APPROXIMATE ANALYTICAL METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

HYTHAM AWAD HAMAD ALKRESHEH

UNIVERSITI SAINS MALAYSIA

2017

APPROXIMATE ANALYTICAL METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

by

HYTHAM AWAD HAMAD ALKRESHEH

Thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

December 2017

ACKNOWLEDGEMENT

In the name of Allah, the most gracious and the most merciful. All praises to Allah for the strengths and His blessing in completing this thesis.

Firstly, I would like to express my sincere appreciation and gratitude to my supervisor, Professor Dr. Ahmad Izani Md. Ismail, for the continuous support of my Ph.D study and related research, for his patience, motivation, encouragement and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better adviser and mentor for my Ph.D study. I would like to thank my co-supervisor, Dr. Amirah Azmi, for her invaluable assistance, suggestions and comments to improve the thesis. I would also like to thank all staff of the School of Mathematical Sciences, Universiti Sains Malaysia (USM), for their various help. I extend my heartfelt thanks to all my friends for their encouragement and helpful discussions throughout this work.

Finally, my deepest gratitude goes to my parents, my wife, my children, my brothers and sisters for their understanding, encouragement and support which made this work possible.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

ADM	Adomian decomposition method
AVITM	Alternative variational iteration transform method
BVP	Boundary value problem
CPU	Central processing unit
DTM	Differential transform method
FDEs	Fractional differential equations
FDTM	Fractional differential transform method
HAM	Homotopy analysis method
HATM	Homotopy analysis transform method
IVP	Initial value problem
MADM	Modified Adomian decomposition method
MSFDTM	Multi-step fractional differential transform method
MSHATM	Multi-step homotopy analysis transform method
MSVITM	Multi-step variational iteration transform method
VIM	Variational iteration method
VITM	Variational iteration transform method

LIST OF SYMBOLS

Greek letters

α, β, τ	Variables
$\Gamma(.)$	Euler's gamma function
$\lambda(\tau)$	Lagrange multiplier (VIM)

Roman letters

$A^n[a,b]$	Set of functions with absolutely continuous derivative of order $n-1$
A_n	Adomian polynomials
$B(p,q)$	Beta function
$C[a,b]$	Set of continuous functions on $[a,b]$
$\mathbb{C}$	The set of complex numbers
$\mathbb{C}$	The set of complex numbers
$D_a^\alpha, \ \alpha \in \mathbb{R}^+$	Riemann-Liouville fractional differential operator
$D_{*a}^\alpha, \ \alpha \in \mathbb{R}^+$	Caputo fractional differential operator
$D_*^\alpha, \ \alpha \in \mathbb{R}^+$	Caputo fractional differential operator when $a = 0$
$E_\alpha(.)$	Mittag-Leffler function
$J_a^\alpha, \ \alpha \in \mathbb{R}^+$	Riemann-Liouville fractional integral operator

$J^{\alpha}, \;\; \alpha \in \mathbb{R}^+$	Riemann-Liouville fractional integral operator when $a = 0$
$H(t)$	Auxiliary function (HAM)
$\hbar$	Auxiliary convergence parameter (HAM)
I	Identity operator
L_p	Lebesgue space
$\mathscr{L}\{.\}$	Laplace transform
$\mathbb{N}$	The set of natural numbers
q	Embedding parameter (HAM)
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of strictly positive real numbers
$R_{\hbar}$	The valid region of convergence (HAM)

Other symbols

$\lceil \alpha \rceil$	Ceiling function, $\lceil \alpha \rceil = min\{z \in \mathbb{Z} : z > \alpha\}$
$\|.\|_{\infty}$	Maximum norm

