SHAPE PRESERVING INTERPOLATION USING RATIONAL CUBIC BALL TRIANGULAR PATCHES

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by

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LIST OF ABBREVIATIONS

2D   Two Dimensional

3D   Three Dimensional

ADC   Analog to Digital Converter

CAD   Computer Aided Design

CAGD Computer Aided Graphic Design

DAC   Digital to Analog Converter

DP   Delgado-Peña

ESR   Erythrocyte Sedimentation Rate

GPRC General Piecewise Rational Cubic

RBF   Radial Basis Function

LIST OF SYMBOLS

C^0 Zero order parametric continuity

C^1 First order parametric continuity

C^2 Second order parametric continuity

G^1 First order geometric continuity

O_2 Oxygen

INTERPOLASI PENGEKALAN BENTUK MENGGUNAKAN TAMPALAN

SEGI TIGA BALL KUBIK NISBAH

ABSTRAK

Interpolasi pengekalan bentuk merupakan bidang yang penting untuk persembahan grafik data bertaburan yang mana paling dikehendaki dalam komputer grafik, pembuatan berbantukan komputer, rekabentuk geometri berbantukan komputer, pemodelan geometri, geologi, meteorologi, dan juga dalam proses fizikal dan kimia. Dalam banyak masalah interpolasi, ciri-ciri bentuk data permukaan yang lazimnya dipertimbangkan adalah kepositifan, keekanadaan dan kecembungan. Oleh itu, tumpuan tesis ini adalah pada paparan grafik permukaan segi tiga data bertaburan yang mempunyai ciri-ciri bentuk positif, ekanada dan cembung, masing-masing. Skim pemeliharaan bentuk akan dipaparkan untuk tampalan segi tiga dengan menggunakan fungsi Ball kubik nisbah dengan parameter-parameter bentuk (fungsi pemberat-pemberat). Ia akan menunjukkan bahawa skim yang dicadangkan ini menghasilkan visual yang menyenangkan apabila parameter yang sesuai dipilih. Terlebih dahulu, setiap set data dalam rantau dua dimensi (x, y) dibahagikan kepada elemen-elemen segi tiga dengan menggunakan kaedah penyegitigaan Delaunay. Permukaan interpolasi data bertaburan adalah gabungan cembung daripada tiga tampalan segi tiga Ball kubik nisbah dengan set sama bagi ordinat sempadan. Syarat-syarat untuk mendapatkan pemeliharaan permukaan kepositifan, keekanadaan dan kecembungan, masing-masing diperoleh pada ordinat Ball dengan parameter bebas untuk mengekalkan ciri-ciri bentuk yang diwariskan daripada data asas. Akhir sekali, hubungan antara asas nisbah Bézier dan asas nisbah Ball akan ditunjukkan dengan

menggunakan formula penukaran. Formula penukaran ditakrifkan dengan menggunakan matriks pekali yang menyediakan transformasi antara kedua-dua jenis fungsi asas bivariat yang mewakili polinomial bivariat yang berdarjah sama dan sebaliknya. Dalam tesis ini, algoritma untuk membina permukaan dengan menukarkan set titik kawalan dan pemberat daripada fungsi nisbah Bézier kepada fungsi nisbah Ball pada permukaan yang sama dengan menggunakan data positif telah dicadangkan. Algoritma ini boleh digunakan sebagai alternatif lain dalam membina permukaan interpolasi selain menggunakan fungsi biasa seperti fungsi segi tiga Bézier nisbah. Beberapa contoh akan dibentangkan secara grafik dalam tesis ini dengan menggunakan fungsi ujian yang diketahui.

SHAPE PRESERVING INTERPOLATION USING RATIONAL CUBIC

BALL TRIANGULAR PATCHES

ABSTRACT

Shape preserving interpolation is an important area for graphical presentation of scattered data where it is most desired in computer graphics, computer aided manufacturing, computer aided geometric design, geometric modeling, geology, meteorology, as well as in physical and chemical process. In many interpolation problems, shape characteristics of the surface data commonly considered are positivity, monotonicity and convexity. Thus, the focus of this thesis is on the graphical displays of triangular surfaces of scattered data which possess positive, monotone and convex shape features, respectively. Shape preserving schemes will be displayed for triangular patches using rational cubic Ball function with free shape parameters (weights function). It will be shown that the proposed scheme is visually pleasing when appropriate parameters are chosen. Firstly, for each data set in two dimensional (2D) region (x, y) is divided into triangular elements using Delaunay triangulation method. The interpolating surface of scattered data is a convex combination of three rational cubic Ball triangular patches with the same set of boundary Ball ordinates. Conditions to obtain positivity, monotonicity and convexity preserving surfaces, respectively, are derived on the Ball ordinates with free parameters in order to preserve the inherited shape characteristics of the underlying data. Finally, a relationship between rational Bézier and rational Ball bases will be shown using conversion formulae. The conversion formulae are defined using coefficient matrices which provide a transformation between the two types of bivariate basis functions representing

bivariate polynomials of the same degree and vice versa. In this thesis, an algorithm to construct a surface by converting a set of control points and weights from rational Bézier function into rational Ball function on the same surface using positive data is proposed. This algorithm can be utilised as another alternative in construction of surface interpolation besides using ordinary function likes rational Bézier triangular function. A number of examples will be presented graphically in this thesis using well-known test functions.

