

**C^1 NON-NEGATIVITY PRESERVING
INTERPOLATION USING CLOUGH-TOCHER
TRIANGULATION**

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**C^1 NON-NEGATIVITY PRESERVING
INTERPOLATION USING CLOUGH-TOCHER
TRIANGULATION**

by

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LIST OF SYMBOLS

T	Triangle
V	Vertices of triangle
u, v, w	Barycentric coordinates
R	Cubic Bézier triangular patch
$d_{i,j,k}$	Bézier ordinates of R
C^0	Zero order parametric continuity
C^1	First order parametric continuity
α_i, β_i	Barycentric coordinates of triangle
V_c	Splitting point
S	Patch produced by macro-triangle T
c_i	Bézier ordinates
$-l$	Lower bound
$P(x)$	Cubic Bézier polynomial curve
Ω	Domain of a function
\hat{c}_i	Bézier ordinates of adjacent triangle
φ_i	Intersection point

INTERPOLASI MENGEKALKAN KETAKNEGATIFAN C^1 DENGAN TRIANGULASI CLOUGH-TOCHER

ABSTRAK

Pada masa kini, visualisasi saintifik merupakan satu cabang penting dalam grafik komputer untuk menggambarkan data saintifik dari fenomena tiga dimensi secara grafik. Pembinaan permukaan biasanya melibatkan penjanaan satu set tampalan permukaan yang lancar dihubungkan bersama dan permukaan harus mewarisi sifat bentuk data tertentu seperti ketaknegatifan. Pembinaan permukaan interpolasi C^1 yang mengekalkan ketaknegatifan kepada data berselerak dipertimbangkan. Data yang diberikan di triangulasi dengan menggunakan triangulasi Delaunay. Permukaan interpolasi kepada data berselerak dibentuk demi demi dengan tampalan segi tiga Bézier. Permukaan dihasilkan menggunakan kaedah pemisahan Clough-Tocher. Syarat cukup untuk ketaknegatifan pada ordinat Bézier diterbitkan bagi memastikan ketaknegatifan tampalan segi tiga Bézier kubik. Set batas bawah baru dicadangkan kepada ordinat Bézier. Nilai awal ordinat Bézier ditentukan oleh data yang diberikan dan kecerunan yang dianggarkan pada data. Ordinat Bézier akan diubah jika perlu dengan mengubah kecerunan pada data supaya ordinat Bézier memenuhi syarat ketaknegatifan. Skema pembinaan permukaan yang mengekalkan ketaknegatifan adalah setempat. Ia membina permukaan interpolasi C^1 untuk data berselerak tertakluk kepada satah kekangan. Beberapa contoh bergrafik dibentangkan.

C^1 NON-NEGATIVITY PRESERVING INTERPOLATION USING CLOUGH-TOCHER TRIANGULATION

ABSTRACT

Nowadays, scientific visualization is an important branch in computer graphics to graphically visualize the scientific data from three dimensional phenomena. The construction of a surface usually involves generating a set of surface patches that smoothly connected together and the surface should inherit certain shape property of the data like non-negativity. The construction of non-negativity preserving C^1 interpolation surface to scattered data is considered. The given data is triangulated using Delaunay triangulation. The interpolating surface to scattered data is piecewise with Bézier triangular patches. The surfaces are produced using the method of Clough-Tocher split. Sufficient non-negativity conditions on the Bézier ordinates are derived to ensure the non-negativity of a cubic Bézier triangular patch. New set of lower bounds is proposed to the Bézier ordinates. The initial values of the Bézier ordinates are determined by the given data and the estimated gradients at the data sites. The Bézier ordinates are adjusted if necessary by modifying the gradients at the data sites so that the Bézier ordinates fulfill the non-negativity conditions. The scheme for constructing the non-negativity preserving surface is local. It constructs C^1 interpolating surface to scattered data subject to constraint plane. Some graphical examples are presented.

CHAPTER 1

INTRODUCTION

1.1 Background Study

Scientific visualization is an important branch in computer sciences. It plays an important role to graphically illustrate the data in Cartesian space such that enable users to analyze, understand and gather important information from the data. When the data are visualized there may be some inherent properties in the data which one wishes to preserve. Among the shape properties that commonly been preserved in the literature, non-negativity preservation is important. A patch defined in Cartesian space is said to be non-negative when the z -coordinate of every point on the patch is greater or equals to zero. There are many phenomena and physical situations where negative values are not physically meaningful, such as rainfall data and concentration of a material.

The problem of non-negativity preservation had been discussed by a number of authors via a variety of methods. In 1991, Goodman *et al.* discussed about the sufficient and necessary condition for the non-negativity preservation of univariate cases. Rational cubic was used in their work.

Chan and Ong (2001) extended the univariate case and derived a sufficient condition of non-negativity for the bivariate case. Sufficient non-negativity condition was expressed as lower bounds to the Bézier ordinates of cubic Bézier

triangular patch. The resulting surface is a convex combination of three cubic Bézier triangular patches. Non-negativity of the surface was preserved by adjusting the first order partial derivatives at the data sites. In Piah *et al.* (2005), a local C^1 range restricted scattered data interpolation scheme was presented. In that paper, more relaxed lower bound was derived to the Bézier ordinates.

