UNIVERSITI SAINS MALAYSIA

Final Examination
2016/2017 Academic Session

May/June 2017

JIM 417 – Partial Differential Equations
[Persamaan Pembezaan Separa]

Duration : 3 hours
[*Masa: 3 jam*]

Please ensure that this examination paper contains **TEN** printed pages before you begin the examination.

Answer **ALL** questions. You may answer either in Bahasa Malaysia or in English.

Read the instructions carefully before answering.

Each question is worth 100 marks.

In the event of any discrepancies, the English version shall be used.

Sila pastikan bahawa kertas peperiksaan ini mengandungi ***SEPULUH*** *muka surat yang bercetak sebelum anda memulakan peperiksaan ini.*

Jawab ***SEMUA*** *soalan. Anda dibenarkan menjawab sama ada dalam Bahasa Malaysia atau Bahasa Inggeris.*

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah

Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai.

1. (a) Show that

$$u(x,y) = e^{-6x}\left(Ae^{4y} + Be^{-9y}\right)$$

where A and B are constant, is a general solution of partial differential equation

$$u_{xx} = u_{yy} + 5u_y\,.$$

Hence, determine the values of A and B if the solution satisfies the initial conditions

$$u(x,0) = e^{-6x}$$
$$u_y(0,0) = 0.$$

(40 marks)

(b) By assuming $u(x,t) = w(x)\sin t$, find a solution of the equation

$$u_{xx} - u_{tt} = x\sin t$$

satisfying the boundary conditions

$$u(0,t) = 0$$
$$u(\ell,t) = 0, \qquad \ell \neq \pi.$$

(30 marks)

(c) Find the general solution of the partial differential equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - 2u = x.$$

(30 marks)

2. A periodic function $f(x)$ is defined as

$$f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \leq x < 2 \end{cases}$$

and

$$f(x) = f(x+2).$$

(a) Sketch the graph of the function f for at least 3 periods.

(10 marks)

(b) Find the Fourier coefficients corresponding to the function f.

(60 marks)

(c) Obtain the corresponding Fourier Series of the function *f*.

(10 marks)

(d) Obtain the sum of the infinite series

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$$

(20 marks)

3. Given the partial differential equation

$$\frac{\partial^2 u}{\partial x^2}+10\frac{\partial^2 u}{\partial x \partial y}+9\frac{\partial^2 u}{\partial y^2}=y\,.$$

(a) Determine the type of the equation.

(10 marks)

(b) Obtain the characteristic equations.

(15 marks)

(c) Use the transformation

$$\xi = y-9x$$
$$\eta = y-x$$

to reduce the equation to canonical form.

(40 marks)

(d) Find the general solution of the partial differential equation.

(35 marks)

4. A chemical diffusion is governed by

$$\frac{\partial C}{\partial t}=k\frac{\partial^2 C}{\partial x^2}-LC$$
$$C(0,t)=C(a,t)=0, \qquad t\geq 0$$
$$C(x,0)=f(x), \qquad 0\leq x\leq a.$$

(a) Use the method of separation of variables to determine the concentration for $C(x,t)$.

(40 marks)

(b) What happens to the concentration as $t \to \infty$?

(20 marks)

(c) What is the concentration $C(x,t)$ if the initial condition is

$$C(x,0) = \cos\left(\frac{\pi x}{a}\right)?$$

(40 marks)

5. Use the method of Laplace transform to solve the following initial-boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}, \qquad (x \le x \le 2,\ t \ge 0)$$

$$u(x,0) = 0, \qquad (0 \le x \le 2)$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \qquad (0 \le x \le 2)$$

$$\frac{\partial u}{\partial x}(2,t) = \frac{F}{E}, \qquad (t \ge 0)$$

where E and F are constants.

Given $\mathscr{L}^{-1}\left\{\dfrac{c\ \sinh\left(\dfrac{sx}{c}\right)}{s^2\cosh\left(\dfrac{sl}{c}\right)}\right\}$

$$= x + \frac{8l}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi x}{2l}\right)\cos\left(\frac{(2n-1)ct}{2l}\right)\right].$$

(100 marks)

1. (a) Tunjukkan bahawa

$$u(x,y) = e^{-6x}\left(Ae^{4y} + Be^{-9y}\right)$$

di mana A dan B adalah pemalar, merupakan penyelesaian am persamaan pembezaan separa

$$u_{xx} = u_{yy} + 5u_y .$$

Seterusnya, tentukan nilai A dan B jika penyelesaian tersebut memuaskan syarat awal

$$u(x,0) = e^{-6x}$$
$$u_y(0,0) = 0.$$

(40 markah)

(b) Dengan menganggap $u(x,t) = w(x)\sin t,$ cari penyelesaian persamaan

$$u_{xx} - u_{tt} = x\sin t$$

yang memuaskan syarat sempadan

$$u(0,t) = 0$$
$$u(\ell,t) = 0, \qquad \ell \neq \pi.$$

(30 markah)

(c) Cari penyelesaian am persamaan pembezaan separa

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - 2u = x.$$

(30 markah)

2. Fungsi berkala $f(x)$ ditakrifkan sebagai

$$f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \le x < 2 \end{cases}$$

dan

$$f(x) = f(x+2).$$

(a) Lakar graf fungsi f untuk sekurang-kurangnya tiga kala.

