UNIVERSITI SAINS MALAYSIA

Final Examination
2015/2016 Academic Session

May/June 2016

JIM 312 – Probability Theory
[Teori Kebarangkalian]

Duration : 3 hours
[Masa: 3 jam]

Please ensure that this examination paper contains **TEN** printed pages before you begin the examination.

Answer **ALL** questions. You may answer either in Bahasa Malaysia or in English.

Read the instructions carefully before answering.

Each question is worth 100 marks.

In the event of any discrepancies, the English version shall be used.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEPULUH** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.*

*Jawab **SEMUA** soalan. Anda dibenarkan menjawab sama ada dalam Bahasa Malaysia atau Bahasa Inggeris.*

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah.

Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai.]

1. (a) Suppose that $f_{x,y}(x,y)=\frac{2}{3}(x+2y), 0\le x\le 1,\ 0\le y\le 1$.

Find $E(X+Y)$.

(25 marks)

(b) Given that two discrete random variables X and Y follow the joint pmf $P_{X,Y}(x,y)=k(x+y)$, for x = 1,2,3 and y = 1,2,3.

(i) Find k.

(ii) Evaluate $P_{Y|x}(1)$ for all values of x.

(50 marks)

(c) Let Y have pdf

$$f_Y(y)=\begin{cases} y, & 0\le y\le 1 \\ 2-y, & 1\le y\le 2 \\ 0, & \text{elsewhere}\end{cases}$$

Find $M_Y(t)$.

(25 marks)

2. (a) Prove that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

(25 marks)

(b) Let $\overline{Y}$ and S denote the sample mean and sample standard deviation, respectively, based on a set of n = 20 measurements taken from a normal distribution with $\mu=90.6$. Find the function $k(S)$ for which:

$P\left(90.6-k(S)\le\overline{Y}\le 90.6+k(S)\right)=0.99$.

(25 marks)

(c) Let $Y_1,Y_2,\ldots,Y_n$ be a random sample of size n from a normal distribution having mean μ and variance σ^2. What is the smallest value of n for which

$$P\left(\frac{S^2}{\sigma^2}<2\right)\ge 0.95 \text{ given } \chi^2_{0.95,8}=15.507.$$

(25 marks)

(d) (i) Evaluate $\int_0^{1.24} e^{-\frac{z^2}{2}} dz$.

(ii) Evaluate $\int_{-\infty}^{\infty} 6e^{-\frac{z^2}{2}} dz$.

(25 marks)

3. (a) An urn contains five red, three orange and two blue balls. Two balls are randomly selected. Let *X* represents the number of red balls selected. Let *Y* be the number of orange balls selected. Find the correlation between *X* and *Y*.

(50 marks)

(b) The moment generating function of *Z* is $M_Z(t) = e^{t^2/2}$. Prove or disprove that $X = aZ + b$ has the same coefficients of skewness and kurtosis of *Z*.

(30 marks)

(c) Given $f(y) = y + \frac{1}{2}$, $0 \le y \le 1$.
Validate or disprove the Markov inequality.

(20 marks)

4. (a) Let *X* be a Poisson random variable with variance λ.

(i) Show that the moment generating function of *X*, $M_X(t) = \exp[\lambda(e^t - 1)]$.

(ii) Compute the moment generating of $\sum_{i=1}^{n} X_i$ where the $X_1, \ldots, X_n$ are independent Poisson random variables with variance λ.

(iii) What is the distribution of $\sum_{i=1}^{n} X_i$?

(20 marks)

(b) Let $Y = \sum_{i=1}^{n} X_i$ where the $X_1, \ldots, X_n$ are independent exponential random variables with mean 1.

(i) Compute the moment generating function of *Y*.
(ii) Compute the exact probability of $P(Y > 200)$.
(iii) Use the Central Limit Theorem to obtain $P(Y > 200)$.

(30 marks)

(c) The joint density function of X and Y is given by

$$f(x,y) = \frac{e^{-yx^2/2}}{\sqrt{2\pi/y}} ye^{-y}, \quad -\infty < x < \infty, \quad y > 0.$$

(i) Find the conditional density of $f(x|y)$.
(ii) Compute $E(X|y)$.
(iii) Compute $\text{Var}(X|y)$.
(iv) Compute $\text{Var}(X)$.

(50 marks)

5. (a) For a negative binomial random variable whose pmf is given by:

$$P_X(k) = P(X - k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \cdots$$

Find $E(X)$ directly by evaluating

$$\sum_{k=r}^{\infty} k \binom{k-1}{r-1} p^r (1-p)^{k-r}.$$

(25 marks)

(b) Use the fact that $(n-1)\frac{S^2}{\sigma^2}$ is a chi square random variable with $n - 1$ df to prove that:

$$\text{Var}(S^2) = \frac{2\sigma^4}{n-1}.$$

(25 marks)

(c) Given that X is a random variable. Prove or disprove $E[X^2] = \{E[X]\}^2$.

(25 marks)

(d) $X_1, \ldots, X_n$ are independent random variables having a common $N(\theta, 1)$ distribution. Let $T(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i$. Show that the conditional distribution of $X_1, \ldots, X_n$ given $T(X_1, \ldots, X_n)$ does not depend upon θ.