CHAPTER 1

INTRODUCTION

Computer Aided Graphic Design (CAGD) is a field that was initially developed to bring the advantage of computers to manufacturing industries of products such as automotive, aerospace and shipbuilding. CAGD has now expanded rapidly and is a branch of applied mathematics that is applied in scientific data visualization, computational geometry, graphics, numerical analysis and vision. In recent years, a few researchers have worked in the area of scientific data visualization, especially in shape preserving interpolation, both for curves and surfaces. According to Friendly [1], scientific visualization concerns the visualization of 3D phenomena such as in the architectural, meteorological, medical and biological fields, where the emphasis is on realistic renderings of volumes, surfaces, illumination sources, and so forth, perhaps with a dynamic (time) component. This situation is considered especially when data arises from some complex mathematical functions or scientific data to visualize graphical insights. This graphical insight provides limited or incomplete information to users to understand the various physical phenomena and data understanding.

1.1   Shape Preserving Interpolation

Shape preservation and interpolation are a fundamental process in scientific visualization for graphical presentation of data. The known data represent only a sample and may not be sufficient to let one visualize the entire entity accurately. Therefore, a sufficiently smooth univariate or bivariate function that interpolates or approximates these data preserving the same characteristic features should be constructed. There are some special characteristic features in the data that are most often used in shape preserving interpolation such as positivity, monotonicity and convexity.

Positivity is one of the shape properties that are most discussed by researchers in this field. There are many physical situation applications where entities only have meaning when the visualization of physical quantity is positive. Meteorological measurement at different weather stations is an important branch of science that concerns the phenomena of the atmosphere, especially as a means of forecasting the weather. The rainfall data are reconstructed graphically [2], where the positivity of the values should be preserved, otherwise negative values are not physically meaningful. Another application of positivity preservation in a real-life problem is measurement of oxygen (O_2) concentration in the Gulf of Mexico. The oxygen concentration is measured after the incident of oil spill and allows one to determine oxygen deficits in the water column resulting in O_2 anomaly that are 0 (no O_2 missing) or positive (indicating O_2 that has been consumed). However, in the absence of the oxygen generating process in the deep waters, the value is never negative [3]. Similarly, when dealing with the levels of gas released in certain chemical processes, concentration of sugar in blood and population

density [4]. All these physical situations are meaningful when they have positive quantities only.

The second shape characteristic that commonly arises in various application areas is the monotonicity, either increasing or decreasing, where the entities only have a meaning when their values are monotone. For example, monotonic function are very commonly used in pricing models in economic and financial application involving demand, supply and pricing [5]. Monotonicity is also involved in medical diagnosis like the level of blood uric acid in patients having gout problem or Erythrocyte Sedimentation Rate (ESR) in cancer patients and rate of dissemination of drug in blood [6]. In engineering fields, monotonicity is one of the key specifications that help engineers to ensure that the production of Digital to Analog Converter (DAC) or Analog to Digital Converter (ADC) meet the characteristics of monotonicity. These devices are considered monotone; when the input to the device increased (decreased), the output must also increase (decrease) accordingly. In general, a non-monotonic DAC is unacceptable [7]. Other examples in other fields include dose-response curves and surfaces in biochemistry and pharmacology, approximation of copulas and quasi-copulas in statistics and design of aggregation operators in multiple criteria decision making and fuzzy logic [6].

Another shape property that is most applied in CAGD is convexity. The clear example could be seen in the manufacturing industry of television screens. The requirement of convexity preservation is very important to construct well-shaped surfaces of the mask that are located in a cathode tube behind the television screen. Other examples are related to car modelling in the automobile industry, aeroplane and ship designs. In the

process of modelling a car, the aesthetic requirements also play an important role in this process apart from considering the technical and physical conditions. For example, an unwanted feature on the car model is that its surface contains wiggles and produces poorly-shaped images. Therefore, it is very important that this visual situation put certain constraints on the shape to produce smooth visual shapes without swing. Other than that, examples of convexity can be viewed in scientific applications such as optimal control, parameter estimation and approximation of function [8].

