BIFURCATION AND TRANSITION PHENOMENA OF MULTIPLE CHARGED MONOPOLE PLUS HALF-MONOPOLE OF THE SU(2) YANG-MILLS-HIGGS THEORY

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by

ZHU DAN

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FENOMENA PENCABANGAN PERALIHAN MONOKUTUB DAN SESETENGAH MONOKUTUB BERCAJ BERGANDA UNTUK TEORI SU(2) YANG-MILLS-HIGGS

ABSTRAK

Monokutub magnet dan multi-monokutub adalah penyelesaian soliton bertopologi dalam tiga dimensi yang timbul apabila simetri SU(2) tak-Abelian dipecah secara spontan oleh medan Higgs. Teori tolok yang boleh menerangkan kewujudan meraka adalah teori SU(2) Yang-Mills-Higgs, yang juga dikenali sebagai model SU(2) Georgi-Glashow. Baru-baru ini, kewujudan penyelesaian monokutub separuh telah dicadangkan dan konfigurasi yang melibatkan satu monokutub separuh dan satu monokutub biasa ‘t Hooft-Polyakov dalam model SU(2) Georgi-Glashow juga dilaporkan walaupun demikian, disebabkan monokutub separuh merupakan bidang penyelidikan yang baru, topik yang berkaitan dengan interaksi antara monokutub dan monokutub separuh adalah sedikit. Dalam tesis ini, kami mengkaji tentang penyelesaian monokutub dengan monokutub separuh dalam teori SU(2) Yang-Mills-Higgs yang mempunyai nilai nombor penggulungan ϕ yang lebih besar, n ($2 \leq n \leq 4$), antara sesuatu jarak terhad malar gandingan Higgs, λ ($0 < \lambda \leq 40$). Penggunaan grid beresolusi (110×100) dalam kaedah berangka kami untuk mendapat penyelesaian juga adalah lebih besar berbanding dengan kajian yang lepas. Matlamat disertasi ini adalah untuk mendapatkan maklumat tentang ciri-ciri konfigurasi monokutub magnet dengan monokutub separuh, untuk mengkaji interaksi antara konstituen melalui fenomena pencabangan dan peralihan penyelesaian, danjuga untuk memperdehi pemahaman lebih mendalam tentang struktur teori tolok dan pada masa yang sama, memperoleh perfahaman yang lebih mendalam tentang struktur teori tolok terlibat. Apabila $n \geq 2$, kami mendapati monokutub menjadi n-monokutub yang bertindih di tempat yang sama. Pada masa yang sama, monokutub separuh yang bertempat di titik asalan menjadi satu n-monokutub separuh yang bertindih. Apabila $n = 2$, penyelesaian berperangai ganjil, dan mencapah selepas $\lambda = 8.00$ dan apabila $n \geq 3$, bertentangan dengan pemerhatian yang didapati pada

konfigurasi pasangan monokutub-anti monokutub (MAP) atau rantai monokutub-anti monokutub (MAC), monokutub tidak bergabung dengan monokutub separuh untuk membentuk gegelung vorteks. Sebaliknya, apabila pemalr gandingan Higgs mencapai nilai peralihan fasa kritikal, λ_t, n-monokutub separuh kekal tidak berubah di titik asalan tetapi gegelung vorteks dibentuk pada n-monokutub yang bertindih. Ini dikenali sebagai fenomena peralihan. Pada masa yang sama, pencabangan dapat diperhatikan apabila $n \geq 3$, yang masa selain penyelesaian asas, satu lagi cabangan penyelesaian baru yang mempunyai tenaga yang lebih tinggi muncul apabila λ mencapai sesuatu nilai kritikal λ_c, dan dalam kes istimewa di mana $n = 4$, satu cabangan penyelesaian baru muncul. Selain itu, untuk $n \geq 2$, wujud suatu nilai batasan bawah kritikal λ_b untuk penyelesaian asas, yang mana tiada penyelesaian boleh didapati apabila $\lambda < \lambda_b$. Perangai ganjil untuk penyelesaian apabila $n = 2$ dan penyelesaian cabang baru yang dijumpai apabila $n = 4$ boleh diatributkan kepada kewujudan monokutub setengah dalam model ini. Dengan mengambilkira penyelidikan sebelum dan membandingkan penyelidikan ini dengan penyelesaian MAP piawai, satu spekulasi yang menarik boleh dibuat, yang mana kewujudan monokutub setengah hanya mempengaruhi penyelesaian dengan nombor penggulungan ϕ, n yang genap. Selain itu, di samping impak besar keatas penyelesaian, monokutub setengah adalah dorman dan tak-aktif, perubahan dalam kuantiti fizik nampaknya disumbangkan hanya oleh monokutub.

BIFURCATION AND TRANSITION PHENOMENA OF MULTIPLE CHARGED MONOPOLE PLUS HALF-MONOPOLE OF THE SU(2) YANG-MILLS-HIGGS THEORY

ABSTRACT

Magnetic monopoles and multimonopoles are three-dimensional topological soliton solutions, which arise when the non-Abelian SU(2) symmetry is spontaneously broken by the Higgs field. The gauge theory describing their existence is the SU(2) Yang-Mills-Higgs theory, which is also known as the SU(2) Georgi-Glashow model. Recently, the existence of half-monopole solutions had been proposed, and a configuration involving a half-monopole and an ordinary 't Hooft-Polyakov monopole within the SU(2) Georgi-Glashow model was also reported. However, since half-monopole is a relatively new field of research, topics regarding the interactions between one-monopoles and half-monopoles are rather scarce. In this thesis, the one-monopole plus half-monopole solution of the SU(2) Yang-Mills-Higgs theory with higher value of ϕ-winding number, n ($2 \leq n \leq 4$) is studied for a range of the Higgs coupling constants, λ ($0 < \lambda \leq 40$), and the resolution of the grids used (110×100) in the numerical method for calculating the solutions is also greater than previous research. The goal of this dissertation is to gain information about the general behaviors and properties of the one-plus-half monopole configuration, to probe the interactions between constituents through phenomena manifested as bifurcations and transitions of solutions, as well as to obtain a deeper understanding of the structure of gauge theories. We noticed that for $n \geq 2$, the one-monopoles become an n-monopole superimposed at the same location. At the same time, the half-monopoles at the origin, in the same manner, becomes a superimposed n-half-monopole. When $n = 2$, the solutions behave strangely and diverge after $\lambda = 8.00$ and when $n \geq 3$, in contrary to the observation in monopole-antimonopole pair (MAP) or monopole-antimonopole chain (MAC) configurations, the one-monopoles do not merge with the half-monopoles to form vortex-rings. Instead, when the Higgs coupling constant reaches a certain critical

phase transition value, λ_t, the n-half-monopole remains unchanged at the origin while vortex-rings were formed among the superimposed n-monopole. This is known as the transition phenomenon. At the same time, bifurcation phenomenon is also observed when $n \geq 3$, where besides the fundamental branch, new branches of solution with higher energies emerge at some critical value of λ, λ_c and in the special case where $n = 4$, a completely new branch of solutions appeared. It is also noticed that for $n \geq 2$, there exists a critical lower bound λ_b for the fundamental branch, for which when $\lambda < \lambda_b$, no solution can be found. The peculiar behavior of the solution when $n = 2$ and the new branch of solution found when $n = 4$ can all be attributed to the existence of half-monopoles within this model. Taking previous researches into account and comparing this research with the standard MAP results, one interesting speculation can be drawn, which is that the existence of half-monopoles seems only affect solutions with even number of the ϕ-winding number, n. Furthermore, despite the huge impact on solutions, half-monopoles appear rather dormant and inactive, the change in physical quantities seems only contributed by one-monopoles.