(10 markah)

(b) Cari koefisien Fourier yang sepadan bagi fungsi f.

(60 markah)

(c) Dapatkan Siri Fourier yang sepadan bagi *f*.

(10 markah)

(d) Dapatkan hasil tambah bagi siri tak terhingga

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$$

(20 markah)

3. Diberi persamaan pembezaan separa

$$\frac{\partial^2 u}{\partial x^2}+10\frac{\partial^2 u}{\partial x \partial y}+9\frac{\partial^2 u}{\partial y^2}=y\,.$$

(a) Tentukan jenis persamaan.

(10 markah)

(b) Dapatkan persamaan cirian.

(15 markah)

(c) Guna transformasi

$$\xi = y-9x$$
$$\eta = y-x$$

untuk menurunkan persamaan tersebut kepada bentuk berkanun.

(40 markah)

(d) Cari penyelesaian am bagi persamaan pembezaan separa tersebut.

(35 markah)

4. Suatu tindakbalas kimia diperihalkan oleh

$$\frac{\partial C}{\partial t}=k\frac{\partial^2 C}{\partial x^2}-LC$$
$$C(0,t)=C(a,t)=0, \qquad t\ge 0$$
$$C(x,0)=f(x), \qquad 0\le x\le a.$$

(a) Guna kaedah pemisahan pembolehubah untuk mencari kepekatan bagi $C(x,t)$.

(40 markah)

(b) Apa akan terjadi kepada kepekatan sekiranya $t \to \infty$?

(20 markah)

(c) Apakah kepekatan $C(x,t)$ sekiranya syarat awal adalah $C(x,0) = \cos\left(\frac{\pi x}{a}\right)$?

(40 markah)

5. Guna kaedah jelmaan Laplace untuk menyelesaikan masalah nilai awal-sempadan berikut:

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}, \qquad (x \le x \le 2,\ t \ge 0)$$
$$u(x,0) = 0, \qquad (0 \le x \le 2)$$
$$\frac{\partial u}{\partial t}(x,0) = 0, \qquad (0 \le x \le 2)$$
$$\frac{\partial u}{\partial x}(2,t) = \frac{F}{E}, \qquad (t \ge 0)$$

di mana *E* dan *F* adalah pemalar.

Diberi $\mathscr{L}^{-1}\left\{\frac{c\sinh\left(\frac{sx}{c}\right)}{s^2\cosh\left(\frac{sl}{c}\right)}\right\}$

$$= x + \frac{8l}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi x}{2l}\right)\cos\left(\frac{(2n-1)ct}{2l}\right)\right].$$

(100 markah)

Formulae

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}.$$

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

with

$$a_o = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\ \ n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx,\ \ n = 1, 2, 3, \ldots$$

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

with

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\ \ n = 1, 2, 3, \ldots$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

with

$$b_n = \frac{2}{L}\int_0^L f(x)\left(\frac{n\pi x}{L}\right)dx, \;\; n = 1, 2, 3, \ldots$$

$$f(x) = \frac{1}{2}\sum_{-\infty}^{\infty} c_n e^{inx}$$

with

$$c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx, \;\; n = 0, \pm 1, \pm 2, \ldots$$

$\frac{d^2y}{dx^2} - \alpha^2 y = 0$ has solution

$$y = Ae^{\alpha x} + Be^{-\alpha x} \text{ or } C \cosh \; \alpha x + D \sinh \; \alpha x.$$

$\frac{d^2y}{dx^2} + \alpha^2 y = 0$ has solution

$$y = A \cos \alpha x + B \sin \alpha x.$$

$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R = 0$ has solution

$$R_n = C_n r^n + \frac{D_n}{r^n}$$

$r\frac{d^2R}{dr^2} + r\frac{dR}{dr} = 0$ has solution

$$R = A + B \ln r.$$

$$\mathscr{L}\left[e^{\alpha t} f(t)\right] = F(s-\alpha).$$

$$\mathscr{L}\{H(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathscr{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$$

$$\mathscr{L}\left[f^n(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n}\left[F(s)\right]$$

$$\mathscr{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$$

$$\mathscr{L}^{-1}\left\{F(s)G(s)\right\} = \int_0^t f(u)\,g(t-u)\,du = f * g$$

Laplace Transforms

$f(t)$	$\mathscr{L}\left\{f(t)\right\} = F(s)$
1	$\frac{1}{s}$
t^n, $n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$t \cos bt$	$\frac{s^2-a^2}{\left(s^2+b^2\right)^2}$
$t \sin bt$	$\frac{2bs}{\left(s^2+b^2\right)^2}$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$

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