There are some families of functions commonly used for interpolation such as polynomials, rational functions, trigonometric functions and exponential functions. The simplest type of interpolation is to use polynomial and it is a well-known fact that any univariate data set can be interpolated by any degree polynomial. In general, the resulting interpolation of curves or surfaces will have more oscillations when the number of data points becomes larger even though that polynomial interpolation generates function of high smoothness. These wiggles are undesirable in many applications because it completely deviates the data from its natural features and need to be eliminated [9]. The curve or surface produced that contains no extraneous wiggles, will make it more readily acceptable to scientists and engineers. Thus, spline is an appropriate method for many problems regarding interpolation defined as piecewise polynomials that are connected in a smooth way. The most commonly used interpolation method is Bézier spline, which in practice, appears to be sufficiently smooth. The spline interpolation method reduces the occurrence of large oscillations due to their relatively low degree if compared to polynomial interpolation. However, the splines cannot completely avoid the unwanted wiggles in certain practical problems. Therefore, it is important to suggest simpler functions in the interpolation

method to enforce shape preservation. In the last few years, there have been important advances in the study of shape preserving interpolation (positivity, monotonicity, convexity) where the representations of curve designs have considered the use of univariate spline [4, 7, 9 – 16] for regular data. Some work [4, 8, 17 – 20] on shape preservation extended their results of univariate interpolation to bivariate interpolation for data arranged on rectangular mesh. Most of them have solved the interpolation problems of regular data; and, only some authors have considered the interpolation of scattered data over a triangular grid. Therefore, bivariate spline using triangular patch is another method that is commonly used to construct the interpolation of scattered data [2, 3, 21 – 38]. This method will be applied to construct interpolation of scattered data and one of the main objectives in this thesis.

The earlier study in a piecewise Bézier-Bernstein polynomial interpolation over a triangle was introduced by Farin [39] in 1986, which is most popular and very commonly used in interpolation surface of scattered data problems. Each triangular patch of the interpolating surface is formed as a convex combination of three cubic Bézier triangular patches with parametric or geometric continuity. Besides using Bézier triangular surface, some other examples of triangular surface models are Said-Ball [40] and Wang-Ball [41]. Early history associated with the cubic Ball curve is introduced by Ball [42] for CONSURF system by British Aircraft Corporation at Warton. From the work of Hu et al. [41], it is known that there are two types of higher degree generalized Ball basis functions and corresponding curves have been derived by Wang [43] and Said [44] independently. Later, Hu et al. [45] provided another generalization and recursive algorithm for the resulting curve, which has been shown to be more efficient than Said [44]. Subsequently, this effort has been continued to

achieve a bivariate basis for both types of Ball bases on triangle where the basis functions are reduced to the univariate basis functions on each side of the triangle. Goodman and Said [40] extended the bivariate generalized Ball surfaces into odd degrees only which is defined as Said-Ball triangular surface. They proved that the recursive algorithm for evaluation and degree raising in [40] is more efficient compared to the de Casteljau algorithm for evaluating a polynomial expressed in the more usual Bézier form. Later, Hu et al. [41] extended the univariate Wang-Ball basis [43] to the bivariate case on a triangle, which is defined as a triangular Wang-Ball surface. They claimed that Wang-Ball curves are much better than Bézier curves and Said-Ball curves in terms of evaluation, degree elevation and degree reduction. Therefore, automatically the recursive algorithm for triangular Wang-Ball surfaces is better than both recursive algorithms for Bézier triangular surfaces and Said-Ball triangular surfaces. Even so, generalized Ball representations by Wang [43] and Said [44] possess the same shape preserving properties as the Bézier representation. Therefore, designers in graphic design should consider the generalized Ball function instead of the Bézier function in the construction of curves and surfaces. The definition of Bézier, Said-Ball and Wang-Ball triangular functions will be discussed in the next chapter.

Interpolation techniques are fundamental in shape preserving curve and surface visualizations, and can be categorized into global methods and local methods [46]. The global methods create the curves or surfaces using all the data points at once. If the shape of the curves or surfaces needs to be modified by adding or deleting the particular data points, the problem must be solved again and the entire surface changes. Global methods are practically limited to small data sets because it is based on a

minimization problem and involve large-scale computations. On the other hand, in the local methods, the interpolation is performed locally. If any one of the data points changes, only a part of the interpolated surface must be re-computed and only influenced by the values from points nearby in the scattered point set. Local methods can treat much larger data sets because they do not need large-scale computations since no optimization procedure has to be performed. Besides that, local methods are less sensitive to data modifications, because they only influence the solution in a restricted area. However, local methods may become quite complex, too, if a smooth result is needed. In this thesis, all the interpolation schemes are local methods, and they are limited to the proposed scheme.

1.2   Interpolation of Scattered Data

Nowadays, the problem of shape preservation by the interpolation method is that it not only aims to preserve the nature of the data but is also concerned to get a smooth surface for pleasing visual display of the data. The types of data can be divided into two cases, namely, regular data and irregular or scattered data. Many shape preserving interpolation schemes have been introduced, which commonly used regular grid. For example, univariate polynomial spline in $[4, 7, 10 - 12, 16, 20, 47 - 55]$. Some of these schemes were extended to bivariate cases to preserve the shape of data over rectangular mesh. In other cases, interpolation of scattered data using the triangulation method is more suitable for the problem of fitting smooth surfaces through a non-uniform distribution of data points. Among examples of the main sources of scattered data in the real applications are geology, meteorology, oceanography, medical imaging, experimental results in sciences and engineering [56].

