EXCEEDANCE PROBLEMS FOR CRITICAL BRANCHING PROCESSES

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A problem of the first excedance of a given level by the family of independent branching processes is considered. The limit Theorems for the number of the first process exceeding some levels in critical cases will be presented. The limit distributions obtained for normalized exceedance time are exponential. Asymptotic formulae for expectations and variances of the number are shown.

Key words: Exceedance, Galton-Watson, Bellman-Harris, critical.

1. Introduction

Consider the evolution and reproduction process of some individuals (or particles). Suppose that each individual lives for a unit time and at the ends of its lifetime, have produce k new offspring with probability P_k, $k \geq 0$. This new individuals also have unit lifetimes and at their death, independent from each other, reproduce a random number of new offspring. The number of individuals $X(t)$ at time $t, t \in N_0 = \{0, 1, 2, \cdots\}$, called the Galton-Watson process that is the size of the population is the main investigation in the theory of branching processes.

Let $X_{ti}, t \in N_0 = \{0, 1, 2, \cdots\}, i \in N = \{1, 2, \cdots\}$ be independent and identically distributed random variables taking values in the set N_0. We define process $X(t), t \in N$ by the relation

$$X(t) = \sum X_{ti}, \quad X(0) = 1$$

Here, X_{ti} denotes the number of descendants of the ith individual existing at time t. Now we consider the sequence of Galton-Watson branching processes $\{X_i(t), i \in N\}$ such that

(a) $X_i(t)$ are independent for any fixed $t \in N_0$ and $X_i(0) = 1$;

(b) $P\{X_i(t) = k \mid X_i(0) = 1\} = P_k, k \in N_0, \sum_{k=0}^{\infty} P_k = 1$.

We define the 'index' process

$$\nu(t) = \min\{k : X_k(t) > \theta(t)\}$$

for a given "level" function $\theta(t)$.

If we interpret $X_i(t), i \in N$, as sizes of different populations existing in different regions of an area, assumptions (a) and (b) mean that life of individuals in different regions are independent and the distribution of the number of direct descendants of one individual is the same for all regions. The process $\nu(t)$ can be considered as the index of the first process in the family which exceeds level $\theta(t)$.

Interested readers can read more on the properties of exceedances of given levels by sequences of independent and identically distributed random variables in Leadbetter (1983), Watts et. al. (1982) and references in there where these properties have been studied widely. The processes of exceedance were considered in Rahimov (1998a) and Rahimov (1998b) where some limit theorems for the index of the first process were obtained for Galton-Watson process.

We denote

$$f(s)=E[s^{X(1)}]=\sum_{k=0}^{\infty}P_k s^k,\quad 0\le s\le 1$$
$$A=f'(1)=E[X],\quad Q(t)=P[X(t)>0],\quad F(t,x)=P[X(t)\le x].$$

Here, we will only consider the case that $A=1$ where the process is called the critical process. The probability of such an exceedance is clearly $1-F[t,\theta(t)]$. In this critical case, all the processes of the set $X_i(t)$ will extinct with probability 1.

Assume that the offspring generating function $f(s)$ is representable in the form of

$$f(s)=s+(1-s)^{1+\alpha}L(1-s) \tag{1}$$

where $\alpha\in(0,1)$ and $L(x)$ is a slowly varying at zero function. Note that if the offspring distribution satisfies (1), it has expectation 1 and the variance may not be finite. Under the assumption (1) [Slack(1967)],

$$\lim_{t\to\infty}P\{Q(t)X(t)\le x\mid X(t)>0\}=G(x),\quad x\ge 0,$$

where $G(x)$ has the Laplace transform

$$\int_0^{\infty}e^{-\lambda x}dG(x)=1-\lambda(1+\lambda^{\alpha})^{-1/\alpha}, \tag{2}$$

and the probability of non-extinction has the following asymptotic behaviour:

$$Q(t)\sim N(t)/t^{1/\alpha},\quad t\to\infty,$$

where $N(t)$ is a slowly varying as $t\to\infty$ function such that

$$[N(t)]^{\alpha}L\left(N(t)/t^{1/\alpha}\right)\to\alpha^{-1}.$$

The following theorem has been proved in Rahimov (1998a).

Theorem 1.1. If $f(s)$ satisfied (1) and $\theta(t)=c/Q(t), c\in(0,\infty)$, then

$$\lim_{t\to\infty}P\{\nu(t)Q(t)\le x\}=1-e^{-(1-G(c))x};\quad x\ge 0$$

and $G(c)$ is defined by (2).

Theorem 1 gives the limit distribution of $\nu(t)$ when $\theta(t)\to\infty$ in such a way that $\theta(t)Q(t)\to c\in(0,\infty)$. One can see that $Q(t)\nu(t)\to\infty$ as $t\to\infty$ in probability if $\theta(t)Q(t)\to\infty$ and the limiting distribution is exponential with parameter 1 if $\theta(t)Q(t)\to 0$.