In 1996, Ong and Wong described a local C^1 scattered data interpolation scheme using the side vertex method for interpolation in triangles. Rational cubic was used along every line segment joining a vertex to the opposite edge of a triangle. The surface curve is subject to the non-negativity conditions given in Goodman *et al.* (1991).

In 2004, Kong *et al.* discussed the problem of range restricted scattered data interpolation where each domain triangle was split into three mini triangles by using the Clough-Tocher splitting method. Lower bounds of non-negativity were derived to the Bézier ordinates. Similar approach can be found in Schumaker and Speleers (2010) but weaker set of sufficient conditions were introduced in order to preserve the non-negativity of the surface.

Lai and Meile (2015) described a C^1 smooth interpolation of non-negative data over scattered locations by using bivariate splines. Constrained minimal energy method is employed to produce the surface. Classic projected gradient algorithm is then used to find the minimizer subject to a simplified non-negative constraint.

In 2018, Karim *et al.* discussed the positivity preserving interpolation to scattered data using cubic Bézier triangular patches. The piecewise triangular surface is constructed by blending method via convex combination of three local

cubic patches. The positivity of the surface was ensured by imposing positivity conditions onto the cubic Bézier ordinates.

Zhu (2018) described a C^2 positivity preserving interpolation using rational splines with local control parameters. The surface is constructed by blending rational boundaries using Boolean sum of quintic interpolating operators. Sufficient data dependent conditions were also derived on the control parameters for generating positivity preserving interpolant to the 3D positive data arranged over a rectangular grid.

Besides preserving the non-negativity property, the surface produced is required to be visually smooth. In order to have smooth surface, continuity problem is concerned. Parametric continuity C^1 is commonly considered as in the papers mentioned above. However Boschioli *et al.* (2011) gave a comparative study on geometric continuous G^1 interpolatory schemes.

1.2 Motivation

In Chan and Ong (2001) and Piah *et al.* (2005), the interpolating surfaces were constructed using blending method. Sufficient conditions for the non-negativity prescribe lower bounds on the Bézier ordinates. These lower bounds are assigned to be negative or zero value. In the works of Kong *et al.* (2004) as well as Schumaker and Speleers (2010), similar lower bounds were imposed onto Bézier ordinates. Besides, there are many works in the literature imposed lower bound of zero value on the Bézier ordinates. This motivates us whether any other lower bound exists for the non-negativity preservation. This brings us to the following objectives.

1.3 Objective of Study

In this thesis, we wish to achieve the following objectives:

- (a) to explore new lower bound of non-negativity for a Clough-Tocher triangular patch;
- (b) to determine sufficient conditions for adjoining Clough-Tocher triangular patches to be C^1 non-negative;
- (c) to build a scheme for generating a C^1 non-negative interpolating surface.

The word “non-negative” in this thesis is referred to the resultant value is greater than or equal to zero.

1.4 Methodology

In this thesis, non-negativity preserving C^1 interpolation to scattered data is considered. Clough-Tocher split is used to construct the interpolating surface. The interpolating surface to scattered data is piecewise with cubic Bézier triangular patches. Sufficient non-negativity condition on the Bézier ordinate is concerned and parametric C^1 continuity is prescribed. The Bézier ordinates are determined such that the surface interpolates the given data and the first order partial derivatives estimated at the data sites. They may be modified in order to obtain a non-negative smooth interpolant.

1.5 Structure of Thesis

This thesis consists of six chapters. Chapter 2 gives preliminary discussion on cubic Bézier triangular patch, parametric C^1 continuity for adjoining cubic Bézier patches and the construction of Clough-Tocher split. Chapter 3 describes the sufficient non-negativity conditions for a Clough-Tocher triangular patch to be non-negative. These sufficient conditions prescribe lower bounds to the Bézier ordinates. Comparison with the work of Schumaker and Speleers (2010) is also presented. Several examples are exhibited.

In Chapter 4, C^1 non-negativity preserving for two adjacent Clough-Tocher patches is considered. Additional non-negativity condition will be derived in conjunction with the C^1 continuity condition. Some examples are presented to support the argument.

In Chapter 5, a local scheme for C^1 non-negativity preserving interpolation to scattered data is presented. The domain of the surface is triangulated with vertices at the given data. Each triangle is then divided into three mini triangles which give three cubic Bézier patches. The Bézier ordinates are modified if necessary to obtain non-negative surface. Graphical examples are presented in Chapter 6 to illustrate the interpolation scheme. Conclusion is provided at the end of this chapter.

CHAPTER 2

CLOUGH-TOCHER SPLIT

Bézier triangular patch is a mathematical model that widely used in Computer Aided Design due to its useful properties in shape design. In this chapter, cubic Bézier triangular patch will be discussed. Continuity between two adjoining Bézier patches will also be discussed. It followed by the Clough-Tocher split and its constructions.

