UNIVERSITI SAINS MALAYSIA

Final Examination
2015/2016 Academic Session

May/June 2016

JIM 317 – Differential Equations II
[Persamaan Pembezaan II]

Duration : 3 hours
[*Masa: 3 jam*]

Please ensure that this examination paper contains **NINE** printed pages before you begin the examination.

Answer **ALL** questions.

Read the instructions carefully before answering.

Each question is worth 100 marks.

In the event of any discrepancies, the English version shall be used.

*Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEMBILAN** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.*

*Jawab **SEMUA** soalan.*

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah.

Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai.

1. The Bessel equation of order one is

$$x^2 y'' + xy' + (x^2 - 1)y = 0.$$

(a) Show that $x = 0$ is a regular singular point.

(20 marks)

(b) Show that the roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$.

(20 marks)

(c) Construct the series solution at $x = 0$ for $r = 1$.

(40 marks)

(d) Show that the series solution converges for all x.

(20 marks)

2. (a) Consider the boundary value problem

$$y'' + 2y' + 2y = -\lambda y, \quad 0 < x < 1,$$

(i) Rewrite the problem as a Sturm-Liouville problem.

(ii) Identify p, q and r.

(40 marks)

(b) Given the Sturm-Liouville problem

$$-(xy')' = \lambda x^{-1} y, \qquad 1 < x < e$$

with the boundary conditions $y(1) = 0,\ y'(e) = 0.$

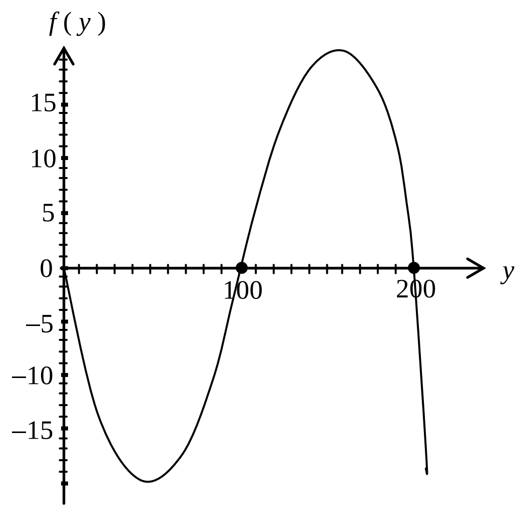
(i) Find all eigenvalues and eigenfunctions.

(ii) Expand the function $f(x) = 1$ in terms of the eigenfunctions.

(60 marks)

3. (a) Let $f(y)=\left(\frac{y}{200}-1\right)\left(1-\frac{y}{100}\right)y$, and consider the autonomous differential equation $\frac{dy}{dt}=f(y)$.

Below is a plot of $f(y)$ vs y.



(i) Draw the phase line and identify each critical point as asymptotically stable or unstable.

(ii) Sketch the equilibrium solutions as well as several solutions in each of the regions separated by equilibrium solutions in the *ty*-plane.

(iii) Suppose the equation models the population of a species. Identify its threshold population and carrying capacity.

(60 marks)

(b) A nonlinear system is described by the differential equations

$$\frac{dx}{dt}=1-y$$
$$\frac{dy}{dt}=x^2-y^2$$

Show that the system has singular points $(1,1)$ and $(-1,1)$.

Determine its nature.

(40 marks)

4. (a) Consider the initial value problem

$$y' = -2y + te^{3t} \qquad y(0) = 0 \qquad 0 \le t \le 1,$$

(i) Use Euler's method with h = 0.5 to approximate the solution at time t = 1 to equation above.

(ii) Given the exact solution to the above initial value problem is:

$$y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}.$$

Determine an error bound for the approximation obtained in (i).

(50 marks)

(b) Consider the following Runge-Kutta method

$$w_0 = y_0$$
for $i = 0.1...., N-1,$
$$k_1 = hf(t_i, w_i),$$
$$k_2 = hf(t_i + \alpha h, w_i + \beta k_1)$$
$$w_{i+1} = w_i + a_1 k_1 + a_2 k_2$$

(i) Show that the above Runge-Kutta method is of order 2 if, for any α,

$$\beta = \alpha, \qquad a_1 = 1 - \frac{1}{2\alpha}, \qquad a_2 = \frac{1}{2\alpha}.$$

(ii) Show that by choosing $\alpha = 1$ in (i), we obtain the modified Euler method.

(50 marks)

1. Persamaan Bessel berperingkat sayu adalah

$$x^2 y'' + xy' + (x^2 - 1)y = 0.$$

(a) Tunjukkan bahawa $x = 0$ adalah titik singular sekata.

(20 markah)

(b) Tunjukkan bahawa punca bagi persamaan indeksan adalah $r_1 = 1$ dan $r_2 = -1$.

(20 markah)

(c) Dapatkan penyelesaian siri di $x = 0$ untuk $r = 1$.

(40 markah)

(d) Tunjukkan bahawa penyelesaian siri berkenaan menumpu untuk semua x.

(20 markah)

2. (a) Pertimbangkan permasalahan nilai batas berikut

$$y'' + 2y' + 2y = -\lambda y, \quad 0 < x < 1,$$

(i) Tulis semula permasalahan itu sebagai masalah Sturm-Liouville.