In this thesis, we are concerned about the interpolation of scattered data. There are several scattered data techniques used in the scattered data interpolation problem such as Shepard's method [57], Radial Basis Function (RBF) methods [58] and triangulation-based method. In this thesis, we apply triangulation techniques to interpolate the scattered data. The idea of this technique is to construct triangulated domain of data points using Delaunay triangulation [59] and generate piecewise construction of the triangular patches with parametric or geometric continuous for each triangle. Each triangular patch is bounded by three curves formed by one or more polynomial or rational surface patches, so that parametric or geometric continuity of the overall surface will be achieved. There are several methods proposed to form these triangular patches, such as polynomial interpolating method [39, 60], discrete-triangular method [61], split-triangle method (known as Clough-Toucher interpolant) [62, 63], Powell-Sabin split [64], the minimum norm network [65 – 67] and convex combination method [22, 68, 69]. In this thesis, we shall concentrate on the convex combination method to generate the interpolating surface. This method is applied to fill the triangular patch without splitting the triangle. The idea behind this method is to develop three triangular patches with each triangular patch constructed by interpolating the boundary data that satisfies the required continuity condition over each triangle boundary. Then, three triangular patches are blended using the convex combination method into a single patch. In that way, the resulting triangular patches will interpolate all the scattered data. This method requires less computation which is advantageous compared to methods in [70] and [71].

The convex combination of surfaces is normally built with rational weight functions, which produce rational interpolant on all faces. There are several weight functions

which are commonly used in the previous works on this method such as a lower degree rational weight function [32, 68, 72], and upper degree rational weight function [22, 31, 69]. Several approaches have been proposed in the previous studies to construct triangular surfaces using the convex combination method. This method was first introduced by Barnhill [73] which considered the combination of interpolation operators consisting of univariate interpolation along lines parallel to the sides of the triangle. Another method is the side-vertex method proposed by Nielson [72] which presented interpolation scheme over each triangular patch defined by the convex combination method. These interpolants are generated by univariate interpolation on boundary edges joining a vertex and its opposite side. The interpolant scheme not only interpolates data values at the vertices but also interpolate to first order derivatives on boundary. One of the famous approaches in [68] and [22] presented the construction of a C^1 cubic Bézier triangular scheme for scattered data interpolation by using a different weight function in the convex combination method. Further, Chang and Said [69] extended this approach to C^2 quantic Bézier triangular patch schemes which need to be derived from second-order partial derivatives at the vertices of the triangle in order to determine Bézier control points for the triangular patch. Most recently, a new method has been presented by Zhang and Cheng [74] to construct C^1 triangular patches that interpolates on boundary curves and cross-boundary slopes by combining four interpolation operators. The interpolation operators include three side-vertex interpolation operators [72] and an interior operator is based on that three quartic curves. In this thesis, we shall review the C^1 side-vertex method proposed by [72] to interpolate the data arranged over the triangular grid to generate the triangular patches defined by the convex combination.

This thesis presents the construction of shape preserving interpolation using triangular patches for positive data, monotone data and convexity data. Thus, we may describe the given functional data as follows:

$$\left(x_i, y_i, z_i \right), \ i = 1, 2, ..., N, \text{ where } z_i \geq 0, \forall i,$$

we wish to construct a C^1 surface $z = F\left(x, y \right)$ such that

$$z_i = F\left(x_i, y_i \right), \ i = 1, 2, ..., N, \text{ and } F\left(x, y \right) \geq 0, \ \forall x, y.$$

Generally, the construction process for generating the triangular surface are summarized as follows:

1. triangulation of the domain,

2. define derivatives at the data points,

3. assign boundary and inner Ball ordinate values for each triangular patch,

4. generate the triangular patches of the surface by using convex combination methods.

1.3 Conversion between Two Types of Functions

As mentioned earlier, there are two generalized Ball curves known as Said-Ball and Wang-Ball curves, named by Hu et al. [41], which were introduced as alternative models to the popular Bézier curve. The effectiveness of the Said-Ball and Wang-Ball curves was described in the work by Hu et al. [45] in 1996. Goodman and Said [40] pointed out that all recursive evaluations of Said-Ball and Wang-Ball curves are faster than the Bézier curve. The advantage of these generalized Ball basis is efficient evaluation of a polynomial at any point, which is better than de Casteljau algorithm for Bernstein form [75].

