UNIVERSITI SAINS MALAYSIA

Final Examination
2016/2017 Academic Session

May/June 2017

JIM 312 – Probability Theory
[Teori Kebarangkalian]

Duration : 3 hours
[Masa: 3 jam]

Please ensure that this examination paper contains **TEN** printed pages before you begin the examination.

Answer **ALL** questions. You may answer either in Bahasa Malaysia or in English.

Read the instructions carefully before answering.

Each question is worth 100 marks.

In the event of any discrepancies, the English version shall be used.

Sila pastikan bahawa kertas peperiksaan ini mengandungi ***SEPULUH*** *muka surat yang bercetak sebelum anda memulakan peperiksaan ini.*

Jawab ***SEMUA*** *soalan. Anda dibenarkan menjawab sama ada dalam Bahasa Malaysia atau Bahasa Inggeris.*

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah.

Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai.

1. (a) Suppose $f_{X,Y}(x,y) = \frac{2}{3}(2x+y), 0 \le x \le 1, 0 \le y \le 1.$ Find $P(X < Y)$.

(25 marks)

(b) Given two discrete random variables X and Y with joint probability mass function

$P(x, y) = k|x + y|$ for $x = 1, 2, 3$ and $y = 1, 2, 3$.

(i) Find k.

(ii) Evaluate $P_{X|y}(3)$ for all values of y.

(50 marks)

(c) Let Y have the probability density function

$$f_Y(y) = \begin{cases} y, & 0 \le y < 1 \\ 2 - y, & 1 \le y \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Find $F_Y(y)$.

(25 marks)

2. (a) An urn contains five red, three orange and two blue balls. Two balls are randomly selected. Let X represents the occasion that two blue balls are selected. Let Y be the occasion that one red ball and one orange ball are selected. Find the correlation between X and Y.

(50 marks)

(b) The moment generating function of X is $m_X(t) = \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$. Prove or disprove that $Y = aX + b$ has the same coefficients of skewness and kurtosis as X.

(30 marks)

(c) X is a random variable with mean μ and variance σ^2. Let k be any value such that it is positive. Use the Markov inequality to derive the Chebyshev's inequality.

(20 marks)

3. (a) (i) Evaluate $\int_{-1.24}^{0} e^{-\frac{z^2}{2}} dz$.

(ii) Evaluate $\int_{-\infty}^{\infty} e^{-\left(\frac{x^2-2x+1}{2}\right)} dx$.

(25 marks)

(b) Find the mean of the geometric random variable directly using its probability mass function.

(25 marks)

(c) Given that X is a random variable with finite variance. Prove or disprove $E[X^2] > \{E[X]\}^2$.

(25 marks)

(d) $X_1, \ldots, X_n$ are independent random variables having a common Poisson(θ) distribution. Let $T\left(X_1, \ldots, X_n\right) = \sum_{i=1}^{n} X_i$. Show that the conditional distribution of $X_1, \ldots, X_n$ given $T\left(X_1, \ldots, X_n\right)$ does not depend upon θ.

(25 marks)

4. (a) Evaluate $\Gamma\left(\frac{3}{2}\right)$.

(25 marks)

(b) The final examination scores of this course over 10 years have a mean and variance of 60 and 64, respectively. The class of 2012 has n = 100 students. The mean score for that year was 58. Is there evidence that this class is under performing?

(25 marks)

(c) (i) Show that the Gamma$\left(\frac{\nu}{2}, 2\right)$ distribution is equivalent to a chi-squared distribution with ν degrees of freedom.

(ii) Suppose X ~ Gamma(4, 2), evaluate $P(X > 2)$.

(50 marks)

5. (a) Suppose IQ scores are normally distributed with a mean of 100 and a standard deviation of 16.

(i) If one person is chosen at random, find the probability that she/he will have an IQ score between 50 and 80.

(ii) If one person is chosen at random, find the probability that she/he will have an IQ score greater than 130.

(iii) Suppose 10 people are selected independently for a study. Let Y denote the number of people in this group with an IQ score over 130. What is the probability that at least 2 people in this group will have an IQ score over 130?

(50 marks)

(b) (i) How long should we expect to have to wait to get a 6 in a sequence of die tosses?

(ii) Let $X_1,\ldots,X_n$ be independent Geometric (p) random variables. Then prove that $Y = X_1 + \cdots + X_n$ has the negative binomial distribution with parameters n and p.

(30 marks)

(c) If $X_1 \sim \text{Gamma}(\alpha_1, \beta)$ and $X_2 \sim \text{Gamma}(\alpha_2, \beta)$ are two independent random variables, and $Y = X_1 + X_2$, then prove that $Y \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$.

(20 marks)

1. (a) Andaikan $f_{X,Y}(x,y) = \frac{2}{3}(2x+y), 0 \le x \le 1, 0 \le y \le 1.$ Cari $P(X < Y)$.