(25 marks)

1. (a) Anggap bahawa $f_{x,y}(x,y) = \frac{2}{3}(x+2y), 0 \le x \le 1,\ 0 \le y \le 1.$

Cari $E(X+Y)$.

(25 markah)

(b) Diberikan dua pembolehubah rawak diskret X dan Y dengan fungsi jisim kebarangkalian tercantum (pmf)

$P_{X,Y}(x,y) = k(x+y)$, untuk x = 1,2,3 dan y = 1,2,3.

(i) Cari k.

(ii) Nilaikan $P_{Y|x}(1)$ untuk semua nilai x.

(50 markah)

(c) Biar Y mempunyai fungsi ketumpatan kebarangkalian (pdf)

$$f_Y(y) = \begin{cases} y, & 0 \le y \le 1 \\ 2-y, & 1 \le y \le 2 \\ 0, & \text{lain-lain} \end{cases}$$

Cari $M_Y(t)$.

(25 markah)

2. (a) Buktikan bahawa $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(25 markah)

(b) Biar $\bar{Y}$ dan S mewakili masing-masing, min sampel dan sisihan piawai sampel, berdasarkan atas set berukuran n = 20 yang diambil daripada taburan normal dengan $\mu = 90.6$. Cari fungsi $k(S)$ yang mana:

$P\left(90.6 - k(S) \le \bar{Y} \le 90.6 + k(S)\right) = 0.99.$

(25 markah)

(c) Biar $Y_1, Y_2, \ldots, Y_n$ menjadi sampel rawak dengan saiz n daripada taburan normal dengan min μ dan varians σ^2. Apakah nilai terkecil untuk n yang mana

$$P\left(\frac{S^2}{\sigma^2} < 2\right) \ge 0.95 \text{ diberi } \chi^2_{0.95,8} = 15.507.$$

(25 markah)

(d) (i) Nilaikan $\int_0^{1.24} e^{-\frac{z^2}{2}}\,dz.$

(ii) Nilaikan $\int_{-\infty}^{\infty} 6e^{-\frac{z^2}{2}}\,dz\,.$

(25 markah)

3. (a) Sebuah balang mengandungi lima biji bola merah, tiga biji bola jingga dan dua biji bola biru. Dua biji bola dipilih secara rawak. Katakan X mewakili bilangan bola merah yang dipilih. Katakan Y mewakili bilangan bola jingga yang dipilih. Cari korelasi di antara X dan Y.

(50 markah)

(b) Fungsi penjana momen bagi Z ialah $M_Z(t) = e^{t^2/2}$. Buktikan atau sangkalkan $X = aZ + b$ mempunyai pekali-pekali kepencongan dan kurtosis yang sama dengan Z.

(30 markah)

(c) Diberikan $f(y) = y + \frac{1}{2}$, $0 \le y \le 1$.
Sahkan atau sangkalkan ketaksamaan Markov.

(20 markah)

4. (a) Katakan X adalah suatu pembolehubah rawak Poisson dengan varians λ.

(i) Tunjukkan bahawa fungsi penjana momen bagi X,
$M_X(t) = \exp[\lambda(e^t - 1)]$.

(ii) Dapatkan fungsi penjana momen bagi $\sum_{i=1}^{n} X_i$ di mana $X_1, \ldots, X_n$ adalah pembolehubah-pembolehubah rawak tak bersandar Poisson dengan varians λ.

(iii) Apakah taburan $\sum_{i=1}^{n} X_i$?

(20 markah)

(b) Katakan $Y = \sum_{i=1}^{n} X_i$ di mana $X_1, \ldots, X_n$ adalah pembolehubah-pembolehubah rawak tak bersandar eksponen dengan min 1.

(i) Dapatkan fungsi penjana momen bagi Y.
(ii) Hitungkan $P(Y > 200)$ yang tepat.
(iii) Gunakan Teorem Had Memusat untuk mendapatkan $P(Y > 200)$.

(30 markah)

(c) Fungsi ketumpatan tercantum bagi X dan Y diberi oleh

$$f(x,y) = \frac{e^{-yx^2/2}}{\sqrt{2\pi/y}} ye^{-y}, \quad -\infty < x < \infty, \quad y > 0.$$

(i) Cari ketumpatan bersyarat $f(x|y)$.
(ii) Hitung $E(X|y)$.
(iii) Hitung $\mathrm{Var}(X|y)$.
(iv) Hitung $\mathrm{Var}(X)$.

(50 markah)

5. (a) Untuk pembolehubah rawak binomial negatif yang mempunyai fungsi jisim kebarangkalian (pmf) diberikan sebagai:

$$P_X(k) = P(X - k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \cdots$$

Cari $E(X)$ secara langsung dengan menilai

$$\sum_{k=r}^{\infty} k \binom{k-1}{r-1} p^r (1-p)^{k-r}.$$

(25 markah)

(b) Gunakan hakikat bahawa $(n-1)\frac{S^2}{\sigma^2}$ ialah pembolehubah rawak khi kuasa dua dengan dk $n - 1$ untuk membuktikan bahawa:

$$\mathrm{Var}(S^2) = \frac{2\sigma^4}{n-1}.$$

(25 markah)

(c) Diberikan X adalah suatu pembolehubah rawak. Buktikan atau sangkalkan $E[X^2] = \{E[X]\}^2$.