Lately, there have been several studies about the conversion for two types of parametric curve, which are the non-rational and rational curves that allows conversion from one curve to the other. Jiang and Wang [76] proposed a relationship between Bézier-Bernstein basis and Delagano-Peña (DP) Ball basis. They proved that these formulae are not only valuable for studying the geometric properties, such as subdivision, of the curves and surfaces constructed by this generalized Ball basis, but also can improve the computational speed of the Bézier curves and surfaces. Tien et al. [77] derived the relationship between Bézier and Said-Ball for non-rational and rational curves of the same degree and vice versa. The formulae that relate to control points and weights were derived using the polar form approach. In contrast, Dejdumrong et al. [78] discussed the relationships between Bézier and Wang-Ball for non-rational and rational curves. They obtained a recursive algorithm for plotting the rational Bézier and Wang-Ball curves from the de Casteljau and Wang algorithm using homogeneous coordinates. Besides that, there are some work on the conversion between two types of parametric surface, such as Chen et al. [79] and Pokavanich et al. [80]. Chen et al. [79] proposed a new type of surface that is called triangular DP surface. They also derived basis conversion formula from triangular DP basis to triangular Bézier for degree three and vice versa. Later, Pokavanich et al. [80] proposed a new algorithm to evaluate Bézier triangular surfaces by converting a set of control points for a given Bézier triangular patch into a new set of Hu's Wang-Ball control points of the same surface. They believed that it has a great future in application of geometric design, especially in computing. In this thesis the similar approach in [80] is used to convert from a Bézier triangular surface into Wang-Ball triangular patch by substituting the relationships between two types of such triangular surfaces.

1.4 Motivation of Study

Shape preserving of data has been an important advancement in the area of scientific visualization for the last two decades. Shape preserving interpolation is a method to interpolate the original data but at the same time still preserve the geometric properties on the constructed interpolants with the shape of the given data being positive, monotone or convex. Ordinary interpolation schemes just guarantee the smoothness of curve or surface but does not focus on the shape characteristics of data, which is also required to inherit the resulting curves or surfaces. To overcome this problem, traditional interpolating schemes need to add extra data points to modify the shape of curves or surfaces. Hence, the efficiency of the shape preserving interpolating scheme needs to be raised, the smoothness of curves or surfaces needs to be retained and the shape of the given data needs to be preserved.

Many researchers have put much effort to propose various rational spline schemes as given in [12, 14, 15, 49, 50] by introducing free parameters to preserve the shape of data. Furthermore, previous citations have extended their schemes to visualize the shape of 3D regular data over rectangular grid surfaces as considered in [18, 19, 20, 48, 81 – 84]. However, in many practical problems, the rectangular mesh is very difficult to fit an interpolation of scattered data. Thus, it is vital to construct the bivariate interpolation function over the triangular domain, which has a great potential of constructing complex shapes. There are some researchers who have made great efforts on the establishments of triangular grid surfaces that addressed the problem of shape preserving interpolant using spline for the scattered data as considered in [31, 32, 36, 85] (positive data), [26, 35, 38, 86] (monotone data) and [25, 87, 88] (convex data). In recent years, the studies in interpolation of shape preservation for scattered

data by using rational spline have received attention in the literature such as rational Bézier function in [24, 27, 33, 35, 37, 38, 86, 89] and a rational trigonometric function [34]. However, most of the earlier works often uses Bernstein-Bézier as a basis function of their scheme in the problem of shape preserving interpolating surfaces. Goodman and Said [90] proved that the generalized Ball basis provided the same kind of shape preserving properties as the Bernstein basis, though to a lesser degree.

Following that, we proposed a Ball function as an alternative scheme to the popular Bézier function. Motivated by the previous studies [25, 26, 32, 33], the proposed scheme in this thesis can preserve the inherent shape feature of data (positivity, monotononicity and convexity) by introducing the weights functions in the definition of the proposed scheme as free shape parameters. Control points are the points that give effect to the shape of the curve or surface. However, weights could be treated as shape parameter to control how much each control point influence the curve or surface. Therefore, we could adjust the shape of the rational Ball triangular patch conveniently by using the shape parameters without changing the control net. The following problems are discussed in this thesis:

Problem 1: Construction of positivity preserving interpolation using rational cubic Ball triangular patches for the visualization of 3D data.

Problem 2: Construction of monotonicity preserving interpolation using rational cubic Ball triangular patches for the visualization of 3D data.

Problem 3: Construction of convexity preserving interpolation using rational cubic Ball triangular patches for the visualization of 3D data.

Problem 4: Development of an algorithm for conversion matrices for Ball and Bézier triangular surfaces.

1.5   Objectives of Study

This thesis generally aim to construct the shape preserving triangular patch by using rational cubic Ball functions.

The specific objectives are:

1. To derive the conditions for positivity, monotonicity and convexity to preserve shape of data.

2. To construct a C^1 interpolation of scattered data using rational cubic Ball triangular patches for positivity preserving, monotonicity preserving and convexity preserving interpolation.

3. To derive the conversion formulae of weights and control points between a rational cubic Ball triangular function and a rational cubic Bézier triangular function.

4. To construct rational cubic Ball triangular patches using the conversion technique.

1.6   Literature Review

Descriptions of the importance of data visualization in this field are discussed in this section. Generally, this study aims to document the contributions of various researchers in this field with a concise explanation of the main findings of the case studies discussed. In many interpolation problems, it is vital that the interpolant preserves shape properties such as positivity, monotonicity and convexity of data points. In general, ordinary interpolating techniques normally ignore all the desired features of the data. Hence, a lot of research towards on the shape preserving interpolation schemes was carried out during the past years.