CHAPTER 1 - INTRODUCTION

1.1 A Brief History of Theoretical Physics

The foundation of modern physics is built upon two major achievements of the 20th century, relativity and quantum mechanics. While both theories of relativity are the masterpiece of Albert Einstein, quantum mechanics is the wisdom from hundreds of scientists and through decades of discoveries, Max Planck, Erwin Schrödinger, Paul Dirac, Enrico Fermi, just to name a few. Relativity deals with physics on a grand scale whereas quantum mechanics focuses on the minuscule structure of a nucleus. The undertaking to combine these two extremely different theories into a single, all-inclusive theory is still one of the most formidable task even to this day.

The first and most successful attempt to combine quantum mechanics and special relativity is quantum electrodynamics (QED) in the 1940s. Though the very first formulation of a quantum theory describing radiation and matter interaction was done by Paul Dirac (Dirac, 1927), not until 1947 when the idea of renormalization was proposed by Hans Bethe (Bethe, 1948) did quantum electrodynamics attain its present, fully-accepted form. QED is one of the most strictly tested theories in physics, the agreement of theoretical predictions of QED and experimental results is within ten parts in a billion (10^{-8}) (Peskin and Schroeder, 1995, p.198). Although combining electrodynamics with special relativity was done perfectly as early as in the 1940s, combining quantum theories with general relativity remains an active field of research nowadays and no decisive conclusion has been drawn or experimental evidences to show a clearer path.

The mathematical formulation of QED shows a kind of group symmetry. A mathematical group is defined as a set with a symmetry operation. In mathematics, any operation uses two elements of the set as input and if the output satisfies 4 conditions (namely, closure, associativity, identity element and inverse element), the set and the operation together form a group, *G*, as denoted in mathematics. The group symmetry exhibited in QED is called U(1) or unitary group of order 1. The set consists of all first order complex unitary matrix (numbers) as elements. As the elements are numbers, the group possesses one additional feature, that is the operations commute. Hence the U(1) group is abelian. The manifestation of a group symmetry in QED is actually the result of selecting a particular gauge (Gauge theory wil be discussed in details in Chapter 2 of this dissertation.).

Inspired by the massive success of QED, the mathematical tool, group theory, was soon applied to other theories. In 1954, Chen-Ning Yang and Robert Mills together published a paper in which they imposed a similar group symmetry upon the isospin doublet in the hope of providing an explanation for the nuclear strong interactions (Yang and Mills, 1954). Unlike $G = \mathrm{U}(1)$ in QED, the type of group symmetry exhibited in their formulation was actually $G = \mathrm{SU}(2)$ or special unitary group of order 2. The set consists of all second order complex unitary matrix with determinant 1 as elements. As the elements in the set are no longer numbers, the commutation feature is not preserved. Thus, the SU(2) group symmetry is non-abelian.

The work of Yang and Mills was shortly abandoned as it was later found to be incomplete on its own. Wolfgang Pauli pointed out that Yang and Mills' theory alone describes a long-distance interaction with massless mediator, which was something he encountered back in 1953 and caused him refraining from publishing his results formally (Straumann, 2000). A massless mediator contradicts the features of the extremely short ranged nuclear strong force which requires massive mediators. This issue was resolved when the concept of spontaneous symmetry breaking and the Higgs mechanism were put forward. In Peter Higgs' formulation, mediators acquire mass through the spontaneous symmetry breaking caused by the Higgs field, a scalar field permeating all space (Higgs, 1964). This theory was later known by the name of Yang-Mills-Higgs (YMH) field theory.

The spontaneous symmetry breaking and Higgs mechanism not only saved Yang-Mills theory, they also reassured theoretical physicists that imposing group symmetry was the right approach to a deeper understanding of the fundamental interactions of our universe. Notable achievements after this were quantum chromodynamics (QCD) and the electroweak theory (EWT). They both went through a series of developments and it's hard to put a date on them. QCD is the correct interpretation of strong interactions and belongs to the group symmetry SU(3). Although Yang-Mills theory is monumental as it provides the mathematical basis for all later theories, it was also an early attempt to describe the strong interactions and it was on the wrong track in that respect. QCD was only possible when quark model was brought forth and when the colour scheme was proposed in order to solve the violation of Pauli exclusion principle appeared in resonance particles (Griffiths, 2008, p.43). The latter, EWT, is a marvelous masterpiece of Sheldon Glashow, Steven Weinberg and Abdus Salam. And for this reason, EWT sometimes is called the Glashow-Weinberg-Salam (GWS) model. In this theory, the coupling constants of both electromagnetic interactions and weak interactions become identical when a critical energy limit is reached (Glashow, 1959; Weinberg, 1967; Salam, 1959), which indicates that two of the fundamental interactions of our universe are just different manifestations of the same interaction called electroweak interaction. In this theory, $G = \mathrm{SU}(2)\times\mathrm{U}(1)$.

The above is the brief history of theoretical physics up until the 1970s and to finalize this section, I would like to quote David J. Griffiths from his book, *Introduction to Elementary Particles* (Griffiths, 2008, p.3):

> This theory - or, more accurately, this collection of related theories, based on two families of elementary particles (quarks and leptons), and incorporating quantum electrodynamics, the Glashow-Weinberg-Salam theory of electroweak processes, and quantum chromodynamics - has come to be called the *Standard Model*.

1.2 Standard Model, Achievements and Flaws

The success of a theory is determined by the experimental verifications of its predictions. In the 1970s, when the majority of the theories, which belongs to the Standard Model, had been proposed, only 3 types of quarks (up, down and strange) and 4 types of leptons (electron, muon and their corresponding neutrinos) were observed experimentally and the idea of generations of matter had not yet been fully appreciated. The first sign of success of the Standard Model could be attributed to the confirmation of neutral current in 1973, shortly after being predicted by the GWS model. This solidified the foundation of electroweak theory, a pillar of the Standard Model that we now know today. And later, the detection of charm quark was made in 1974, which completed the first two generations of matter.