(25 markah)

(b) Diberikan dua pembolehubah rawak diskrit X dan Y yang mempunyai fungsi jisim tercantum $P(x, y) = k|x + y|$ bagi $x = 1, 2, 3$ dan $y = 1, 2, 3$.

(i) Cari k.

(ii) Nilaikan $P_{X|y}(3)$ untuk semua nilai y.

(50 markah)

(c) Biar Y mempunyai fungsi ketumpatan kebarangkalian

$$f_Y(y) = \begin{cases} y, & 0 \le y < 1 \\ 2-y, & 1 \le y \le 2 \\ 0, & \text{di tempat lain.} \end{cases}$$

Cari $F_Y(y)$.

(25 markah)

2. (a) Sebuah balang mengandungi lima biji bola merah, tiga biji bola jingga dan dua biji bola biru. Dua biji bola dipilih secara rawak. Katakan X mewakili dua biji bola biru dipilih. Katakan Y mewakili sebiji bola merah dan sebiji bola jingga dipilih. Cari korelasi di antara X dan Y.

(50 markah)

(b) Fungsi penjana momen bagi X ialah $m_X(t) = \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$. Buktikan atau sangkalkan $Y = aX + b$ mempunyai pekali kepencongan dan pekali kurtosis yang sama dengan X.

(30 markah)

(c) X ialah suatu pembolehubah rawak yang mempunyai min μ dan varians σ^2. Biar k mengambil sebarang nilai positif. Guna ketaksamaan Markov untuk menerbit ketaksamaan Chebyshev.

(20 markah)

3. (a) (i) Nilaikan $\int_{-1.24}^{0} e^{-\frac{z^2}{2}} dz$.

(ii) Nilaikan $\int_{-\infty}^{\infty} e^{-\left(\frac{x^2-2x+1}{2}\right)} dx$.

(25 markah)

(b) Cari min pembolehubah rawak geometri secara terus dengan menggunakan fungsi jisim kebarangkalian pembolehubah tersebut.

(25 markah)

(c) Diberikan X adalah suatu pembolehubah rawak yang mempunyai varians terhingga. Buktikan atau sangkalkan $E[X^2] > \{E[X]\}^2$.

(25 markah)

(d) $X_1, \ldots, X_n$ adalah pembolehubah-pembolehubah rawak tak bersandar yang mempunyai taburan sepunya Poisson(θ). Andaikan $T\left(X_1, \ldots, X_n\right) = \sum_{i=1}^{n} X_i$. Tunjukkan taburan bersyarat $X_1, \ldots, X_n$ diberikan $T\left(X_1, \ldots, X_n\right)$ tidak bergantung kepada θ.

(25 markah)

4. (a) Nilaikan $\Gamma\left(\frac{3}{2}\right)$.

(25 markah)

(b) Skor peperiksaan akhir kursus ini selama 10 tahun mempunyai min 60 dan varians 64. Kelas 2012 ada n = 100 pelajar. Min skor tahun tersebut adalah 58. Terdapatkah bukti bahawa pelajar di dalam kelas ini kurang cerdas?

(25 markah)

(c) (i) Tunjukkan taburan Gamma$\left(\frac{\nu}{2}, 2\right)$ setara dengan taburan khi-kuasa dua dengan darjah kebebasan ν.

(ii) Sekira X ~ Gamma(4, 2), nilaikan $P(X > 2)$.

(50 markah)

5. (a) Andaikan skor IQ tertabur secara normal dengan min 100 dan sisihan piawai 16.

(i) Jika seorang dipilih secara rawak, cari kebarangkalian skor IQ akan berada di antara 50 dan 80.

(ii) Jika seorang dipilih secara rawak, cari kebarangkalian skor IQ akan melebihi 130.

(iii) Sepuluh orang daripada kumpulan ini dipilih secara tak bersandar untuk suatu kajian. Andaikan Y mewakili bilangan mereka yang dipilih mempunyai skor IQ melebihi 130. Berapakah kebarangkalian sekurang-kurangnya 2 orang yang terpilih ini mempunyai skor IQ melebihi 130?

(50 markah)

(b) (i) Berapa lamakah kita harus menunggu untuk mendapatkan keputusan 6 di dalam turutan lemparan dadu?

(ii) Andaikan $X_1, \ldots, X_n$ sebagai pembolehubah-pembolehubah Geometri (p) yang tak bersandar. Buktikan bahawa $Y = X_1 + \cdots + X_n$ mempunyai taburan binomial negatif berparameterkan n dan p.

(30 markah)

(c) Jika $X_1 \sim \text{Gamma}(\alpha_1, \beta)$ dan $X_2 \sim \text{Gamma}(\alpha_2, \beta)$ adalah dua pembolehubah rawak yang tak bersandar, dan $Y = X_1 + X_2$, maka buktikan $Y \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$.