1.6.1 Positivity preserving interpolation

At the beginning of the previous study, a number of the positivity scheme have been proposed as univariate positivity preserving interpolants, for example, quadratic spline [91], cubic spline [92, 93, 94], quartic spline [95], quintic spline [93], rational function [96]. The positivity preserving interpolation in [91, 94] could be achieved by inserting one or two extra knots where the shape of the curves is not preserved, or by modifying the given derivative values to ensure that the condition acquired are satisfied [92, 93, 95]. Other than that, positivity preserving could be achieved by introducing the weight functions of the rational spline as a free shape parameter that is used to generate the desired curves as required [96].

In recent years, a number of work on positivity preserving using bivariate cases interpolating data on rectangular [4, 19, 48, 97, 98] or over triangular meshes [2, 3, 27 – 29, 32, 33, 36, 38, 85, 99, 100] have been considered by extending the corresponding results from the univariate cases. Among the preliminary studies, this problem have been found in [97] to derive positivity sufficient conditions on first partial derivatives and mixed partial derivatives to preserve the shape of positive data by using the key results that were obtained in [92] for the univariate case. The surface interpolation was generated using a piecewise bi-cubic Hermite interpolant addressing the positivity problems by generating interpolants subject to linear constrains as lower and upper bounds through a set of regular data points arranged over rectangular grid.

The problem of shape preservation of surface (positivity, monotononicity and convexity) on scattered data have received less attention compared to shape preservation on regular data. Therefore, we focused on shape preserving of surface

data defined on scattered data using triangular patch which were motivated by earlier works in [25, 26, 32]. The first approach to construct the surface of scattered data interpolant is the Shepard's method as considered in [101] and [102]. Both papers, visualized positive scattered data to positivity constraints using the modified quadratic Shepard method where the positivity was attained by scaling the basic functions as far as it was needed. The second approach is to triangulate data points, which leads to surface piecewise construction and commonly used by researchers nowadays.

In 1994, Mulansky and Schmidt [99] introduced a constrained C^1 positivity preserving scheme using a quadratic spline on the Powell-Sabin triangulation of the data sites. The sufficient conditions were derived to fulfil the range conditions resulting in a solvable system of linear inequalities with the gradients as parameters and is separated with respect to data sites. The selection of the interpolant is based on a fit-and-modify approach or the minimization of a suitable objective function.

Subsequently, other ways are seen in [29, 31, 32,] and [36], which have considered positive features in scattered data interpolation by using the cubic Bézier triangle patch method when the data provided is positive. The main difference between studies in [29, 31, 32,] and [36] are the way they introduced the calculation of the sufficient conditions on Bézier ordinates in each triangle to preserve the positivity of data. The study by [32] suggested more relaxed conditions and easier to calculate compared to [31] and [36]. If the Bézier ordinates fail to satisfy the derived lower bounds to the Bézier ordinates, then the gradients at data points are re-calculated to be consistent with the positivity conditions. [31] and [36] also extended their interpolation scheme to the construction of a range restricted interpolating surface of the form $z = C(x, y)$

where $C\left(x,y\right)$ is a constant, linear, quadratic or cubic polynomial as a lower or upper constraint. The work of [29] considered a similar approach adopted in [32] to develop the construction of C^1 bivariate interpolant using the cubic Ball function. The sufficient conditions were imposed on 10 Ball ordinates to ensure the positivity on all triangular patches.

Following this, some researchers extended their studies in the construction of interpolating surface of positive data for the higher degrees such as quartic in [2] and [85] where sufficient conditions are derived on Bézier ordinates using triangular Bernstein-Bézier basis functions of degree five with C^2 continuity and G^1 continuity, respectively. Their studies are motivated by the earlier works of [23] and [32]. The interpolating surface in [85] is formed as a single quartic Bézier triangular patch, where the interpolant is positive everywhere according to the positivity conditions derived. They also extended their positivity preserving interpolants to the range restricted data by putting upper and lower constraints to the interpolant surfaces. There are many methods discussing positivity preserving with range restricted interpolation using bivariate functions that can be discovered in the literature [31, 36, 84, 97] and [103] for more detailed explanations since this problem is not the main objective in this thesis. In other studies such as [2], the interpolating surface is formed as a convex combination of quartic Bézier triangular patches, where the resulting surface must be positive everywhere that satisfies the positivity condition and achieve C^2 continuity for all sides of the triangles. The actual data for average monthly rainfall data in Peninsular Malaysia was used in this study to construct the visualization of 3D rainfall interpolant. The advantage of the proposed scheme can also be used to

approximate the amount of rainfall at intermediate locations within the convex hull of the triangulation domain.