(25 markah)

(d) $X_1, \ldots, X_n$ adalah pembolehubah-pembolehubah rawak tak bersandar yang mempunyai taburan sepunya $N(\theta, 1)$. Andaikan $T(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i$. Tunjukkan taburan bersyarat $X_1, \ldots, X_n$ diberikan $T(X_1, \ldots, X_n)$ tidak bergantung kepada θ.

(25 markah)

List of Formulas

1. $F_Y(t) = F_X(g^{-1}(t))$

2. $F_Y(t) = 1 - F_X(g^{-1}(t))$

3. $P\left(\left|X - \mu_X\right| \geq a\sigma_X\right) \leq \frac{1}{a^2}$

4. $E\left(G\left(X\right)\right) = \sum_x G\left(x\right) f\left(x\right)$ or $\int_x G\left(x\right) f\left(x\right) dx$

5. $f\left(x\right) = \binom{n}{x} p^x \left(1-p\right)^{n-x}, \quad x = 0,\ 1,\ \ldots,\ n$

6. $f\left(x\right) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}, \quad x = 0,\ 1,\ \ldots,\ n;\ K < N$

7. $f\left(x\right) = \left(1-p\right)^{x-1} p, \quad x = 1,\ 2,\ 3,\ \ldots$

8. $f\left(x\right) = \binom{x-1}{r-1} p^r \left(1-p\right)^{x-r}, \quad x = r,\ r+1,\ r+2,\ \ldots;\ r = 2,\ 3,\ 4,\ \ldots$

9. $f\left(x\right) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0,\ 1,\ 2,\ \ldots$

10. $f\left(x\right) = \lambda e^{-\lambda x}, \quad x > 0\ ;\ E\left(X\right) = 1/\lambda;\ \text{Var}\left(X\right) = 1/\lambda^2;\quad m\left(t\right) = \frac{\lambda}{\lambda - t}$

11. $\Gamma\left(n\right) = \int_0^\infty x^{n-1} e^{-x} dx = \left(n-1\right)!$

12. $f\left(x\right) = \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma\left(\alpha\right)} e^{-\lambda x}, \quad x > 0\ ;\ m\left(t\right) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha$

13. $m(t) = E\left(e^{tX}\right) = \sum_{x} e^{tx} f(x)$ or $\int_{x} e^{tx} f(x)\, dx$

14. $m(t_1, t_2) = E\left(e^{t_1 X_1 + t_2 X_2}\right)$

15. $\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$

16. $\rho(X, Y) = \dfrac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$

17. $$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 \times 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}, \quad -\infty < x < \infty, -\infty < y < \infty.$$

18. $$f(x \mid y) = \frac{1}{\sigma_x\sqrt{2\pi\left(1-\rho^2\right)}} \exp\left\{-\frac{1}{2\left(1-\rho^2\right)\sigma_x^2} \times \left[x - \mu_x \rho \frac{\sigma_x}{\sigma_y}\left(\text{y-}\mu_{\text{y}}\right)\right]^2\right\}, \quad -\infty < x < \infty.$$

19. $$m(t_1, t_2) = \exp\left[t_1\mu_x + t_2\mu_y + \frac{1}{2}\left(t_1^2\sigma_x^2 + 2\rho t_1 t_2 \sigma_x \sigma_y + t_2^2\sigma_y^2\right)\right]$$

20. $$f(x) = \frac{\Gamma\left[(n+1)/2\right]}{\Gamma(n/2)\sqrt{\pi n}}\left(1 + \frac{x^2}{n}\right)^{-(n-1)/2}, \quad -\infty < x < \infty$$

21. $$T = \frac{Z}{\sqrt{V/n}}$$

22. $$f(x) = \frac{\Gamma\left[(m+n)/2\right]}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{2}\right)^{m/2} \frac{x^{(m-2)/2}}{\left[1 + (m/n)x\right]^{(m+n)/2}}, \quad x > 0$$

23. $$F = \frac{U/m}{V/n}$$

24. $\gamma_1 = \dfrac{E\left[(X-\mu)^3\right]}{\left(E\left[(X-\mu)^2\right]\right)^{3/2}}.$

25. $\gamma_2 = \dfrac{E\left[(X-\mu)^4\right]}{\left(E\left[(X-\mu)^2\right]\right)^{2}}.$

26. $P\{X \ge a\} \le \dfrac{E[X]}{a}.$

27. Let X be of finite mean, μ and variance, σ^2 then $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ where $X_1, X_2, \ldots, X_n$ is a random sample of X.

28. $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

29. $f(x \mid y) = \dfrac{f(x,y)}{f(y)}.$

30. $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty. \; m(t) = \exp\left\{\mu t + \dfrac{\sigma^2 t^2}{2}\right\}.$

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