2.1 Cubic Bézier Triangular Patch

Consider a triangle T with vertices $V_i = (x_i, y_i)$, for $i = 1, 2, 3$, and the barycentric coordinates (u, v, w) such that any point $V = (x, y)$ on T can be written as

$$V = uV_1 + vV_2 + wV_3 ,$$

with $u + v + w = 1$ and $u, v, w \geq 0$. A cubic Bézier triangular patch R on T is defined as (Farin, 1996)

$$R(u, v, w) = \sum_{\substack{i+j+k=3 \\ i,j,k \geq 0}} d_{i,j,k} B_{i,j,k}^3(u, v, w) \quad (2.1)$$

where $d_{i,j,k}$ denote the Bézier ordinates of R and $B_{i,j,k}^3(u, v, w)$ are the Bernstein polynomials of degree 3 defined by

$$B_{i,j,k}^3(u, v, w) = \frac{3!}{i!j!k!} u^i v^j w^k$$

with integers $i, j, k \geq 0$ and $i + j + k = 3$. The Bézier ordinates $d_{i,j,k}$ can be presented with the associated Bézier points in Cartesian space by

$$\left(\frac{i}{3}x_1 + \frac{j}{3}x_2 + \frac{k}{3}x_3, \frac{i}{3}y_1 + \frac{j}{3}y_2 + \frac{k}{3}y_3, d_{i,j,k} \right).$$

The distribution of the points is shown in Figure 2.1. A triangular control net is obtained when the Bézier points are joined orderly with linear segments, see Figure 2.2.

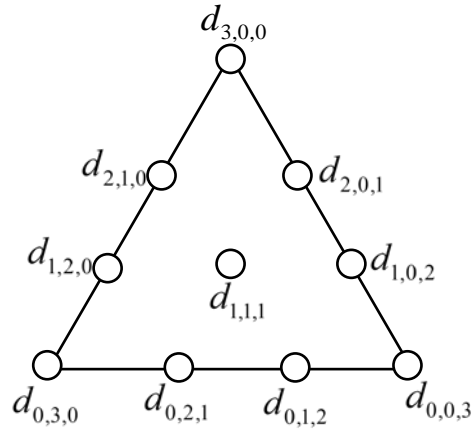


Figure 2.1 Bézier points of a cubic Bézier triangular patch.

We should note that the cubic Bézier triangular patch $R(u, v, w)$ defined in (2.1) is a bivariate function with the coefficients $d_{i,j,k} \in \mathbb{R}$, $i + j + k = 3$ and $i, j, k \geq 0$. Hence the patch can also be indicated as $R(x, y)$. The coefficients $d_{i,j,k}$ are used to control the shape of the patch. Some important properties of the cubic Bézier triangular patch will be described. They are listed as follows.

a) Endpoint interpolation

The cubic Bézier triangular patch $R(u, v, w)$ interpolates the Bézier ordinates at the three vertices of T , that is

$$\begin{aligned} R(V_1) &= d_{3,0,0}, \\ R(V_2) &= d_{0,3,0}, \\ R(V_3) &= d_{0,0,3}. \end{aligned} \tag{2.2}$$

b) Cubic Bézier boundary curve

The boundary curves of the patch R are absolutely determined by the boundary Bézier ordinates $d_{i,j,k}$, where at least one of the i , j , or k is zero. For instance, the boundary curve along the edge V_2V_3 (i.e. $u = 0$) is

$$\begin{aligned} R(0, v, w) &= \sum_{\substack{i+j+k=3 \\ i,j,k \geq 0}} d_{i,j,k} B_{i,j,k}^3(0, v, w) \\ &= \sum_{j=0}^3 d_{0,j,3-j} \frac{3!}{j!(3-j)!} v^j (1-v)^{3-j}, \quad 0 \leq v \leq 1. \end{aligned}$$

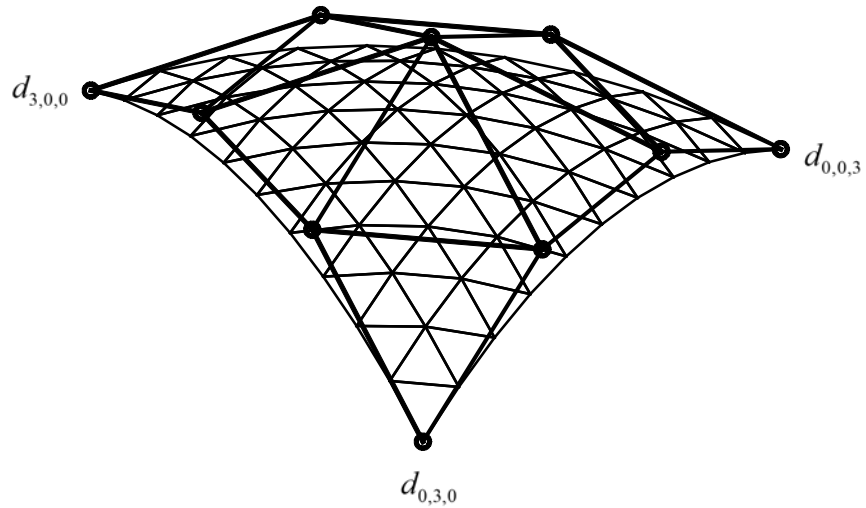


Figure 2.2 Cubic Bézier patch and its control net.

c) Convex hull property

The Bézier triangle lies completely within the convex hull of its Bézier net. This is because of (2.1) is a convex combination with respect to the Bézier points where the Bernstein functions have the properties

$$B_{i,j,k}^3(u, v, w) \geq 0 \quad \text{and} \quad \sum_{\substack{i+j+k=3 \\ i,j,k \geq 0}} B_{i,j,k}^3(u, v, w) = 1.$$

Moreover, if all the Bézier ordinates are positive, then the Bézier patch R is positive.

