UNIVERSITI SAINS MALAYSIA

Final Examination
2015/2016 Academic Session

May/June 2016

JIM 414 – Statistical Inference
[Pentaabiran Statistik]

Duration: 3 hour
[Masa: 3 jam]

Please ensure that this examination paper contains **EIGHT** printed pages before you begin the examination.

Answer **ALL** questions. You may answer either in Bahasa Malaysia or in English.

Read the instructions carefully before answering.

Each question is worth 100 marks.

In the event of any discrepancies, the English version shall be used.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi ***LAPAN*** *muka surat yang bercetak sebelum anda memulakan peperiksaan.*

Jawab ***SEMUA*** *soalan. Anda dibenarkan menjawab sama ada dalam Bahasa Malaysia atau Bahasa Inggeris.*

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah.

Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai.]

1. (a) Let $f(x)=\begin{cases} \frac{1}{3}, & -1<x<2 \\ 0, & \text{elsewhere} \end{cases}$,

be the pdf X. Find the cdf and pdf of $Y=X^2$.

(50 marks)

(b) Let X_1, X_2, X_3 be iid common pdf $f(x)=\begin{cases} e^{-x}, & X>0 \\ 0, & \text{elsewhere} \end{cases}$.

Find the joint pdf of $Y_1=\dfrac{X_1}{X_2}, Y_2=\dfrac{X_3}{(X_1+X_2)}$ and $Y_3=X_1+X_2$. Are Y_1, Y_2, Y_3 mutually independent?

(50 marks)

2. (a) Let $X_1, X_2, \ldots, X_n$ represent a random sample from a distribution having the following pdf:

$$f(x;\theta)=\begin{cases} e^{-(x-\theta)}, \ \theta \le x<\infty, & -\infty<\theta<\infty, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the maximum likelihood estimator $\hat{\theta}$ of θ.

(50 marks)

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(0,\theta)$ distribution. We want to estimate the standard deviation $\sqrt{\theta}$. Find the constant c so that $Y=c\sum_{i=1}^{n}|X_i|$ is an unbiased estimator of $\sqrt{\theta}$ and determine its efficiency.

(50 marks)

3. (a) In an opinion poll, data X is modelled by Binomial($n;\ p$) distribution, $0 \le p \le 1$.

(i) Find the maximum likelihood estimator of p.

(ii) Calculate the maximum likelihood estimate of p when n = 500 and X = 200.

(20 marks)

(b) $X_1, ..., X_n$ is a random sample from a Beta(1, β) distribution.

(i) Find the method of moments estimator for β.
(ii) Show that (i) is biased.
(iii) Suppose an estimator $g\left(\bar{X}\right)$ such that $E\left[g\left(\bar{X}\right)\right] = \beta$ was found. Would this estimator, $g\left(\bar{X}\right)$, be the best unbiased estimator? If not, is there a way that this estimator can be used to find the minimum variance unbiased estimator? Justify your answer.

(50 marks)

(c) $X_1, ..., X_n$ is a random sample from a $N(\mu, \sigma^2)$ distribution.

(i) Find sufficient statistics for (μ, σ^2).
(ii) Are they also complete?

(30 marks)

4. (a) Show that the following distributions are of exponential class.

(i) The Beta (α, β) distribution.
(ii) The chi-square distribution with n degrees of freedom.

(30 marks)

(b) Suppose X_1 and X_2 are independent and identically distributed as Uniform($\theta - 1$, $\theta + 1$). Let $Y_1 = \min(X_1, X_2)$, $Y_2 = \max(X_1, X_2)$, $T = \frac{1}{2}\left(Y_1 + Y_2\right)$ and $U = \frac{1}{2}\left(Y_2 - Y_1\right)$. Determine which statistics are minimally sufficient and which statistics are ancillary.

(30 marks)

(c) $X_1, ..., X_n$ is a random sample from a $N(\mu, \sigma^2)$ distribution. Find the most powerful test of size $\alpha = 0.05$ for H_0: The population sampled from is the $N(0, 1)$ distribution versus H_1: The population sampled from is the $N(1, 1)$ distribution.

(40 marks)

5. (a) Let X_1, X_2 be iid with common distribution having the pdf
$$f(x) = \begin{cases} e^{-x}, 0 < x < \infty \\ 0, \quad \text{elsewhere} \end{cases}.$$
Show that $Z = \dfrac{X_1}{X_2}$ has an *F*- distribution.