(20 markah)

List of Formulas

1. $F_Y(t) = F_X(g^{-1}(t))$

2. $F_Y(t) = 1 - F_X(g^{-1}(t))$

3. $P\left(\left|X - \mu_X\right| \geq a\sigma_X\right) \leq \dfrac{1}{a^2}$

4. $E\left(G\left(X\right)\right) = \sum_x G\left(x\right) f\left(x\right)$ or $\int_x G\left(x\right) f\left(x\right) dx$

5. $f\left(x\right) = \binom{n}{x} p^x \left(1-p\right)^{n-x}, \quad x = 0,\ 1,\ \ldots,\ n$

6. $f\left(x\right) = \dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}, \quad x = 0,\ 1,\ \ldots,\ n;\ K < N$

7. $f\left(x\right) = \left(1-p\right)^{x-1} p, \quad x = 1,\ 2,\ 3,\ \ldots$

8. $f\left(x\right) = \binom{x-1}{r-1} p^r \left(1-p\right)^{x-r}, \quad x = r,\ r+1,\ r+2,\ \ldots;\ r = 2,\ 3,\ 4,\ \ldots$

9. $f\left(x\right) = e^{-\lambda} \dfrac{\lambda^x}{x!}, \quad x = 0,\ 1,\ 2,\ \ldots$

10. $f\left(x\right) = \lambda e^{-\lambda x}, \quad x > 0\ ;\ E\left(X\right) = 1/\lambda;\ \mathrm{Var}\left(X\right) = 1/\lambda^2;\quad m\left(t\right) = \dfrac{\lambda}{\lambda - t}$

11. $\Gamma\left(n\right) = \int_0^\infty x^{n-1} e^{-x} dx = \left(n-1\right)!$

12. $f\left(x\right) = \dfrac{\lambda^\alpha x^{\alpha-1}}{\Gamma\left(\alpha\right)} e^{-\lambda x}, \quad x > 0\ ;\ m\left(t\right) = \left(\dfrac{\lambda}{\lambda - t}\right)^\alpha$

13. $m(t) = E\left(e^{tX}\right) = \sum_{x} e^{tx} f(x)$ or $\int_{x} e^{tx} f(x)\, dx$

14. $m(t_1, t_2) = E\left(e^{t_1 X_1 + t_2 X_2}\right)$

15. Cov(X, Y) = $E[(X - \mu_X)(Y - \mu_Y)]$

16. $\rho(X, Y) = \dfrac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$

17. $$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 \right.\right.$$
$$\left.\left. \times\ 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}, \quad -\infty < x < \infty, -\infty < y < \infty.$$

18. $$f(x \mid y) = \frac{1}{\sigma_x\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{1}{2(1-\rho^2)\sigma_x^2}\right.$$
$$\left. \times\left[x - \mu_x \rho\frac{\sigma_x}{\sigma_y}(\text{y-}\mu_\text{y})\right]^2\right\}, \quad -\infty < x < \infty.$$

19. $m(t_1, t_2) = \exp\left[t_1\mu_x + t_2\mu_y + \dfrac{1}{2}\left(t_1^2\sigma_x^2 + 2\rho t_1 t_2 \sigma_x\sigma_y + t_2^2\sigma_y^2\right)\right]$

20. $f(x) = \dfrac{\Gamma\left[(n+1)/2\right]}{\Gamma(n/2)\sqrt{\pi n}}\left(1 + \dfrac{x^2}{n}\right)^{-(n-1)/2}$, $-\infty < x < \infty$

21. $T = \dfrac{Z}{\sqrt{V/n}}$

22. $f(x) = \dfrac{\Gamma\left[(m+n)/2\right]}{\Gamma(m/2)\Gamma(n/2)}\left(\dfrac{m}{2}\right)^{m/2} \dfrac{x^{(m-2)/2}}{\left[1 + (m/n)x\right]^{(m+n)/2}}$, $x > 0$

23. $F = \dfrac{U/m}{V/n}$

24. $\gamma_1 = \dfrac{E\left[(X-\mu)^3\right]}{\left(E\left[(X-\mu)^2\right]\right)^{3/2}}.$

25. $\gamma_2 = \dfrac{E\left[(X-\mu)^4\right]}{\left(E\left[(X-\mu)^2\right]\right)^{2}}.$

26. $P\{X \ge a\} \le \dfrac{E[X]}{a}.$

27. Let X be of finite mean, μ and variance, σ^2 then $\bar{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ where $X_1, X_2, \ldots, X_n$ is a random sample of X.

28. $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

29. $f(x \mid y) = \dfrac{f(x, y)}{f(y)}.$

30. $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty.$ $m(t) = \exp\left\{\mu t + \dfrac{\sigma^2 t^2}{2}\right\}.$

31. $P\{X \ge a\} \le \dfrac{E[X]}{a}.$

32. $\sum_{k=0}^{\infty} ar^k = \dfrac{a}{1-r}$, for $|r| < 1$.

33. $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}.$

34. $f(x) = \dfrac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}, x \ge 0.$ $m_X(t) = \left(\dfrac{1}{1-2t}\right)^{\frac{k}{2}}, t < \dfrac{1}{2}.$

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