2.2 C^1 Continuity for Cubic Bézier Triangular Patches

Consider two cubic Bézier triangular patches that defined on two adjacent domain triangles $V_1V_2V_3$ and $V_2V_3V_4$ respectively. Let $d_{i,j,k}$ and $\hat{d}_{i,j,k}$ denote the corresponding Bézier ordinates as shown in the Figure 2.3. Suppose the cubic Bézier patches are defined as

$$R(u, v, w) = \sum_{\substack{i+j+k=3 \\ i,j,k \geq 0}} d_{i,j,k} B_{i,j,k}^3(u, v, w)$$

and

$$\hat{R}(r, s, t) = \sum_{\substack{i+j+k=3 \\ i,j,k \geq 0}} \hat{d}_{i,j,k} B_{i,j,k}^3(r, s, t)$$

where (u, v, w) and (r, s, t) are the barycentric coordinates with respect to each triangle $V_1V_2V_3$ and $V_2V_3V_4$. Along the common edge V_2V_3 , we have $u = 0$ and $r = 0$. Assuming $v = s$ and $w = t$,

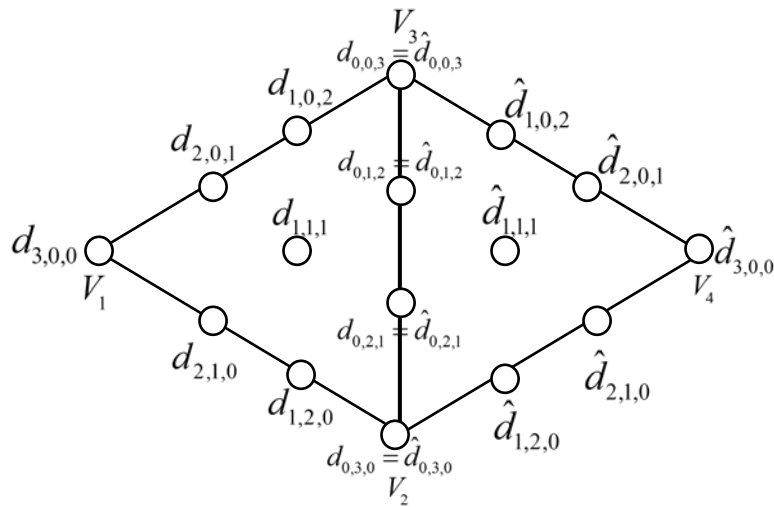


Figure 2.3 A pair of cubic Bézier triangular patches.

The cubic Bézier patches R and \hat{R} are joined with C^0 parametric continuity across the common edge V_2V_3 if and only if

$$R(0, v, w) = \hat{R}(0, s, t).$$

To satisfy this condition, we required that all the Bézier ordinates along the common edge are set to be

$$d_{0,j,k} = \hat{d}_{0,j,k}, \quad \text{for } j, k \geq 0 \text{ and } j+k=3. \quad (2.3)$$

Furthermore, to achieve C^1 tangential continuity along the common edge V_2V_3 between the two patches, the partial derivatives along the edge V_2V_3 must be

$$\frac{\partial R}{\partial x}(V_2V_3) = \frac{\partial \hat{R}}{\partial x}(V_2V_3),$$

$$\frac{\partial R}{\partial y}(V_2V_3) = \frac{\partial \hat{R}}{\partial y}(V_2V_3).$$

The above necessary and sufficient conditions lead to (Farin,1996)

$$\hat{d}_{1,j,k} = \beta_1 d_{1,j,k} + \beta_2 d_{0,j+1,k} + \beta_3 d_{0,j,k+1}, \quad \text{for } j, k \geq 0, j+k=2, \quad (2.4)$$

where $\beta_1 + \beta_2 + \beta_3 = 1$ and $V_4 = \beta_1 V_1 + \beta_2 V_2 + \beta_3 V_3$.

However, an ordinary cubic Bézier triangle may not be sufficient to handle C^1 continuity across all three edges of its domain triangle due to the limited degree of freedoms. A method of surface construction is required such that the surface generated fulfills the C^1 smoothness conditions along all the three boundaries. The method named Clough-Tocher split interpolation (Clough & Tocher, 1965) is used to solve the smoothness problem along the patch boundaries. The detail of construction will be discussed in next section.

2.3 Clough-Tocher Split

Consider a triangle T with vertices V_1 , V_2 and V_3 , which is called macro-triangle. The macro-triangle T is split into 3 mini triangles which are referred to be micro-triangles. Let the splitting point be denoted by $V_C = (x_C, y_C)$. In this study the incenter is taken to be the splitting point for the macro-triangle T . The incenter of triangle T is determined by taking the intersection of the angle bisectors of three vertices of the triangle, see Figure 2.5. It is formulated by

$$V_C = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right) \quad (2.5)$$

where a , b and c are the length of the edges V_2V_3 , V_3V_1 and V_1V_2 respectively.