(25 marks)

(b) Let Y_1 denote the minimum of a random sample of size n from a distribution that has pdf
$$f(x) = \begin{cases} e^{-(x-\theta)}, & \theta < x < \infty \\ 0, & \text{elsewhere.} \end{cases}$$
Let $Z_n = n(Y_1 - \theta)$. Investigate the limiting distribution of Z_n.

(25 marks)

(c) X_1, ..., X_n is a random sample from a population whose probability mass function is $p(x) = \theta^{x}(1 - \theta)^{1-x}$, $x = 0, 1$, $0 < \theta < 1$. Show that $T(X_1, \ldots X_n) = \sum_{i=1}^{n} X_i$ is a sufficient statistic. Determine whether it is also complete.

(25 marks)

(d) X_1, ..., X_n is a random sample from a Poisson(λ) distribution. Construct an approximate likelihood ratio test for H_0: $\lambda = 1$ versus H_1: $\lambda \neq 1$.

(25 marks)

1. (a) Biar $f(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & \text{di tempat lain,} \end{cases}$

adalah fungsi ketumpatan kebarangkalian *X*. Cari fungsi taburan longgokan dan fungsi ketumpatan kebarangkalian dari $Y = X^2$.

(50 markah)

(b) Biar X_1, X_2, X_3 adalah fungsi ketumpatan kebarangkalian tertabur secara secaman $f(x) = \begin{cases} e^{-x}, & X > 0 \\ 0, & \text{di tempat lain.} \end{cases}$

Cari fungsi ketumpatan kebarangkalian tercantum bagi $Y_1 = \dfrac{X_1}{X_2}, Y_2 = \dfrac{X_3}{(X_1 + X_2)}$ dan $Y_3 = X_1 + X_2$.

Apakah Y_1, Y_2, Y_3 saling tak bersandar?

(50 markah)

2. (a) Biar $X_1, X_2, \ldots, X_n$ mewakili suatu sampel rawak daripada taburan yang mempunyai fungsi ketumpatan berikut:

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)}, & \theta \le x < \infty, \quad -\infty < \theta < \infty, \\ 0, & \text{di tempat lain.} \end{cases}$$

Cari pengganggar kebolehjadian maksimum $\hat{\theta}$ bagi θ.

(50 markah)

(b) Biar $X_1, X_2, \ldots, X_n$ adalah sampel rawak daripada taburan $N(0,\theta)$. Kita mahu mengganggarkan sisihan piawai $\sqrt{\theta}$. Cari pemalar *c* supaya $Y = c\sum_{i=1}^{n} |X_i|$ ialah penggangar saksama untuk $\sqrt{\theta}$ dan tentukan kecekapan *Y*.

(50 marks)

3. (a) Di dalam suatu tinjauan pendapat, data *X* dimodel oleh taburan Binomial (*n; p*), $0 \le p \le 1$.

(i) Cari penganggar kebolehjadian maksimum bagi *p*.
(ii) Hitungkan anggaran kebolehjadian maksimum bagi *p* apabila *n* = 500 dan *X* = 200.

(20 markah)

(b) $X_1, ..., X_n$ adalah sampel rawak daripada taburan Beta(1, β).

(i) Cari penganggar kaedah momen bagi β.
(ii) Tunjukkan (i) adalah tidak saksama.
(iii) Andaikan terdapat penganggar $g\left(\bar{X}\right)$ supaya $E\left[g\left(\bar{X}\right)\right]=\beta$.

Adakah penganggar ini, $g\left(\bar{X}\right)$, akan menjadi penganggar saksama yang terbaik? Jika tidak, adakah jalan yang membolehkan penggunaan penganggar ini untuk mendapatkan penganggar saksama bervarians minimum? Jelaskan jawapan anda.

(50 markah)

(c) $X_1, ..., X_n$ adalah suatu sampel rawak daripada taburan $N(\mu, \sigma^2)$.

(i) Cari statistik-statistik cukup bagi (μ, σ^2).
(ii) Adakah statistik-statistik tersebut juga lengkap?

(30 markah)

4. (a) Tunjukkan bahawa taburan-taburan berikut adalah ahli kelas eksponen.