Besides using bivariate splines in interpolation of scattered data, visualization of positive data using bivariate rational splines has also received attention from researchers by introducing the weights in the definition of functions as free shape parameters to refine the shape of surfaces as desired. In 2011, Hussain and Hussain [33] proposed a rational cubic Bernstein-Bézier scheme to interpolate positive scattered data. The proposed positivity condition by [33] are derived in the same way as in [32] where the sufficient condition is derived on the Bézier ordinates to preserve the shape of positive data. The main difference between their works is that the proposed scheme by Hussain and Hussain [33] involves weight function in their definition which has advantage for user to modify the shape of data if the Bézier ordinates in any triangular patch does not meet the requirement of the designated lower bounds. Initial values of boundary Bézier ordinates are calculated in the same way as in [32], but not for initial values of inner Bézier ordinates which are calculated by adopting the local scheme concept in [22] in order to satisfy C^1 continuity requirement on all edges of the triangle. In 2014, Hussain et al. [37] continued their work on the rational quartic Bernstein-Bézier interpolation scheme for positive scattered data. This scheme has 15 Bézier ordinates and 15 weight functions compared to the previous scheme in [33]. Due to the high number of weight functions in this scheme, it gives advantage to the user by providing degrees of freedom for refinement of the surface shape, if required. There are three free parameters (weight functions at each vertex of triangle) in [37], while the remaining parameter are constrained by these free parameters. This is in contrast to [33] where all the weight functions are defined

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as a free shape parameter to enhance the resulting shape of surface and preserve the shape of positive data. Both developed schemes in [33] and [37] are local with C^1 continuity and applicable for data accompanied with derivatives or not.

The authors in [24, 27, 34] and [89] considered another approach to preserve positive [24, 27, 89] and monotone [89] scattered data by arranging over a triangular domain. All of these studies discussed the problem of a C^1 piecewise function by using the Nielson side vertex method [72] for interpolation over a triangle. Delaunay triangulation [59] is used to triangulate the domain of scattered data. The study by [89] proposed the scheme of a cubic interpolant with one parameter to interpolate along each edge of each triangular patch and along the line segments joining a vertex to the opposite edge. Other literature such as [24] derived data dependent constraints on shape parameters to preserve the shape of the positive scattered data but does not provide free parameters for shape refinement; and in contrast to [27] and [34], they constructed a rational cubic Bézier and a rational trigonometric basis function with more than one free parameter, respectively. Both authors introduced four free parameters in the definition of the interpolant to generate the interpolating surface. Out of these four parameters, two parameters are constrained to preserve the shape of positive data. Meanwhile, the remaining two parameters are free for users to make modifications on the surface as desired. Thus, with additional free parameters, it could give more advantage to obtain visually pleasing surface compared with one free parameter as given in [89] and no free parameters in [24] which does not give the freedom to users to refine the shape of the data.

Hussin et al. [38] discussed two shape properties which is positivity and monotonicity of triangular surface data. The side vertex scheme triangulates initial data to interpolate the given data over each triangle. In their studies, they have introduced rational function with three parameters at each boundary and radial curve. Therefore, there are 18 parameters in each triangular patch, where six of these parameters are derived from data dependent constraints to preserve the shape of positive and monotone data and the remaining 12 parameters are free for users to modify the shape as required. The proposed scheme is also used if data are given with derivatives or otherwise.

This thesis worked on contributions towards this progress. A C^1 bivariate interpolation on positive data is constructed using rational cubic Ball function which has similar basis function as in [29]. The proposed scheme that involves weight function in the interpolant scheme can give advantage for users to make surface's shape refinement but still preserve the positive shape which are not considered in [31, 32] and [36]. The interpolant surface not only visualizes the shape preservation but also provides a smooth visual image as it is an important concern by researchers nowadays. The computation of positivity condition is more relaxed and simpler as compared to existing methods. Besides that, the construction of surface also local and fulfil the C^1 continuity for each boundary of triangular patch.

1.6.2 Monotonicity preserving interpolation

The other fundamental shapes of data that are most often considered by researchers in shape preserving interpolation is monotonicity. In 1998 and 2000, the concept of monotonicity preservation on rectangular and triangular surfaces have been introduced

in Floater and Peña [104, 105] by characterizing types of monotonicity preservation, namely, axially monotone [104, 105], strongly monotone [105] and monotone [105] by using Bernstein polynomial in both cases, four-sided and three-sided patches. It was proven that the Bernstein basis on rectangular surfaces and Bézier triangles are axially monotonicity preserving and even monotonicity preserving as mentioned in Delgado and Peña [106]. Subsequently, Delgado and Peña [106] displayed some examples which show that rational Bézier surfaces are not axially monotonicity preserving, and that surfaces generated by the tensor product of rational bases do not fulfil the stronger properties of monotonicity preservation, in spite of the fact that they are axially monotonicity preserving. They also proved that surfaces generated by rational Bézier functions with weights on a triangle are not axially monotonicity preserving unless all weights in the functions are equal to one.

In the univariate case, previous literatures that highlighted the problem of monotonicity preserving approximation and interpolation can be found in [6, 54, 107, 108] and the reference therein. In the bivariate case, most of the earlier work focus on rectangular mesh (essentially grounded on tensor product or partially blended) using various types of polynomial spline and rational spline interpolant and deal with the problem shape preservation of monotone data [19, 20, 81 – 83].