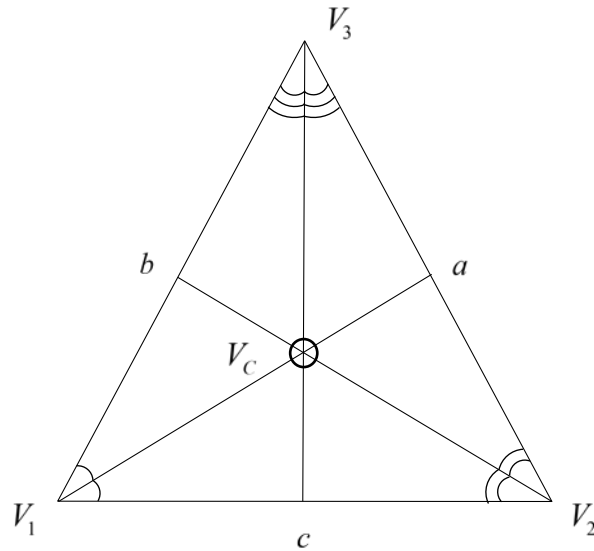


Figure 2.4 Incenter of a triangle.

Beside the incenter, a common choice named barycenter can also be used as alternate splitting point where it is defined as the point of intersection of the lines joining from each vertex to the median of the opposite edge. The splitting point as barycenter is defined by (Kong et al., 2004)

$$V_C = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right).$$

With the splitting point V_C , the macro-triangle T is subdivided into three micro-triangles $V_C V_1 V_2$, $V_C V_2 V_3$ and $V_C V_3 V_1$. On each micro-triangle a triangular patch is generated such that the patch joins C^1 continuously to the other two triangular patches. Here, in order to produce C^1 smooth surface especially across the interior edges $V_C V_1$, $V_C V_2$, $V_C V_3$, and the exterior edges $V_1 V_2$, $V_2 V_3$, $V_3 V_1$, cubic Bézier triangle in (2.1) is used to fit each micro-triangle. The corresponding Bézier ordinates are shown in Figure 2.5.

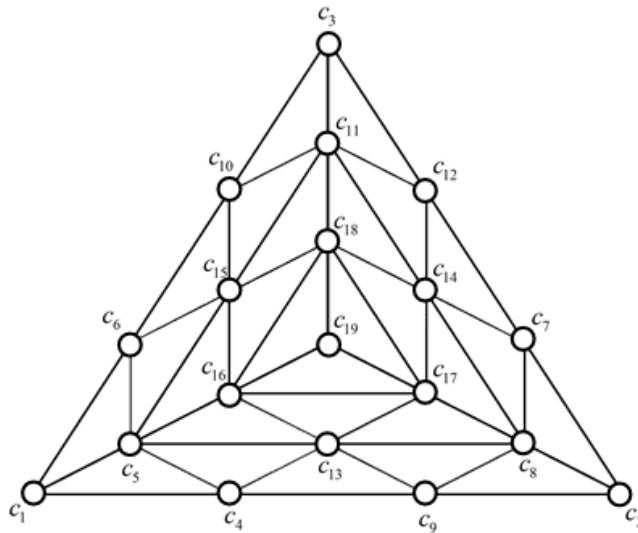


Figure 2.5 Bézier ordinates of Clough-Tocher macro-element.

Let $S(x, y)$ be the patch produced on the macro-triangle T . It consists of three cubic Bézier patches which satisfy the C^1 continuity conditions described in Section 2.2, that is the analogs of (2.3) and (2.4). Based on the endpoint property

(2.2) of Bézier patch the values of c_1, c_2, c_3 are the surface values S at the vertices of T , that is

$$\begin{aligned} c_1 &= S(V_1), \\ c_2 &= S(V_2), \\ c_3 &= S(V_3). \end{aligned} \tag{2.6}$$

The values of the rest of Bézier ordinates c_4, c_5, \dots, c_{12} are obtained by using the partial derivatives defined on the three vertices. Suppose the partial derivatives at each vertex $V_i, i=1, 2, 3$, are denoted as $S_x(V_i)$ and $S_y(V_i)$ along the x and y directions respectively. The related Bézier ordinates are calculated by

$$\begin{aligned} c_4 &= \frac{(x_2 - x_1)S_x(V_1) + (y_2 - y_1)S_y(V_1)}{3} + S(V_1), \\ c_5 &= \frac{(x_C - x_1)S_x(V_1) + (y_C - y_1)S_y(V_1)}{3} + S(V_1), \\ c_6 &= \frac{(x_3 - x_1)S_x(V_1) + (y_3 - y_1)S_y(V_1)}{3} + S(V_1), \\ c_7 &= \frac{(x_3 - x_1)S_x(V_2) + (y_3 - y_1)S_y(V_2)}{3} + S(V_2), \\ c_8 &= \frac{(x_C - x_2)S_x(V_2) + (y_C - y_2)S_y(V_2)}{3} + S(V_2), \\ c_9 &= \frac{(x_1 - x_2)S_x(V_2) + (y_1 - y_2)S_y(V_2)}{3} + S(V_2), \\ c_{10} &= \frac{(x_1 - x_3)S_x(V_3) + (y_1 - y_3)S_y(V_3)}{3} + S(V_3), \end{aligned}$$

$$c_{11} = \frac{(x_C - x_3)S_x(V_3) + (y_C - y_3)S_y(V_3)}{3} + S(V_3) ,$$

$$c_{12} = \frac{(x_2 - x_3)S_x(V_3) + (y_2 - y_3)S_y(V_3)}{3} + S(V_3) . \quad (2.7)$$

The inner Bézier ordinates c_{13} , c_{14} , c_{15} are determined in such a way that the surface S is C^1 continuous along the exterior edges V_1V_2 , V_2V_3 , V_3V_1 of T . In this study, we estimate these ordinates using the method in Goodman and Said (1994), in which the normal derivative of the patch is required to vary linearly along the boundary. We denote by $e_{i,j}$ the side vector from V_i to V_j , in Cartesian system, refer to Figure 2.6.