Then the Standard Model set out on a road of success and triumph. Tau lepton was detected using the Stanford Positron Electron Asymmetric Rings (SPEAR) with a series of experiments conducted between 1974 and 1977 (Okun, 1980, p.103; Perl et al., 1975). In 1977, bottom quark was detected at Fermilab (Herb et al., 1977). The mediators of the weak force (W^+, W^- and Z) were all discovered in 1983 at CERN.

After that, things went quite for a decade as the precision and operating energy of the then-top-of-the-line labs were not enough to detect or produce the remaining particles predicted by the Standard Model. With the construction of new detectors like those used in the Collider Detector at Fermilab (CDF), D0 experiment (D0 or D∅) and Direct Observatrion of the Nu Tau (DONUT) at Fermilab, top quark was detected in 1995 (Abe et al., 1995; Abachi et al., 1995) and tau neutrino was detected in 2000, which completed the 3 generations of quarks and leptons. All of these breathtaking achievements culminated in 2012 when the last piece of the puzzle, the Higgs boson, was finally detected.

These discoveries of the past 40 years clearly indicate that the Standard Model is indeed the correct interpretation of our universe. But no one would say that it is the final picture as there are still lots of questions left unexplained and unanswered by the Standard Model. The most obvious problem is that only three of the four fundamental forces are being described by the Standard Model, the fourth one, gravity, is not accounted for. Another problem into which the Standard Model offers no insight is the matter-antimatter asymmetry or the baryon asymmetry problem. The world we live in is made of baryonic matter, if the Big Bang created the same amount of matter and antimatter, where did all the antimatter go? The Standard Model also provides no explanation for dark matter and dark energy, something we now know that made up roughly 95 percent of our universe's total mass energy.

On the other hand, the Standard Model itself is also not as elegant as physicists hope it could be. It contains around 20 unrelated, arbitrary constants (Blumhofer and Hutter, 1997) whose value can only be determined through experiments and the theoretical explanations on such constants are completely inadequate (Cahn, 1996). The Standard Model thus received a lot of criticism from an aesthetic point of view.

All of these flaws and insufficiencies hint at the possiblity of an even more all-inclusive theory. A few tries have already been made in the past several decades and some of them will be briefly discussed in the next section.

1.3 Beyond the Standard Model

Looking back at history, physics is all about unification. Back in the 19^{th} century, James Clerk Maxwell unified electric and magnetic forces. Then following the huge success of combining nuclear weak force with electromagnetic force in the 1960s, it is natural to try including nuclear strong force into the picture as well. Though in the Standard Model, the theory responsible for describing strong force, QCD, is already an important constituent, it is presented only as a parallel to the EWT and no unification has been made.

1.3.1 The Grand Unified Theory

Mathematically, in EWT, $G = \mathrm{SU}(2)\times\mathrm{U}(1)$ and in QCD, $G = \mathrm{SU}(3)$. Thus, a theory incorporating both of them must show $G = \mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)$ at least or possesses an even higher order symmetry which includes $\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)$ as a subgroup. Historically, such a theory has come to be called a Grand Unified Theory or GUT for short.

In 1974, Howard Georgi and Sheldon Glashow attempted to construct a GUT (Georgi and Glashow, 1974) and their theory is sometimes called the Georgi-Glashow (GG) model. The GG model is based on the smallest, simplest simple Lie group (The mathematical details of a simple Lie group will not be discussed here as it is outside of the scope of this dissertation. Interested readers are encouraged to refer to advanced textbooks on abstract algebra.) that contains the Standard Model, $G = \mathrm{SU}(5)$. In their model, under some extremely high energy, the SU(5) symmetry is spontaneously broken into smaller symmetry groups, those possessed by the Standard Model. This is very reminiscent of, as it should be, what we have seen in EWT in which under certain energy limit the SU(2)$\times$U(1) symmetry is broken into smaller symmetry group which is U(1).

Apart from incorporating strong force into the picture, expressing the strong force as a different manifestation of a single, unified interaction, the GG model predicted one other phenomenon, the proton decay (Griffiths, 2008, p.33). When the SU(2)$\times$U(1) symmetry is broken, 4 force mediating particles are produced, W^+, W^-, Z and γ. By the same token, when SU(5) symmetry is broken, several other force carrying particles, the X bosons (Cheng and Li, 1983, p.437), are produced and they provide ways for protons to decay. In addition, allowing protons to decay breaks the conservation law of baryon number and theoretically solves the baryon asymmetry problem mentioned in the previous section. In the GG model, the proton half-life is predicted to be at least 10^{30} years (Griffiths, 2008, p.406). In 1998, a research team conducted a 414-days-long experiment to detect proton decay. Though not a single sign of the decay was detected, they managed to push the lower bound of proton half-life and the result they obtained back then was at least 1.6×10^{33} years (Abe et al., 1998). This obviously vetoes the GG model, but it's not the only GUT we have at hand.

One of the major difficulties when constructing a GUT is the choice of symmetry group. There are simply too many of them and we don't know which one truely describes our universe. Except the SU(5) used by the GG model, there are other symmetry groups that have been tested, some of them are: SO(10) (special orthogonal group of order 10, consists of all orthogonal matrices of order 10 with determinant 1), SO(16), SU(8), Sp(8) (symplectic group of order 8, consists of all symplectic matrices of order 8), etc. The proton half-lifes predicted by these models are slight different from each other, but they all lie within the range of 10^{30} to 10^{36} years (Nath and Perez, 2007). Since the experiment results indicate the lower bound of proton half-life should not be lower than 10^{33} years, simpler GUTs, like the GG model, have been ruled out.

Although there are still possible GUT candidates, the lack of hard experimental evidence and aesthetic elegance put the whole thing into question for some physicists. They resort to a different approach and hence comes the Supersymmetry (SUSY).

1.3.2 Supersymmetry (SUSY)

GUTs, being an enlarged version of the Standard Model, share both its features and defects. The fourth fundamental force, gravity, is still being excluded and even more arbitrary parameters have been added to them. Also, problems regarding dark matter and dark energy are still not addressed in almost all versions of GUT. Interestingly enough, nearly all these conundrums are marvelously solved by the introduction of SUSY, if it's eventually proven to be correct.

Physics has always been dealing with internal symmetries that link closely related subatomic particles, like the color symmetry that links red, blue and green charges and flavor symmetry that links 6 different types of quark. Physical systems are invariant under rotations within the corresponding space (color space, flavor space, etc.). In the 1960s, Japanese physicist Hironari Miyazawa put forward an idea that could be counted as the prototype of SUSY (Miyazawa, 1966, 1968). He proposed a kind of symmetry that transcends normal internal symmetry and links mesons and baryons together. Since obviously the symmetry, if there is any, is extremely broken, his work was mostly ignored at the time. A few year later, Miyazawa's original idea was generalized to all particles linking fermions with bosons, most notably in Wess and Zumino's work (Wess and Zumino, 1974). In that particular symmetry space, a physical state representating a fermion turns into a boson after rotation (and vice versa). The system remains invariant, the fermion and boson are just different manifestations of a single state. Invariance of this kind is called SUSY (Griffiths, 2008, p.412).