2. Bellman-Harris Process

We consider the process of evolution and reproduction process as before. Each of these individuals, independently of others, has a random life length L and at the end of its life, produces a random number X of new individuals. We assume the random variables L and X are independent and we also allow lifetime L to be random variables with arbitrary distribution that is $G(u)=P(L\le u)$. Beside independent and identically distributed, the pairs (L,X) are different for different individuals and that a realized individual is born at the death of his mother. If we denote $X(t)$ to be the number of individuals at time $t, t\in[0,\infty)$ then $X(t)$ is called the Bellman-Harris process.

The Bellman-Harris process has been considered by many authors [Rahimov (1995), Athreya & Ney (1972)]. We consider the sequence of the Bellman-Harris process $X_i(t), i \in N = \{1, 2, \cdots\}, t \in [0, \infty)$ with the same assumption (a) and (b) except that the process considered is Bellman-Harris process.

It is known that if $A = 1, f''(1) = \sigma^2 < \infty, \int_0^\infty t dG(t) = \mu < \infty$ and $t^2[1 - G(t)] \to 0$ as $t \to \infty$ then,

$$\lim_{t\to\infty} P\left\{\frac{X(t)}{t} \le x \mid X(t) > 0\right\} = 1 - e^{-\left(\frac{2\mu}{\sigma^2}x\right)}; \quad x \ge 0 \tag{3}$$

Theorem 2.1. *If* $A = 1, \theta(t) = ct, c \in [0, \infty)$ *then*

$$\lim_{t\to\infty} P[Q(t)\nu(t) \le x] = 1 - \exp\{e^{-\left(\frac{2\mu}{\sigma^2}c\right)}\}x; \quad x \ge 0$$

Proof: We obtain the following relation for the probability generating function $H(t, s)$ of $\nu(t)$:

$$H(t,s) = \frac{s\{1 - F[t, \theta(t)]\}}{1 - s\, F[t, \theta(t)]}. \tag{4}$$

Putting $s = e^{-\lambda Q(t)}$ and using (4) above with $\theta(t) = ct$, we find the Laplace transform of $Q(t)\nu(t)$:

$$E[e^{-\lambda Q(t)\nu(t)}] = H\left(t, e^{-\lambda Q(t)}\right) = \frac{e^{-\lambda Q(t)}[1 - F(t, ct)]}{1 - e^{-\lambda Q(t)} F(t, ct)}.$$

Since $\frac{1 - F(t, ct)}{Q(t)} = \frac{P[X(t) > ct, X(t) > 0]}{P[X(t) > 0]} = P\left[\frac{X(t)}{t} > c \mid X(t) > 0\right]$, , using the limit theorem in (3), we obtain

$$\lim_{t\to\infty} \frac{1 - F(t, ct)}{Q(t)} = e^{-\frac{2\mu}{\sigma^2}c} \tag{5}$$

On the other hand, it follows the formula $e^{-\alpha} = 1 - \alpha + o(\alpha), \alpha \to 0$ that

$$e^{-\lambda Q(t)} = 1 - \lambda Q(t)(1 + o(1)) \tag{6}$$

Thus, from relations (4), (5) and (6), we have that

$$\lim_{t\to\infty} H\left(t, e^{-\lambda Q(t)}\right) = \frac{e^{-\frac{2\mu}{\sigma^2}c}}{\lambda + e^{-\frac{2\mu}{\sigma^2}c}}. \tag{7}$$

The last expression is the Laplace transform of the exponential distribution with parameter $e^{-\frac{2\mu}{\sigma^2}c}$. Hence, the assertion of the theorem follows from (7) and the continuity theorem for Laplace transform [Feller (1971)].

We conclude the discussion with remarks on the asymptotic behaviour of some moments of the process $\nu(t)$. Using (4), we find the expectation and the variance of the process.

$$E[\nu(t)] = H'(t,1) = \frac{1}{1 - F[t, \theta(t)]}.$$

$$Var[\nu(t)] = \frac{F[t, \theta(t)]}{\{1 - F[t, \theta(t)]\}^2}.$$

For the Galton-Watson process [see Rahimov (1998a)], if the offspring distribution satisfies (1) and $\theta(t) = c / Q(t)$, then

$$E[\nu(t)] \sim \frac{1}{1 - G(c)} [Q(t)]^{-1},$$

$$Var[\nu(t)] = \frac{G(c)}{[1 - G(c)]^2} [Q(t)]^{-2}, \quad t \to \infty$$

where $G(c)$ is as defined in ( ).

For the Bellman-Harris model, if $f''(1) = \sigma^2 < \infty$, $\int_0^\infty t dG(t) = \mu < \infty$, $t^2[1 - G(t)] \to 0$ as $t \to \infty$ and $\theta(t) = ct$, then

$$E[\nu(t)] \sim \frac{1}{e^{-\frac{2\mu}{\sigma^2} c}} [Q(t)]^{-1},$$

$$Var[\nu(t)] = \frac{1 - e^{-\frac{2\mu}{\sigma^2} c}}{(e^{-\frac{2\mu}{\sigma^2} c})^2} [Q(t)]^{-2}, \quad t \to \infty$$

The study of the processs $\nu(t)$ in the case when $\theta(t)$ has different asymptotic behaviour and for different processes such as subcritical and supercritical is another topic for further investigation.

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