KAEDAH ANALISIS HAMPIRAN UNTUK PERSAMAAN PEMBEZAAN BIASA PERINGKAT PECAHAN

ABSTRAK

Dalam beberapa dekad yang lalu, populariti dan kepentingan topik persamaan pembezaan pecahan (PPP) telah kian meningkat. Ini disebabkan terutamanya oleh hakikat bahawa alat-alat kalkulus pecahan didapati lebih berkesan dan praktikal daripada alat-alat yang berkait dengan kalkulus klasik untuk pemodelan beberapa fenomena dalam bidang sains dan kejuruteraan. Sebagai contoh, PPP telah berjaya digunakan untuk masalah biologi, kimia dan biokimia, fizik, perubatan, teori kawalan, kewangan dan ekonomi. Oleh itu, terdapat keperluan yang kian meningkat untuk mencari teknik penyelesaian yang cekap dan tepat untuk persamaan pembezaan sedemikian. Walau bagaimanapun, adalah sukar untuk mendapat penyelesaian analisis yang tepat untuk PPP pada umumnya. Akibatnya, kaedah analisis hampiran dan berangka memainkan peranan yang penting untuk mengenal pasti penyelesaian persamaan-persamaan ini dan meneroka aplikasi mereka. Dalam kajian ini, tumpuan diberikan kepada kaedah analisis hampiran. Kaedah-kaedah ini termasuk: Kaedah penguraian Adomian (KPA), kaedah berubah pembezaan pecahan (KBPP), kaedah analisis homotopi (KAH), dan kaedah lelaran bervariasi (KLB). Objektif utama tesis ini adalah untuk membangunkan, menganalisis dan menggunakan kaedah-kaedah ini untuk mencari penyelesaian analisis hampiran untuk beberapa kes PPP biasa yang linear dan bukan linear. Kes-kes ini merupakan masalah nilai awal, masalah nilai sempadan dan sistem-sistem masalah nilai awal PPP. Untuk KPA, kelas-kelas baru polinomial Adomian telah dibangunkan.

Penggunaan kelas yang dicadangkan memberikan penyelesaian anggaran yang lebih tepat untuk PPP bukan linear. Algoritma baru juga telah dicadangkan untuk mencari penyelesaian anggaran masalah nilai sempadan PPP secara langsung tanpa keperluan untuk menyelesaikan sistem persamaan aljabar bukan linear pada setiap langkah penyelesaian seperti dalam KPA piawai. Algoritma yang dicadangkan telah diuji melalui dua contoh. Hasil yang diperoleh menunjukkan kecekapan kaedah ini untuk memberikan anggaran penyelesaian dengan cara yang lebih mudah daripada KPA piawai dengan ketepatan anggaran penyelesaian yang baik. Untuk KBPP, pendekatan piawai telah berjaya digunakan untuk menyelesaikan masalah nilai awal khas yang disebut persamaan Abel. Pengubahsuaian berdasarkan gabungan KBPP piawai dengan polinomial Adomian juga telah digunakan untuk menyelesaikan masalah nilai sempadan dan jenis sistem pecahan khas yang disebut sistem kaku. Untuk KAH dan KLB, pengubahsuaian berdasarkan penggabungan kaedah ini dengan penjelmaan Laplace telah berjaya digunakan untuk menyelesaikan masalah nilai awal dan sempadan. Selain itu, kaedah pelbagai langkah berdasarkan penggabungan ini telah dibangunkan untuk menyelesaikan sistem PPP tanpa perlu mengira kamiran pecahan atau terbitan pecahan pada mana-mana peringkat penyelesaian. Beberapa contoh ujian diberikan untuk mengilustrasi kaedah-kaedah yang dicadangkan dan pengubahsuaiannya. Keputusan yang diperolehi oleh kaedah-kaedah ini dibandingkan antara satu sama lain dan juga dengan kaedah lain dalam literatur terbuka.