Since SUSY links fermions with bosons, it assigns a partner to each and every particle known to the Standard Model. Each particle and its partner (called superpartners and collectively known as sparticles) share exactly the same quantum numbers, except their spins. Particles and sparticles should have the same mass and some of the superpartners of relatively light particles like electrons should easily be detected. The reason behind why selectrons (superpartner of electrons) are never detected could be due to the SUSY of our universe is severely broken by the spontaneous symmetry breaking mechanism and superpartners like selectrons gain an enormous amount of mass energy through interacting with the Higgs field, thus rendering them impossible to be produced by our current equipments. There are other possibilites, however, especially if gravity is brought into the picture (Griffiths, 2008, p.412).

All of these seems like a sleight of hand. They are, if not for their deeper theoretical implications. As SUSY does not belong to the scale of this dissertation, its theoretical achievements will be just briefly mentioned here in this section. Among them, there are four major accomplishments of SUSY that worth mentioning.

First of all, SUSY could naturally incorporate the fourth natural force, gravity, into its scheme, the resulting theory is called Supergravity (SUGRA), details of this theory can be found in Van Nieuwenhuizen's publication (Van Nieuwenhuizen, 1981). Theories like this, incorporating all four natural forces, are referred to as Theory of Everything, or ToE for short.

The second major achievement of SUSY could be attributed to the theoretical fact that all three running coupling constants of electromagnetic force, weak force and strong force can perfectly converge at a single point when energy reaches some critical value (Griffiths, 2008, p.406). Something all GUTs are trying to do but failed. This marvelous convergence is so elegant that most scientists believe that there is no coincidence and it's a clear indication that SUSY must be correct.

There is another problem left unanswered and that has haunted the Standard Model since the very beginning, the so-called, Hierarchy Problem. The values of coupling constants we measured in experiments are their effective values, after the renormalization process (Bethe, 1948). However, the real values of some coupling constants are vastly different from their effective values and require a huge amount of fine-tunings. This is the dilemma the Hierarchy Problem addresses. Even though lots of scientists devoted a large portion of their careers trying to solve this conundrum, no one could offer an answer to the fact that while the coupling constants of the other three forces are relatively close, gravity is 10^{24} times weaker than weak force (Hughes, 2005). In SUSY, on the other hand, divergent parts occurred in the renormalization process cancelled out naturally between superpartners and do not require fine-tunings at all (Haber, 2013).

Last but not least, among the huge amount of new particles predicted by different versions of SUSYs, some of them, known collectively as the neutralinos (Griffiths, 2008, p.416), could be considered as candidates of the WIMPs (weakly interacting massive particles) we need to solve the dark matter problem.

Indeed, SUSY looks promising, but despite all these theoretical achievements, there is no exprimental evidence whatsoever to prove whether SUSY is right or wrong, slowly but surely time will tell. Except SUSY, there are other approaches as well, superstring theory, multiverse, loop quantum gravity, just to name a few. Interestingly enough, nearly all of them share one thing in common, that is they require the existence of one particular thing, magnetic monopoles.

1.4 Magnetic Monopoles

The word "magnetic monopole" was coined by early physicists even before James Clerk Maxwell's unification of electricity and magnetism. The concept was put forward in order to give an explanation to the naturally magnetized nature of lodestones. It was believed that magnetic monopoles carrying different charges (corresponding to north and south pole) accumulate at opposite sides of a lodestone, forming the so-called magnetic fluids and thus give lodestones a magnetized nature.

Of course, this idea was quickly vetoed by a better understanding of electromagnetism in the nineteenth century when French physicist André-Marie Ampère discovered the circuital law. Then the word "magnetic monopole" was rarely seen in the physics community until Paul A. M. Dirac brought it back under the spotlight.

In 1931, Dirac published a paper (Dirac, 1931) in which he demonstrated that the existence of even a single magnetic monopole, given that the form of Maxwell's equations is intact, would force all the electric charges in the entire universe to be quantized. This is called the Dirac quantization condition. Even though the quantization of electric charges is a well-observed phenomenon, it is only a necessary condition and thus logically does not prove the existence of magnetic monopoles, but a lack of proper explanations as to why all electric charges in our universe are quantized have led lots of physicists to believe that magnetic monopoles must exist.

Another compelling theoretical evidence is the aesthetically pleasing form which the Maxwell's equations exhibit when magnetic monopoles are incorporated (Moulin, 2002), as tabulated in Table 1.1, this symmetric form exhibited by the new formulation rendered the classical Maxwell's equations artificial and unnatural.

Table 1.1: A comparison between formulations of Maxwell's equations with or without magnetic monopoles in SI units

Name of Laws	without magnetic monopoles	with magnetic monopoles
Gauss's Law	$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\varepsilon_0}$	
	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = \mu_0 \rho_m$
Faraday's Law	$-\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}$	$-\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + \mu_0 \mathbf{J_m}$
Ampère's Law	$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J_e}$	

Finally in 1974, the first topologically smooth monopole solution was proposed independently by Gerard 't Hooft and Alexander Polyakov ('t Hooft, 1974; Polyakov, 1974). The 't Hooft-Polyakov monopole is similar to Dirac monopole, but possesses a finite total energy and no singularities (Details of 't Hooft-Polyakov monopole will be discussed in Chapter 3). The mathematical formulation of 't Hooft-Polyakov monopoles and the methodologies involved later became the cornerstone of a vast majority of the researches done in this field, including the model described in this dissertation.

On top of all the above, almost all versions of GUTs and candidates of ToEs predict the existence of magnetic monopoles and the masses predicted are seem to be very model-dependent. Hence, the experimental confirmation of magnetic monopoles become vital as it could show us as to which GUTs or ToEs are on the right track and which should be discarded. Sadly, ever since Dirac's paper brought this mysteriously rare particle back to the spotlight, tons of systematic and thorough searches have been performed, but all attemps returned null results. However, just as the string-theorist, Joseph Polchinski, once said, "The existence of magnetic monopoles seems like one of the safest bets that one can make about physics not yet seen." (Polchinski, 2003), magnetic monopoles, with its theoretical feasibilities and the state of being the logically natural next step of modern physics, remain the most long-waited particles in the wake of the discovery of Higgs bosons. Here, waits the future of modern physics.

1.5 Objective and Research Gap

The objective of this research is to study the interactions between multi-monopoles and multiple half-monopoles over a large range of the Higgs coupling constant by investigating the bifurcation and transition phenomena, which are the results or manifestations of the interactions between constituents of the one-plus-half monopole configuration. This particular goal is chosen out of the consideration that half-monopoles are themselves still a relatively new concept and a study regarding their interactions with the more commonly known one-monopoles, 't Hooft-Polyakov monopoles in particular, will surely shine some light into the nature of this type of exotic particles. Currently, the newly found one-plus-half monopole configuration is the only platform in the field which made this effort possible. Finally, the scope of this research concerns only the numerical aspect of the solutions found, specifically, the trending behaviours. Physical quantities such as total energy, magnetic dipole moment, pole separation, Higgs modulus, magnetic charge density, energy density, magnetic charge of the system are plotted, analyzed and discussed in this research.