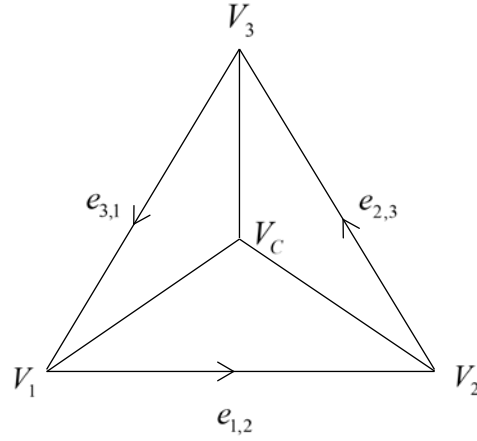


Figure 2.6 Notation of a macro-triangle.

The inner ordinates are computed by (Goodman & Said, 1994)

$$c_{13} = \frac{c_8 + c_5 - c_1 - c_9 + 2c_4 + h_1(c_1 + 3c_9 - 3c_4 - c_2)}{2} ,$$

$$c_{14} = \frac{c_8 + c_{11} - c_2 - c_{12} + 2c_7 + h_2(c_2 + 3c_{12} - 3c_7 - c_3)}{2} ,$$

$$c_{15} = \frac{c_{11} + c_5 - c_3 - c_6 + 2c_{10} + h_3(c_3 + 3c_6 - 3c_{10} - c_1)}{2},$$

(2.8)

where

$$h_1 = -\frac{e_{1,2} \bullet e_{C,1}}{e_{1,2} \bullet e_{1,2}}, \quad h_2 = -\frac{e_{2,3} \bullet e_{C,2}}{e_{2,3} \bullet e_{2,3}}, \quad h_3 = -\frac{e_{3,1} \bullet e_{C,3}}{e_{3,1} \bullet e_{3,1}},$$

and “ \bullet ” denotes the dot product of two vectors.

Lastly, the remaining Bézier ordinates are then determined using the C^1 continuity condition analogous to (2.4) by the formulae

$$c_{16} = \alpha_1 c_5 + \alpha_2 c_{13} + \alpha_3 c_{15},$$

$$c_{17} = \alpha_1 c_{13} + \alpha_2 c_8 + \alpha_3 c_{14},$$

$$c_{18} = \alpha_1 c_{15} + \alpha_2 c_{14} + \alpha_3 c_{11},$$

$$c_{19} = \alpha_1 c_{16} + \alpha_2 c_{17} + \alpha_3 c_{18},$$

(2.9)

where α_i , $i = 1, 2, 3$, satisfy $V_C = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Note that the surface S interpolates the Bézier ordinate c_{19} at the splitting point V_C , i.e. $S(V_C) = c_{19}$. In the next chapter, sufficient nonnegativity conditions will be imposed onto the Bézier ordinates c_i , $i = 1, 2, \dots, 19$, such that the surface generated preserves the nonnegativity feature of the given data.

CHAPTER 3

NON-NEGATIVITY CONDITION ON A CLOUGH-TOCHER TRIANGULAR PATCH

This chapter derives the condition for a Clough-Tocher triangular patch to be non-negative. A non-negative patch means any point on the patch is greater or equals to zero, that is the patch lies above xy -plane. A lower bound is used to restrict the Bézier ordinates such that the patch preserves non-negativity. This lower bound is obtained based on the initial work of Schumaker et al. (2010), in which is used as a guideline on deriving a more general and relaxed condition. Simple examples will be illustrated in the end of this chapter.

3.1 Sufficient Nonnegativity Condition for a Clough-Tocher Triangular Patch

In 2001, Chan and Ong derived sufficient conditions for a cubic Bézier triangular patch to be non-negative where a lower bound was imposed on the Bézier ordinates of the Bézier patch. We are interested to construct a lower bound for a Clough-Tocher macro-triangle defined in Section 2.3 to be non-negative. Firstly, we denote $-l$ as lower bound where $l \geq 0$. Given that $c_1, c_2, c_3 \geq 0$ where c_1, c_2, c_3 are associated to the vertices V_1, V_2, V_3 respectively as a triangular patch to be constructed must be non-negative at the vertices of triangle T . Motivated by the work in Chan and Ong (2001), consider a cubic Bézier polynomial curve

$$P(x) = A(1-x)^3 + 3B(1-x)^2x + 3C(1-x)x^2 + Dx^3,$$

where A, B, C, D indicate the Bézier ordinates, $0 \leq x \leq 1$ and $A, D > 0$

Setting $B = C = -l$

$$\begin{aligned} P(x) &= A(1-x)^3 - 3l(1-x)^2 x - 3l(1-x)x^2 + Dx^3 \\ &= A(1-3x+3x^2-x^3) - 3l(1-2x+x^2)x - 3l(1-x)x^2 + Dx^3 \\ &= (D-A)x^3 + 3(A+l)x^2 - 3(A+l)x + A. \end{aligned}$$

The idea from Chan and Ong (2001), shows that the minimum of $P(x)$ occurs at the real roots of $(D-A)x^2 + 2(A+l)x - (A+l) = 0$ where $x \in [0,1]$.