(ii) Tentukan p, q dan r.

(40 markah)

(b) Diberi masalah Sturm-Liouville

$$-(xy')' = \lambda x^{-1} y, \qquad 1 < x < e$$

dengan syarat-syarat sempadan

$$y(1) = 0, \ y'(e) = 0.$$

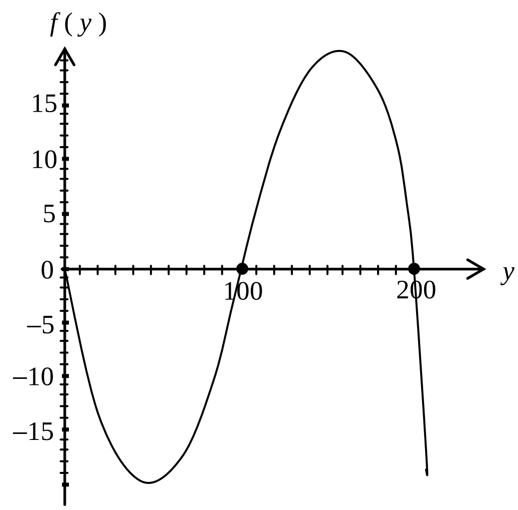
(i) Cari semua nilai eigen dan fungsi eigen.

(ii) Kembangkan fungsi $f(x) = 1$ dalam sebutan fungsi eigennya.

(60 markah)

3. (a) Katakan $f\left(y\right)=\left(\frac{y}{200}-1\right)\left(1-\frac{y}{100}\right)y$, dan pertimbangkan persamaan pembezaan autonomous $\frac{dy}{dt}=f\left(y\right)$.

Di bawah adalah plot bagi $f\left(y\right)$ lawan y.



(i) Lukiskan garisan fasa dan kenalpasti setiap titik kritikal sebagai stabil atau tak stabil secara asimptotik.

(ii) Lakarkan penyelesaian keseimbangan dan juga beberapa penyelesaian dalam setiap kawasan yang dipisahkan oleh penyelesaian keseimbangan dalam satah-ty.

(iii) Andaikan persamaan berkenaan memodelkan populasi suatu spesis. Kenalpasti populasi takat dan kapasiti pembawa.

(60 markah)

(b) Suatu sistem tak linear diperihalkan oleh persamaan pembezaan

$$\frac{dx}{dt}=1-y$$
$$\frac{dy}{dt}=x^2-y^2$$

Tunjukkan bahawa sistem tersebut mempunyai titik singular $(1,1)$ dan $(-1,1)$. Tentukan jenisnya.

(40 markah)

4. (a) Pertimbangkan masalah nilai awal

$$y' = -2y + te^{3t} \qquad y(0) = 0 \qquad 0 \le t \le 1,$$

(i) Gunakan kaedah Euler dengan h = 0.5 untuk mencari penyelesaian pada t = 1 kepada persamaan di atas.

(ii) Diberi penyelesaian sebenar kepada masalah nilai awal di atas adalah:

$$y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}.$$

Tentukan batas ralat untuk anggaran yang diperoleh dalam (i).

(50 markah)

(b) Pertimbangkan kaedah Runge-Kutta berikut

$$w_0 = y_0$$
untuk $i = 0.1...., N-1,$
$$k_1 = hf(t_i, w_i),$$
$$k_2 = hf(t_i + \alpha h, w_i + \beta k_1)$$
$$w_{i+1} = w_i + a_1 k_1 + a_2 k_2$$

(i) Tunjukkan bahawa kaedah Runge-Kutta di atas adalah peringkat 2 sekiranya, untuk sebarang α,

$$\beta = \alpha, \qquad a_1 = 1 - \frac{1}{2\alpha}, \qquad a_2 = \frac{1}{2\alpha}.$$

(ii) Tunjukkan bahawa dengan memilih $\alpha = 1$ di dalam (i), kita akan memperoleh kaedah Euler terubahsuai.

(50 markah)

Appendix

Trigonometry identities

$$\sin^2 x + \cos^2 x = 1$$

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 2\cos^2 x - 1$$

$$= 1 - 2\sin^2 x$$

Power series repserentation of elementary functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Sturm-Liouville problem

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx} \right] - q(x)y + \lambda r(x)y = 0 \qquad (a < x < b)$$

$$\alpha_1 y(a) - \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0$$

Eigenfunction expansions

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

where

$$c_n = \frac{\int_a^b f(x) y_n(x) r(x)\, dx}{\int_a^b [y_n(x)]^2 r(x)\, dx}$$

Euler method

$$y_{n+1} = y_n + h \cdot f(x_n, y_n) \quad \text{where} \quad f(x_n, y_n) = \frac{dy}{dx}$$

Improved Euler method

Predictor: $u_{n+1} = y_n + h \cdot f(x_n, y_n)$

Corrector: $y_{n+1} = y_n + h \cdot \frac{1}{2}\left[f(x_n, y_n) + f(x_{n+1}, u_{n+1}) \right]$

Rungke-Kutta method

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

where $k_1 = f(x_n, y_n)$

$$k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1)$$

$$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2)$$

$$k_4 = f(x_n + h, y_n + hk_3)$$

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