1.6 Dissertation Outline

This dissertation is divided into 6 chapters. The mathematical framework upon which all modern particle physics are based on, gauge field theory, is discussed in detail in Chapter 2. A review of monopole solutions is given in Chapter 3. Theoretical details about constructing the one-plus-half monopole solutions, the numerical method employed in this research and the physical quantities investigated are presented in Chapter 4. Results and discussions are in Chapter 5 and some comments and future research suggestions are saved for the last chapter.

CHAPTER 2 - GAUGE FIELD THEORY

2.1 Introduction to Gauge Theory

Gauge theories are the theories that describe literally all elementary particle interactions in modern physics. The word "gauge" (German "eich") was coined by German physicist Hermann Weyl and first appeared in his paper in 1929 (Weyl, 1929). Its meaning can be taken as "scale" or "measure". Technically, it refers to the mathematical formalism used to regulate the redundant degrees of freedom in the Lagrangian. Even though gauge theories are notoriously mathematically heavy, the method used to construct a gauge theory is rather simple.

We are interested in transformations made to constituents of the Lagrangian that leave it unchanged, in other words, gauge transformations. The Lagrangian is said to be invariant under these transformations and their specific mathematical form is the gauge. If the transformations do not depend on spatial coordinates, then the invariance involved is referred to as the global gauge invariance (German "eichinvarianz"). A gauge theory is then constructed by demanding the global gauge invariance to hold locally, that is, to require the transformations to depend on spatial-temporal coordinates and at the same time, leaves the Lagrangian untouched. This is called the local gauge invariance. Furthermore, gauges of a particular Lagrangian form a Lie group which is referred to as the gauge group. Generators of this group generate fields which are called gauge fields and the field quanta associated to these fields are the gauge bosons.

It is now clear that Maxwell's unifying theory of electricity and magnetism also exhibits local gauge invariance with electromagnetic four-potential as the gauge field and photon being the only gauge boson. It manifests a U(1) symmetry just as QED does and can be taken as the very first and simplest gauge theory in the history of physics.

Mathematically, gauge theories can be classified into two categories, Abelian gauge theory and Non-Abelian gauge theory according to the commutative property of their underlying operation (as discussed in section 1.1).

2.2 Abelian Gauge Theory

In electrostasis, if we have an electric potential, V, the electric field, $\mathbf{E}$, can be obtained through $\mathbf{E} = -\nabla V$. The gradient indicates that the electric field is directly related to the change in electric potential. That is to say, if the electric potential, V, transforms according to $V \to V' = V + C$, where C is some constant, then, the electric field, $\mathbf{E}$, stays the same. The transformation made to V is precisely a type of gauge transformation. In the following subsections, we will discuss the global and local gauge invarianc to show mathematically that the gauge transformation made to electric potential is what we called, an Abelian gauge transformation, and at the same time, gain some insights into gauge theories.

2.2.1 Global Gauge Invariance

In quantum field theory, there are three Lagrangian densities that are of the utmost importance. The first one being the Klein-Gordon Lagrangian density (Griffiths, 2008, p.355), it describes a scalar field, $\phi(x,y,z,t)$, with its quanta having spin-0:

$$\mathscr{L}\left(\phi, \partial_\mu \phi\right) = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2} m^2 \phi^2. \tag{2.1}$$

The second one is the Dirac Lagrangian density (Griffiths, 2008, p.355). It describes a fermionic field with its quanta having spin-$\frac{1}{2}$ and are represented by the Dirac bispinor, ψ:

$$\mathscr{L}\left(\psi, \partial_\mu \psi\right) = i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi, \tag{2.2}$$

where m is the mass of the field quanta, γ^μ is the Dirac matrices, $\overline{\psi}$ stands for the adjoint spinor and is defined as:

$$\overline{\psi} \equiv \psi^\dagger \gamma^0, \tag{2.3}$$

here, the dagger stands for the Hermitian conjugate and γ^0 is the zeroth Dirac matrix. Lastly, the third one is the Proca Lagrangian density (Griffiths, 2008, p.356) describing a vector field and is represented by a potential four-vector, $A^\mu(V,\mathbf{A})$, with its quanta having spin-1:

$$\mathscr{L}\left(A^{\mu},\partial_{\mu}A^{\nu}\right)=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}m^{2}A^{\nu}A_{\nu}, \tag{2.4}$$

where $F^{\mu\nu}$ is the field strength tensor and is defined as:

$$F^{\mu\nu}\equiv\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}, \tag{2.5}$$

or, it can be equally represented by matrix:

$$F^{\mu\nu}=\begin{bmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -B_z & B_y\\ E_y & B_z & 0 & -B_x\\ E_z & -B_y & B_x & 0\end{bmatrix}. \tag{2.6}$$

All the above equations are given in natural units.

Once all the cards are on the table, we can start constructing a gauge theory using these building blocks. Now, consider the Dirac Lagrangian density, if we make a transformation, G, to ψ, $\psi\rightarrow\psi'=G\psi$, then the corresponding adjoint spinor field would transform according to:

$$\overline{\psi}\rightarrow\overline{\psi}'=(G\psi)^{\dagger}\gamma^{0}=\psi^{\dagger}G^{\dagger}\gamma^{0}. \tag{2.7}$$

In gauge theory, we are interested with the invariance of Lagrangian. In this case, the product $\overline{\psi}\psi$ must satisfy the following criterion when being transformed according to G:

$$\overline{\psi}\psi\rightarrow(\overline{\psi}\psi)'=\psi^{\dagger}G^{\dagger}\gamma^{0}G\psi=\overline{\psi}\psi. \tag{2.8}$$

Currently, we take G as a number, thus $G^{\dagger}G$ can be brought together and if $G^{\dagger}G=1$ is satisfied, $\overline{\psi}\psi$ is invariant under the transformation. At this point, it's obvious that G is precisely the group U(1) and all elements in G can be expressed as $G = e^{i\theta}$ and this is called the phase factor.

Thus, transformations of the form, $G = e^{i\theta}$, made to the Dirac Lagrangian density manifest what we called the global gauge invariance as the parameter θ in this case is independent of position and time. U(1) is clearly Abelian, but only global gauge invariance alone does not construct a gauge theory, that's where local gauge invariance comes in.