In 1985, Beatson and Ziegler [81] interpolated monotone regular data that are arranged over a rectangular grid using a C^1 piecewise quadratic spline where each rectangle in rectangular grid is subdivided into 16 triangles. Necessary and sufficient constraints on functional and derivative values are derived to ensure the monotonicity preservation of the given data. However, there are disadvantages in the developed scheme [81]; it

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is not applicable to data with derivatives. In the same year, Carlson and Fritch [82] extended a univariate case [108] to a bivariate piecewise bi-cubic interpolation scheme for monotone data arranged on rectangular mesh. In their study, necessary and sufficient conditions on first partial derivatives and mixed partial derivative were derived to preserve monotonicity.

Hussain and Hussain [20] visualized monotone data in the view of monotone curve and rectangular surface with two shape parameters. Then, by extending to partially blended rational bi-cubic function (Coons patches), simple constraints are derived on free parameters to preserve the shape of monotone data. These schemes are local but there is no freedom for designers to do the curve and surface modification. For that reason, Hussain et al. [17] developed partially blended rational bi-cubic function with 16 shape parameters extended from the General Piecewise Rational Cubic (GPRC) function. Dependent sufficient constraints derived eight constraint shape parameters to preserve the monotonicity, while the remaining are free parameters. Unfortunately, scheme [17] is computationally expensive because it has many free parameters. Therefore, a rational bi-cubic function with six free parameters in each rectangular patch which extended from a piecewise rational cubic function to preserve the monotonicity of 3D monotone data was presented in [83]. These free parameters are arranged with two of them as constraint parameters for retaining shape of monotone data, while the remaining free parameters are for improvements, if necessary. This scheme is not only local but also computationally economical and visually pleasant.

Besides using rational cubic spline in shape preserving interpolation, the study in [19] has used the greater degree such as rational quartic spline. Liu et al. [19] constructed

positivity and monotonicity preserving with bi-quartic rational interpolation spline surface over rectangular domain including the classical bi-cubic Coons surface as a special case to preserve for positive and monotone surface data. They provided a set of simpler basis functions, where only four parameters are used in their method to represent the positive or monotonic surface data from [4, 20] as a comparison. Moreover, shape preserving of positive and monotone surfaces could be visualized by selecting the appropriate parameters of the spline.

The common approach to preserve monotone data in a bivariate case is based on the triangulation method by a piecewise construction of the interpolant which are suitable for dealing with scattered data such as in [26, 35, 38, 86, 109]. Han and Schumaker's [109] scheme was based on conversion of scattered data to regular data by applying the Sibson split of each rectangle into four triangles. This conversion is generated in new data sites. The necessary and sufficient conditions on functional and derivative values at these new data sites were calculated by the scattered data interpolant to preserve monotone surfaces. Then, these conditions were applied to construct a new algorithm for fitting a monotone surface by reducing scattered data to gridded data. However, the only shortcoming of this method is when the system of N scattered data points reduced to order-N^2 rectangles and some of the rectangles were very small in one or both directions as mentioned by [109].

Saaban et al. [26] developed sufficient conditions on the gradients at the data points in order to preserve the shape of monotonic scattered data. The definition of monotonicity used in [26] is taken from Floater and Peña [105]. The initial estimation of gradients at data points were derived by using a similar approach as in Goodman et al. [110] and

then modified if necessary, to fulfil the monotonicity conditions in certain directions. The interpolating surface is constructed by piecewise cubic Bézier triangular mesh. However, the resulting surface is only C^0 continuity on each triangle boundary, which is not enough to have a smooth surface. The smoothness of surfaces remains an important issue in the shape preserving interpolation. Therefore, many attempts have been made to produce a smooth monotonicity interpolant as presented in [35, 38, 86]. Studies by Hussain and Hussain [35], Hussain et al. [38] and Sarfraz et al. [86] described a bivariate C^1 interpolant using the rational cubic Bézier function to preserve the shape of monotone scattered data arranged over triangular domain with respect to a certain direction. The interpolating surfaces in all of these proposed schemes have been constructed by taking a convex combination of three side vertex interpolants to generate triangular patches by yielding to 12 free parameters in [35, 38] and 24 free parameters in [86] for each triangular patch to refine the shape of monotone surfaces. In practice, interpolation schemes with higher number of parameters such as [86] are more flexible than [35, 38] to improve the shape of the monotone surface. By taking the definition of monotonicity in Floater and Peña [105] and as used in [26], simple sufficient conditions were derived on these free parameters to visualize the shape of monotone scattered data in a given direction. In each case study, [35, 38] and [86] have introduced different number of constraint parameters such as six, nine and 12, respectively, to preserve the shape of 3D monotone scattered data while the remaining are left free for user's choice to achieve a smooth surface interpolation as desired. The advantages of all these local schemes have the same degree of smoothness C^1 which give improvements compared to previous work by [26]. In addition, schemes [35, 38, 86] are also acceptable to both rectangular and triangular grids. This is comparable to Han and Schumaker [109] that required arrangement in a rectangular