1,1)$ with respect to the weight function $w(x) = (1-x^2)^{-1/2}$.

(d) Confirm that $\tilde{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right)$, for each $k = 1,2,\ldots,n$ are the zeros of the Chebyshev polynomial $T_n(x)$ which is of degree $n \geq 1$.

(e) Use the zeros of the Chebyshev polynomial $T_3(x)$ to construct the Lagrange interpolating polynomial of degree 2 for the function $f(x) = \ln(x+2)$ on the interval $[-1,1]$.

[100 marks]

4. Polinomial Chebyshev $\{T_n(x)\}$ ditakrifkan seperti berikut. Untuk $x \in [-1,1]$,

$$T_n(x) = \cos(n\cos^{-1} x), \qquad \textit{untuk setiap } n \geq 0.$$

(b) Tunjukkan bahawa
(i) $T_0(x) = 1$;
(ii) $T_1(x) = x$;

(b) Untuk $n \geq 1$, tunjukkan bahawa

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x).$$

(PETUNJUK: Gunakan penggantian $\theta = \cos^{-1} x$ *dan identiti trigonometri*

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\cos(a-b) = \cos a \cos b + \sin \sin b$$

ke atas $T_{n+1}(\theta)$ *dan* $T_{n-1}(\theta)$*).*

(c) *Tunjukkan bahawa polinomial Chebyshev berortogon pada* $(-1,1)$ *tertakluk kepada fungsi pemberat* $w(x) = (1-x^2)^{-1/2}$.

(d) *Sahkan bahawa* $\tilde{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right)$, *untuk setiap* $k = 1,2,\ldots,n$ *adalah pensifar polinomial Chebyshev* $T_n(x)$ *berdarjah* $n \geq 1$.

(e) *Gunakan pensifar polinomial Chebyshev* $T_3(x)$ *untuk membina polinomial interpolasi Lagrange berdarjah 2 untuk fungsi* $f(x) = \ln(x+2)$ *pada selang* $[-1,1]$.

[100 markah]

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