2.2.2 Local Gauge Invariance

Now, if the phase factor depends on the position four-vector, x^{μ}. The second term in equation (2.2) stays the same under the transformation $G(x,\, t) = e^{i\theta(x,\, t)}$ (For simplicity, $(x,\, t)$ will be omitted when the context is clear). The first term, however, splits into two parts:

$$i\overline{\psi}\gamma^{\mu}\partial_{\mu}\left(e^{i\theta}\psi\right) = -\overline{\psi}\gamma^{\mu}\left(\partial_{\mu}\theta\right)e^{i\theta}\psi + i\overline{\psi}\gamma^{\mu}e^{i\theta}\partial_{\mu}\psi. \tag{2.9}$$

For reasons shall become clear later, we introduce a new variable, $\lambda(x,\, t)$, by pulling a factor of $-q$ out of θ, that is:

$$\lambda \equiv -\frac{1}{q}\theta, \tag{2.10}$$

now the Lagrangian density changes in the following way when it is being transformed by G:

$$\mathscr{L} \to \mathscr{L}' = \mathscr{L} + (q\overline{\psi}\gamma^{\mu}\psi)\,\partial_{\mu}\lambda. \tag{2.11}$$

This extra term is clearly not zero under normal circumstances and to maintain the Lagrangian density's invariance, we are obliged to add an additional term to $\mathscr{L}$ in order to soak up the extra term in equation (2.11):

$$\mathscr{L} = \left(i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\overline{\psi}\psi\right) - (q\overline{\psi}\gamma^{\mu}\psi)A_{\mu}, \tag{2.12}$$

here, A_{μ} transforms, under the influence of G, according to:

$$A_{\mu} \to A'_{\mu} = A_{\mu} + \partial_{\mu}\lambda. \tag{2.13}$$

This way, when a local gauge transformation is applied to the newly modified $\mathscr{L}$, both the original $\mathscr{L}$ (the two terms in the first parentheses in equation (2.12)) and A_{μ} pick up an extra term and they cancel each other. The invariance of $\mathscr{L}$ is thus restored and judging from the way A_μ transforms, it is yet another four-vector and as we shall see later, A_μ is precisely the electromagnetic potential four-vector.

The concept of covariant derivative needs to be introduced here before we go any further. Note that the steps we've shown thus far to restore the invariance of $\mathscr{L}$ is equivalent to replacing all ∂_μ with:

$$D_\mu \equiv \partial_\mu + iqA_\mu. \tag{2.14}$$

D_μ is called the covariant derivative and the technique of replacing all ∂_μ with D_μ in order to convert a globally invariant Lagrangian density into a locally invariant one is called the minimal coupling rule (Griffiths, 2008, p.360). Thus, the locally invariant Lagrangian density can be also be written as:

$$\mathscr{L} = i\overline{\psi}\gamma^\mu D_\mu \psi - m\overline{\psi}\psi. \tag{2.15}$$

While trying to restore the gauge invariance of $\mathscr{L}$, the new four-vector, A_μ, is inevitably introduced into $\mathscr{L}$, either by following the steps shown from the beginning of this section or by invoking the minimal coupling rule mentioned just now. However, a new term cannot be simply added to the Lagrangian density without considering its effects. Otherwise, it is just a mathematical construct. In our case, an additional term signifying the physics of A_μ must be added to $\mathscr{L}$ as well and at the same time, it must not spoil the overall invariance we are trying to maintain. As A_μ is a four-vector. Naturally, we look to the Proca Lagrangian density, equation (2.4).

In this case, it can be shown that the first term in Proca Lagrangian density is invariant under the transformation G, but the second term is not. So, in order to maintain the locally invariant properity of $\mathscr{L}$, we must set the mass of the field, m, to zero and it is fairly clear now that this particular particle of spin-1 and possesses zero mass is exactly the photon, the only gauge boson of this particular gauge theory and finally, the complete Lagrangian density reads:

$$\mathscr{L} = \left(i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\overline{\psi}\psi\right) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - (q\overline{\psi}\gamma^{\mu}\psi)A_{\mu}. \tag{2.16}$$

Now we can see that the last two terms in the above equation reproduce the Maxwell Lagrangian density and we can identify the current density as:

$$J^{\mu} \equiv q\overline{\psi}\gamma^{\mu}\psi. \tag{2.17}$$

Equation (2.16) is invariant under the local gauge transformation G, which is also Abelian. The transformation condition described in equation (2.13) is the general form of the change in electric potential, $V \to V' = V + C$, mentioned in the beginning. Furthermore, equation (2.16) is also clearly the Lagrangian density for QED, which describes the interaction of two fields, a Dirac field and a Maxwell field. The first two terms belong to the original Dirac Lagrangian density, describing particles of spin-$\frac{1}{2}$. The third term describes photons and the last term depicts an all-permeating massless vector field, which is exactly the electromagnetic field. All of QED can be obtained from this equation.

As seen from the above, demanding the global U(1) gauge invariance of the Dirac Lagrangian density to hold locally (This will be referred to as "the principle of local gauge invariance" for the rest of this dissertation.) generates QED. This is a breathtaking achievement and is done by the simplest of the simplest gauge theories, the Abelian ones. Gauge theories are not some particular physics theory, they are a powerful mathematical tool at our disposal. In the example above, we used a Dirac Lagrangian density to demonstrate, but we can equally well use a Klein-Gordon Lagrangian density, apply the same procedure and another Abelian gauge theory will be produced. In the next section, we will discuss the more general non-Abelian gauge theories.

2.3 Non-Abelian Gauge Theory of SU(2)

In previous sections, our starting point was only one Dirac Lagrangian density with no other interactions presented. Now, if there are two spin-$\frac{1}{2}$ fields interacting with each other, the new Lagrangian density is nothing but the sum of two Dirac Lagrangian densities:

$$\mathscr{L} = \left(i\overline{\psi}_1\gamma^\mu\partial_\mu\psi_1 - m_1\overline{\psi}_1\psi_1\right) + \left(i\overline{\psi}_2\gamma^\mu\partial_\mu\psi_2 - m_2\overline{\psi}_2\psi_2\right). \tag{2.18}$$

The above equation can be compactified if we introduce a two-component column vector:

$$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.19}$$

and the corresponding adjoint spinor matrix is: $\overline{\psi} = (\overline{\psi}_1\ \overline{\psi}_2)$. Then equation (2.18) can be compactly written as:

$$\mathscr{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - M\overline{\psi}\psi, \tag{2.20}$$

where M is the mass matrix:

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{2.21}$$

In particular, if the mass difference between m_1 and m_2 are negligable, then:

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = mI, \tag{2.22}$$

and thus equation (2.20) can be expressed as:

$$\mathscr{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi, \tag{2.23}$$

which is exactly the same as equation (2.2) except the spinors, adjoint spinors and masses become matrices.

The structural similarities between equation (2.2) and equation (2.23) are only made possible if the particles presented in the theory have negligable mass difference, just like protons and neutrons, which is precisely from where Chen-Ning Yang and Robert Mills got their inspirations.

In the previous sections, we see that demanding the principle of local gauge invariance to hold true on one Dirac Lagrangian density generates QED. Similarly, the combination of two Dirac Lagrangian densities with the principle of local gauge invariance constructs the entire SU(2) Yang-Mills theory, the theory which shaped modern physics. Although the SU(2) Yang-Mills theory alone is physically impossible and does not describe any real physical process, its importance and position in the history of physics is widely acknowledged and appreciated. In the following subsections, we are going to discuss the global and local gauge invariance of this theory.