If $D = A$,

$$P(x) = 3(A+l)x^2 - 3(A+l)x + A.$$

The derivatives

$$P'(x) = 6(A+l)x - 3(A+l),$$

$$P''(x) = 6(A+l).$$

then $P'(x) = 0$ gives the extremum point as $x = \frac{1}{2}$.

$$\text{Note that } P\left(\frac{1}{2}\right) = 3(A+l)\left(\frac{1}{2}\right)^2 - 3(A+l)\left(\frac{1}{2}\right) + A$$

$$= \frac{1}{4}(A-3l)$$

and

$$P''\left(\frac{1}{2}\right) = 6(A+l) > 0.$$

since $A > 0$ and $l \geq 0$.

From here, we conclude that the minimum of $P(x)$ is $\frac{1}{4}(A-3l)$ when $D = A$.

That is, $P(x) \geq \frac{1}{4}(A-3l)$ for $\forall x \in [0,1]$. Moreover, the minimum $P\left(\frac{1}{2}\right) = 0$ if $l = \frac{A}{3}$. Obviously the curve $P(x) \geq 0$ if l is chosen to be $l \leq \frac{A}{3}$. For the case

$D \neq A$, we obtained the general result

$$P(x) \geq \frac{1}{4}(\min\{A, D\} - 3l) \geq 0, \quad \forall x \in [0,1] \quad \text{where } l \geq \frac{\min\{A, D\}}{3}.$$

Based on the above observation on cubic Bézier curve, we can apply this to the three sides of the triangle T , such that the corresponding boundary curves are non-negative,

$$\text{let } l = \frac{\min\{c_1, c_2, c_3\}}{3}. \quad (3.1)$$

By setting the boundary ordinates in Figure (2.5) to be $c_4, c_9, c_6, c_{10}, c_7, c_{12} \geq -l$, the three boundaries of the Clough-Tocher macro-triangle will be non-negative.

Next, we wish to derive the lower bound for the rest of Bézier ordinates, starting with c_5, c_8, c_{11} as followed. Inspired from the method used in Schumaker *et al.* (2010), we require that $c_5, c_8, c_{11} \geq 0$. From the C^1 continuity condition of a Clough-Tocher triangular patch, we have

$$c_5 = \alpha_1 c_1 + \alpha_2 c_4 + \alpha_3 c_6,$$

$$c_8 = \alpha_1 c_9 + \alpha_2 c_2 + \alpha_3 c_7,$$

$$c_{11} = \alpha_1 c_{10} + \alpha_2 c_{12} + \alpha_3 c_3.$$

where $0 < \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

We wish to derive a suitable lower bound l such that $c_5, c_8, c_{11} \geq 0$.

Since $c_4, c_6 \geq -l$,

$$\begin{aligned} c_5 &= \alpha_1 c_1 + \alpha_2 c_4 + \alpha_3 c_6 \\ &\geq \alpha_1 c_1 - (\alpha_2 + \alpha_3)l \\ &= \alpha_1 c_1 - (1 - \alpha_1)l. \end{aligned} \tag{3.2}$$

In order to obtain non-negative c_5 , let

$$\alpha_1 c_1 - (1 - \alpha_1)l \geq 0$$

thus
$$l \leq \frac{\alpha_1 c_1}{1 - \alpha_1}.$$

Similarly, by $c_7, c_9, c_{10}, c_{12} \geq -l$, we obtain

$$\begin{aligned} c_8 &= \alpha_1 c_9 + \alpha_2 c_2 + \alpha_3 c_7 \\ &\geq \alpha_2 c_2 - (\alpha_1 + \alpha_3)l \\ &= \alpha_2 c_2 - (1 - \alpha_2)l. \end{aligned} \tag{3.3}$$

and

$$c_{11} = \alpha_1 c_{10} + \alpha_2 c_{12} + \alpha_3 c_3$$

$$\begin{aligned}
&\geq \alpha_3 c_3 - (\alpha_1 + \alpha_2) l \\
&= \alpha_3 c_3 - (1 - \alpha_3) l.
\end{aligned} \tag{3.4}$$

These give $l \leq \frac{\alpha_2 c_2}{1 - \alpha_2}$ and $l \leq \frac{\alpha_3 c_3}{1 - \alpha_3}$ that ensure $c_8, c_{11} \geq 0$ respectively.

After having these three additional conditions on l , the new bound for all boundary ordinates will be determined by

$$l = \min \left\{ \frac{\alpha_1 c_1}{1 - \alpha_1}, \frac{\alpha_2 c_2}{1 - \alpha_2}, \frac{\alpha_3 c_3}{1 - \alpha_3}, \frac{\min \{c_1, c_2, c_3\}}{3} \right\}. \tag{3.5}$$

It can be easily proven that this bound l ensures c_5, c_8 and c_{11} are non-negative (see Appendix).

Now, we are ready to determine the lower bound for c_{13}, c_{14} and c_{15} .

Let

$$c_{13} \geq -a_1 l \tag{3.6}$$

$$c_{14} \geq -a_2 l \tag{3.7}$$

$$c_{15} \geq -a_3 l. \tag{3.8}$$

where $a_1, a_2, a_3 \geq 0$.

To solve for a_1, a_2, a_3 , we can use the conditions $c_{16}, c_{17}, c_{18} \geq 0$ that implemented in the work of Schumaker *et al.* (2010).