(i) Taburan Beta(α, β).
(ii) Taburan khi-kuasa dua dengan darjah kebebasan n.

(30 markah)

(b) Andaikan X_1 dan X_2 adalah tak bersandar dan tertabur secara secaman Seragam(θ–1, θ + 1). Andaikan $Y_1 = \min(X_1, X_2)$, $Y_2 = \text{maks}(X_1, X_2)$, $T=\frac{1}{2}\left(Y_1+Y_2\right)$ dan $U=\frac{1}{2}\left(Y_2-Y_1\right)$. Tentukan statistik yang mencukupi secara minimum dan statistik manakah yang sampingan.

(30 markah)

(c) $X_1, ..., X_n$ adalah sampel rawak daripada taburan $N(\mu, \sigma^2)$. Cari ujian paling berkuasa bersaiz α = 0.05 bagi H_0: Populasi yang disampel bertaburan $N(0, 1)$ lawan H_1: Populasi yang disampel bertaburan $N(1, 1)$.

(40 markah)

5. (a) Biar X_1, X_2 adalah tertabur secara secaman daripada taburan yang mempunyai fungsi ketumpatan kebarangkalian

$$f(x) = \begin{cases} e^{-x}, & 0 < x < \infty \\ 0, & \text{di tempat lain.} \end{cases}$$

Tunjukkan bahawa $Z = \dfrac{X_1}{X_2}$ mempunyai taburan *F*.

(25 markah)

(b) Biar Y_1 menandakan sampel rawak minimum berukuran *n* dari taburan yang mempunyai fungsi ketumpatan kebarangkalian:

$$f(x) = \begin{cases} e^{-(x-\theta)}, & \theta < x < \infty \\ 0, & \text{di tempat lain.} \end{cases}$$

Biar $Z_n = n(Y_1 - \theta)$. Selidiki taburan penghad dari Z_n.

(25 markah)

(c) X_1, ..., X_n adalah sampel rawak daripada taburan yang mempunyai fungsi jisim kebarangkalian $p(x)=\theta^{x}(1-\theta)^{1-x}$, $x = 0,1$, $0 < \theta < 1$. Tunjukkan $T(X_1, \dots X_n) = \sum_{i=1}^{n} X_i$ adalah suatu statistik cukup. Tentukan sama ada statistik tersebut juga adalah lengkap.

(25 markah)

(d) X_1, ..., X_n adalah sampel rawak daripada taburan Poisson(λ). Binakan ujian nisbah kebolehpercayaan hampiran bagi H_0: $\lambda = 1$ lawan H_1: $\lambda \neq 1$.

(25 markah)

Formulas

1. $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, 2, \ldots, n$.

2. $f(x;\alpha,\beta) = \dfrac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \leq x \leq 1. \quad B(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.
   $\Gamma(n) = (n-1)!$

3. $f\left(x;\mu,\sigma^2\right) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty.$

4. $\prod_{i=1}^{n} f\left(x_i;\theta\right) = g_1\left[u_1\left(x_1, x_2, \ldots, x_n;\theta\right)\right] H\left(x_1, x_2, \ldots, x_n\right)$

5. $f\left(x;\theta\right) = h(x)g(\theta)\exp\left(\eta(\theta)T(x)\right)$

6. $f\left(x;a,b\right) = \dfrac{1}{b-a}, a < x < b.$

7. $f_{X_{(k)}}(x) = \dfrac{n!}{(k-1)!(n-k)!}\left[F(x)\right]^{k-1}\left[1-F(x)\right]^{n-k} f(x).$

8. $f_{X_{(j)},X_{(k)}}(x,y) = \dfrac{n!}{(j-1)!(k-j-1)!(n-k)!}\left[F(x)\right]^{j-1}\left[F(y)-F(x)\right]^{k-j-1}\left[1-F(y)\right]^{n-k} f(x)f(y).$

9. $f_{X_{(1)},\ldots X_{(n)}}\left(x_1,\ldots x_n\right) = n!\, f(x_1)\cdots f(x_n)$ where $x_1 \leq x_2 \leq \ldots \leq x_n$.

10. Let X be of finite mean, μ and variance, σ^2 then $\bar{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ where $X_1, X_2, \ldots, X_n$ is a random sample of X.

11. $P(X = x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, 2, \ldots$ $m(t) = \exp[\lambda(e^t - 1)]$.

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