2.3.1 Global Gauge Invariance of SU(2) Yang-Mills Theory

Just as equation (2.2) is globally invariant under a transformation of the form, $G = e^{i\theta}$, a similar global gauge transformation can be applied to equation (2.23). This time, the new transformation G takes the form, $G = e^{iH}$, where H is a 2×2 matrix. The column matrix ψ transforms like $\psi \to \psi' = G\psi$. For the row matrix $\overline{\psi}$, treat γ^0 as a number as it goes into each element of the row matrix. It transforms according to:

$$\begin{aligned} \overline{\psi} \to \overline{\psi}' = \begin{pmatrix} \overline{\psi_1} & \overline{\psi_2} \end{pmatrix}' = \begin{pmatrix} \psi_1^\dagger \gamma^0 & \psi_2^\dagger \gamma^0 \end{pmatrix}' = \left[\begin{pmatrix} \psi_1^\dagger & \psi_2^\dagger \end{pmatrix} \gamma^0 \right]' \\ = \left(\psi^\dagger \gamma^0 \right)' = (G\psi)^\dagger \gamma^0 = \psi^\dagger G^\dagger \gamma^0 = \psi^\dagger \gamma^0 G^\dagger = \overline{\psi} G^\dagger. \end{aligned} \tag{2.24}$$

Obviously, $\overline{\psi}\psi$ is invariant if $G^\dagger G = I$. Thus, G belongs to U(2).

Now suppose G is also Hermitian and any 2×2 Hermitian matrices can be expressed as (Griffiths, 2008, p.362):

$$H = \theta I + \tau \cdot \mathbf{a}, \tag{2.25}$$

where θ is any real number, τ is a vector-like construct made of Pauli matrices and $\boldsymbol{a}$ is any real vector. Then, G can be written as $G = e^{i\theta}e^{i\tau\cdot\mathbf{a}}$. We've already seen $e^{i\theta}$ belongs to U(1). So now, we are more interested in the second factor.

We want to calculate the determinant of matrix $e^{i\tau\cdot\boldsymbol{a}}$ and in order to do so, we first pull out a factor of $-\frac{1}{2}$ out of $\boldsymbol{a}$ for reasons shall become clear later. So, $\mathbf{a}$ = -$\frac{1}{2}\mathbf{b}$ and matrix $e^{i\tau\cdot\boldsymbol{a}}$ becomes $e^{-\frac{i}{2}\tau\cdot\boldsymbol{b}}$, then expand the matrix:

$$e^{-\frac{i}{2}\tau\cdot\mathbf{b}} = 1 + \left(-i\frac{\tau\cdot\mathbf{b}}{2}\right) + \frac{1}{2}\left(-i\frac{\tau\cdot\mathbf{b}}{2}\right)^2 + \frac{1}{3!}\left(-i\frac{\tau\cdot\mathbf{b}}{2}\right)^3 + ... \tag{2.26}$$

Now, multiplication of the form $(\tau\cdot\boldsymbol{a})(\tau\cdot\boldsymbol{b})$ can be easily calculated using summation notations:

$$\begin{aligned}(\tau\cdot\mathbf{a})(\tau\cdot\mathbf{b}) &= \sum_{i,j}\tau_i a_i \tau_j b_j = \sum_{i,j} a_i b_j (\tau_i\tau_j) = \sum_{i,j} a_i b_j (\delta_{ij} + i\varepsilon_{ijk}\tau_k) \\ &= \sum_{i,j} a_i b_j \delta_{ij} + i\sum_{i,j}\varepsilon_{ijk} a_i b_j \tau_k = \mathbf{a}\cdot\mathbf{b} + i\tau\cdot(\mathbf{a}\times\mathbf{b}).\end{aligned} \tag{2.27}$$

Then, in our case, the second term in the expansion becomes:

$$\frac{1}{2}\left(-i\frac{\tau\cdot\mathbf{b}}{2}\right)^2 = -\frac{1}{2}\left(\tau\cdot\frac{\mathbf{b}}{2}\right)\left(\tau\cdot\frac{\mathbf{b}}{2}\right) = -\frac{1}{2}\left[\frac{\mathbf{b}}{2}\cdot\frac{\mathbf{b}}{2} + i\tau\cdot\left(\frac{\mathbf{b}}{2}\times\frac{\mathbf{b}}{2}\right)\right] = -\frac{1}{2}\left(\frac{b}{2}\right)^2, \tag{2.28}$$

and thus the expansion can be simplified to:

$$\begin{aligned} e^{-\frac{i}{2}\tau\cdot\mathbf{b}} &= 1 - \frac{i\tau\cdot\mathbf{b}}{2} - \frac{1}{2}\left(\frac{b}{2}\right)^2 + \frac{1}{3!}i(\tau\cdot\mathbf{b})\frac{b^2}{2^3} + ... \\ &= \left[1 - \frac{1}{2}\left(\frac{b}{2}\right)^2 + \frac{1}{4!}\left(\frac{b}{2}\right)^4 - ...\right] - \frac{i\tau\cdot\mathbf{b}}{b}\left[\frac{b}{2} - \frac{1}{3!}\left(\frac{b}{2}\right)^3 + ...\right] \\ &= \cos\left(\frac{b}{2}\right) - i(\hat{b}\cdot\tau)\sin\left(\frac{b}{2}\right), \end{aligned} \tag{2.29}$$

here $\hat{b}$ is a unit vector. Then we express the matrix in its traditional block form, first calculate $\hat{\mathbf{b}} \cdot \tau$:

$$\hat{\mathbf{b}} \cdot \tau = \hat{b}_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \hat{b}_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \hat{b}_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} \hat{b}_z & \left(\hat{b}_x - i\hat{b}_y\right) \\ \left(\hat{b}_x + i\hat{b}_y\right) & -\hat{b}_z \end{pmatrix}, \tag{2.30}$$

then the matrix $e^{-\frac{i}{2}\tau\cdot\boldsymbol{b}}$ can be expressed as (Note that the first term in equation (2.29) is actually a matrix):

$$e^{-\frac{i}{2}\tau\cdot\mathbf{b}} = \cos\left(\frac{b}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\left(\frac{b}{2}\right)\begin{pmatrix} \hat{b}_z & \left(\hat{b}_x - i\hat{b}_y\right) \\ \left(\hat{b}_x + i\hat{b}_y\right) & -\hat{b}_z \end{pmatrix}$$
$$= \begin{pmatrix} \left(\cos\frac{b}{2} - i\hat{b}_z\sin\frac{b}{2}\right) & -i\left(\hat{b}_x - i\hat{b}_y\right)\sin\frac{b}{2} \\ -i\left(\hat{b}_x + i\hat{b}_y\right)\sin\frac{b}{2} & \left(\cos\frac{b}{2} + i\hat{b}_z\sin\frac{b}{2}\right). \end{pmatrix} \tag{2.31}$$