From (2.9) and (3.6) - (3.8),

$$c_{16} = \alpha_1 c_5 + \alpha_2 c_{13} + \alpha_3 c_{15}$$

$$\begin{aligned}
&\geq \alpha_1 c_5 - \alpha_2 a_1 l - \alpha_3 a_3 l \\
&= \alpha_1 c_5 - l(\alpha_2 a_1 + \alpha_3 a_3),
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
c_{17} &= \alpha_1 c_{13} + \alpha_2 c_8 + \alpha_3 c_{14} \\
&\geq -\alpha_1 a_1 l + \alpha_2 c_8 - \alpha_3 a_2 l \\
&= \alpha_2 c_8 - l(\alpha_1 a_1 + \alpha_3 a_2),
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
c_{18} &= \alpha_1 c_{15} + \alpha_2 c_{14} + \alpha_3 c_{11} \\
&\geq -\alpha_1 a_3 l - \alpha_2 a_2 l + \alpha_3 c_{11} \\
&= \alpha_3 c_{11} - l(\alpha_1 a_3 + \alpha_2 a_2).
\end{aligned} \tag{3.11}$$

Since Inequalities (3.9) - (3.11) involves Bézier ordinates c_5, c_8 and c_{11} , we should consider their minimum values $c_5 = 0$, $c_8 = 0$ and $c_{11} = 0$ as shown below.

Case 1

Suppose $c_5 = 0$, we have (3.9) as

$$c_{16} \geq -l(\alpha_1 a_1 + \alpha_3 a_3)$$

For $c_{16} \geq 0$ to be true, $l = 0$ or $\alpha_1 a_1 + \alpha_3 a_3 = 0$.

Let $l > 0$ and $a_1 = a_3 = 0$. For c_{17} and c_{18} to be non-negative too, we substitute condition of a_1 and a_3 into (3.10) and (3.11), hence

$$\alpha_2 c_8 - l(\alpha_1 a_1 + \alpha_3 a_2) \geq 0$$

$$\alpha_2 c_8 - l(\alpha_3 a_2) \geq 0$$

$$a_2 \leq \frac{\alpha_2 c_8}{\alpha_3 l},$$

and

$$\alpha_3 c_{11} - l(\alpha_1 a_3 + \alpha_2 a_2) \geq 0$$

$$\alpha_3 c_{11} - l(\alpha_2 a_2) \geq 0$$

$$a_2 \leq \frac{\alpha_3 c_{11}}{\alpha_2 l}.$$

We notice that a_2 must satisfy both inequalities above, therefore,

$$a_2 \leq \min \left\{ \frac{\alpha_2 c_8}{\alpha_3 l}, \frac{\alpha_3 c_{11}}{\alpha_2 l} \right\}. \text{ Thus when } c_5 = 0, \text{ the B\u00e9zier ordinates } c_{16}, c_{17}, c_{18} \geq 0 \text{ if}$$

either one below holds

$$(i). \quad l = 0$$

$$(ii). \quad l > 0, \quad a_1 = 0, \quad a_2 \leq \min \left\{ \frac{\alpha_2 c_8}{\alpha_3 l}, \frac{\alpha_3 c_{11}}{\alpha_2 l} \right\}, \quad a_3 = 0. \quad (3.12)$$

Case 2

Next, we let $c_8 = 0$. The Inequality (3.10) gives

$$c_{17} \geq -l(\alpha_1 a_1 + \alpha_3 a_2).$$

For $c_{17} \geq 0$ to be true, $l = 0$ or $\alpha_1 a_1 + \alpha_3 a_2 = 0$.

Consider $l > 0$ and $a_1 = a_2 = 0$. In order to obtain $c_{16}, c_{18} \geq 0$, Inequalities (3.9)

and (3.11) lead to

$$\alpha_1 c_5 - l(\alpha_2 a_1 + \alpha_3 a_3) \geq 0$$

$$\alpha_1 c_5 - l(\alpha_3 a_3) \geq 0$$

$$a_3 \leq \frac{\alpha_1 c_5}{\alpha_3 l},$$

and

$$\alpha_3 c_{11} - l(\alpha_1 a_3 + \alpha_2 a_2) \geq 0$$

$$\alpha_3 c_{11} - l(\alpha_1 a_3) \geq 0$$

$$a_3 \leq \frac{\alpha_3 c_{11}}{\alpha_1 l}.$$

Therefore, when $c_8 = 0$ to ensure $c_{16}, c_{17}, c_{18} \geq 0$, either one must be true

(i). $l = 0$

(ii). $a_1 = 0, a_2 = 0, a_3 \leq \min \left\{ \frac{\alpha_1 c_5}{\alpha_3 l}, \frac{\alpha_3 c_{11}}{\alpha_1 l} \right\}$ where $l > 0$. (3.13)

Case 3

Let $c_{11} = 0$.

Substitute into (3.11), we have

$$c_{18} \geq -l(\alpha_1 a_3 + \alpha_2 a_2).$$

For $c_{18} \geq 0$ to be true, $l = 0$ or $\alpha_1 a_3 + \alpha_2 a_2 = 0$ holds.

Let $l > 0$ and $a_2 = a_3 = 0$. Then for $c_{16}, c_{17} \geq 0$,

(3.9) and (3.10) give