And now we are in a position to calculate the determinant:

$$\det\left(e^{-\frac{i}{2}\tau\cdot\mathbf{b}}\right) = \left(\cos\frac{b}{2} - i\hat{b}_z\sin\frac{b}{2}\right)\left(\cos\frac{b}{2} + i\hat{b}_z\sin\frac{b}{2}\right) + \sin^2\frac{b}{2}\left(\hat{b}_x - i\hat{b}_y\right)\left(\hat{b}_x + i\hat{b}_y\right)$$
$$= \cos^2\frac{b}{2} + \hat{b}_z^2\sin^2\frac{b}{2} + \sin^2\frac{b}{2}\left(\hat{b}_x^2 + \hat{b}_y^2\right) = \cos^2\frac{b}{2} + \sin^2\frac{b}{2}\left(\hat{b}_x^2 + \hat{b}_y^2 + \hat{b}_z^2\right)$$
$$= \cos^2\frac{b}{2} + \sin^2\frac{b}{2} = 1. \tag{2.32}$$

Up until now, we've shown equation (2.23) exhibits a U(2) global gauge invariance and it can be factored out into a U(1) factor plus another one expressed by $e^{i\tau\cdot\boldsymbol{a}}$. In equation (2.32) we proved the determinant of $e^{i\tau\cdot\boldsymbol{a}}$ is 1 and $G^{\dagger}G = 1$ shows that $e^{i\tau\cdot\boldsymbol{a}}$ is unitary. These indicate that $e^{i\tau\cdot\boldsymbol{a}}$ belongs to SU(2). Thus, equation (2.23) is not only invariant under the larger U(2) global gauge transformations, but also invariant under the smaller SU(2) global guage transformations. Next, we are going to show how SU(2) local gauge invariance is achieved.

2.3.2 Local Gauge Invariance of SU(2) Yang-Mills Theory

Suppose $\boldsymbol{a}$ in $e^{i\tau\cdot\boldsymbol{a}}$ now depends on the position four-vector, x^{μ}. Once again, we redefine a new variable, λ, by pulling out a factor of $-q$ out of $\boldsymbol{a}$:

$$\lambda \equiv -\frac{1}{q}\mathbf{a}, \tag{2.33}$$

here, q is a coupling constant analogous to the electric charge. Then, the local SU(2) gauge transformation, G, now takes the form:

$$G = e^{-iq\tau\cdot\lambda}. \tag{2.34}$$

We'll focus on $\partial_{\mu}\psi$ only as it is the only factor that will affect the invariance. Now, apply the local gauge transformation G to $\partial_{\mu}\psi$:

$$\partial_{\mu}(G\psi) = G\partial_{\mu}\psi + \left(\partial_{\mu}G\right)\psi, \tag{2.35}$$

then, invoke the minimal coupling rule mentioned in section 2.2.2 to replace all ∂_{μ} with D_{μ}, in this case, the covariant derivative takes the form:

$$D_{\mu} \equiv \partial_{\mu} + iq\tau\cdot\mathbf{A}_{\mu}, \tag{2.36}$$

here, $\mathbf{A}_{\mu}$ is a vector-like construct made of three four-vectors, that is, similar to the case in section 2.2.2, but rather than one, there are three new fields introduced into the Lagrangian dentisy. Alternatively, they are called the Yang-Mills fields.

The minimal coupling rule obliterates the second term in equation (2.35). So, $D_{\mu}(G\psi) = GD_{\mu}\psi$ and in order for this to hold true, $\mathbf{A}_{\mu}$ must satisfy a certain rule and to find it, we go from $D_{\mu}(G\psi) = GD_{\mu}\psi$, write out the covariant derivates long-hand:

$$\left(\partial_{\mu} + iq\tau\cdot\mathbf{A}'_{\mu}\right)\psi' = G\left(\partial_{\mu} + q\tau\cdot\mathbf{A}_{\mu}\right)\psi, \tag{2.37}$$

where primed terms indicates they were already transformed by G, like $\psi' = G\psi$.

Together with equation (2.35), equation (2.37) now becomes:

$$\left(\partial_\mu G\right)\psi + G\left(\partial_\mu \psi\right) + iq\left(\tau \cdot \mathbf{A}'_\mu\right) G\psi = G\left(\partial_\mu \psi\right) + iqG\left(\tau \cdot \mathbf{A}_\mu\right)\psi. \tag{2.38}$$

Cancel the $G(\partial_\mu \psi)$ on each side of the equation and multiply G^{-1} on the right to obtain a condition for $\tau \cdot \mathbf{A}'_\mu$:

$$\tau \cdot \mathbf{A}'_\mu = G(\tau \cdot \mathbf{A}_\mu)G^{-1} + i\frac{1}{q}(\partial_\mu G)G^{-1}. \tag{2.39}$$

From this point on, to find the exact solution is extremely formidable. The approximate transformation rule in the limiting case of very small $|\lambda|$, however, is rather straightforward and as a finite gauge transformation is bulit upon infinitesimal ones, finding the approximate transformation is equivalent to finding the exact one. Now, expand the relative matrices and keep only the first-order terms:

$$G \approx 1 - iq\tau \cdot \lambda, G^{-1} \approx 1 + iq\tau \cdot \lambda, \partial_\mu G \approx -iq\tau \cdot \partial_\mu \lambda. \tag{2.40}$$

In this approximation, equation (2.39) becomes:

$$\tau \cdot \mathbf{A}'_\mu \approx \tau \cdot \mathbf{A}_\mu + iq\left[\tau \cdot \mathbf{A}_\mu, \tau \cdot \lambda\right] + \tau \cdot \partial_\mu \lambda, \tag{2.41}$$

the square bracket stands for commutator and we've already shown in the previous section that:

$$\left(\tau \cdot \mathbf{a}\right)\left(\tau \cdot \mathbf{b}\right) = \mathbf{a} \cdot \mathbf{b} + i\tau \cdot \left(\mathbf{a} \times \mathbf{b}\right). \tag{2.42}$$

So the commutator $[\tau \cdot \mathbf{A}_\mu, \tau \cdot \lambda]$ becomes:

$$\begin{aligned}\left[\tau \cdot \mathbf{A}_\mu, \tau \cdot \lambda\right] &= \left(\tau \cdot \mathbf{A}_\mu\right)\left(\tau \cdot \lambda\right) - \left(\tau \cdot \lambda\right)\left(\tau \cdot \mathbf{A}_\mu\right) \\ &= i\tau \cdot \left(\mathbf{A}_\mu \times \lambda\right) - i\tau \cdot \left(\lambda \times \mathbf{A}_\mu\right) = 2i\tau \cdot \left(\mathbf{A}_\mu \times \lambda\right).\end{aligned} \tag{2.43}$$