APPROXIMATE ANALYTICAL METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

ABSTRACT

During the past few decades, the subject of fractional differential equation (FDEs) has gained considerable popularity and importance. This is mainly due to the fact that the tools of fractional calculus are found to be more effective and practical than the corresponding ones of classical calculus in the modeling of several phenomena in science and engineering. For example, fractional equations have been successfully applied to problems in biology, chemistry and biochemistry, physics, medicine, control theory, finance and economics. Hence, there is a growing need to find efficient and accurate solution techniques of such differential equations. However, exact analytical solutions of FDEs are generally difficult to obtain. As a consequence, approximate analytical and numerical methods play an important role to identify the solutions of these equations and explore their applications. In this study, the focus is on approximate analytical methods. These methods include: Adomian decomposition method (ADM), fractional differential transform method (FDTM), homotopy analysis method (HAM) and variational iteration method (VIM). The main objective of this thesis is to develop, analyse and apply these methods to find the analytical approximate solutions for some cases of linear and nonlinear ordinary FDEs. These cases are initial value problems, boundary value problems and systems of initial value problems of FDEs. For the ADM, new classes of Adomian polynomials have been developed. The use of the proposed classes gives more accurate approximate solution for nonlinear FDEs.

Also a new algorithm has been proposed to find the approximate solution of boundary value problems of FDEs directly without the need to solve systems of nonlinear algebraic equations at each step of the solution as in the standard ADM. The proposed algorithm has been tested through two examples. The obtained results showed the efficiency of this method to provide the approximate solutions in an easier way than standard ADM with good accuracy of approximate solutions. For FDTM, the standard approach has been successfully applied to solve special kind of initial value problems called Abel equations. Also a modification based on a combination of the standard FDTM with Adomian polynomials has been used to solve boundary value problems and special kind of fractional systems called stiff systems. For HAM and VIM, modifications based on incorporating of these methods with Laplace transform have been successfully applied to solve initial and boundary value problems. Also, multi-step methods based on this incorporation have been developed to solve systems of FDEs without the need to calculate fractional integral or fractional derivatives at any stage of the solution. Several test examples are given to illustrate the proposed methods and its modifications. The obtained results by these methods are compared with each other and also with other methods in the open literature.

CHAPTER 1

INTRODUCTION

1.1 Research Introduction

Fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer order differentiation and n-fold integration (Podlubny, 1998). Fractional differential equations, which can be considered as the generalization form of the classical differential equations, have gained considerable importance during the last few decades due to their applications in various fields of science and engineering. Many experimental data highlighted that the state of a physical system not depends only on its current state but also depends on its historical states. Therefore, because the integer order differential operator is a local operator, the classical models cannot give the best description of the realistic behavior. Since the fractional derivative operators have nonlocal properties, the differential equations with fractional operators have been successfully used in the description of many physical phenomena and become more and more popular (Hesameddini et al., 2016; Guo et al., 2012).

Fractional differential equations can be used to model many of problems in mechanics, viscoelasticity, physics, biology, engineering, fluid flow and many others. It is worth mentioning that the first application of fractional derivative was presented in 1823 by Abel (Oldham and Spanier, 1974; Miller and Ross, 1993), who applied fractional derivative to the solution of an integral equation that arises in the formulation of the tautochrone problem. This problem deals with the determination of the shape of the curve such that the required time of descent of a mass sliding down along the curve under the action of gravity with ignoring of the friction is independent of the starting position.

There are several definitions to the fractional derivative of order $\alpha > 0$. The two definitions, that most commonly used are the Riemann-Liouville and Caputo. These two definitions are in general non-equivalent. The difference between them is in the order of evaluation. However, the two definitions are equivalent under their homogeneous initial conditions. In this thesis, we use the Caputo fractional derivative because it allows integer order initial and boundary condition to be included in the formulation of the problems.

In general, exact analytical solution of fractional differential equations usually is not available, especially for nonlinear problems. Thus numerical and analytical techniques have been used to obtain the approximate solution for such problems. Although numerical approximate methods are applicable to a wide range of practical cases, approximate analytical methods provide highly accurate solutions and a deep physical insight. One of the important advantages of the approximate analytical methods is the ability to provide an analytical representation of the solution, which gives better information of the solution over the time interval. On the other hand, the numerical methods provide solutions in numerical and discretized form, which makes it somewhat complicated in achieving a continuous representation. The focus of this thesis is to study and develop analytical methods for the solution of initial and boundary value problems as well as systems of initial value problems of ordinary fractional differential equations.

1.2 Motivation

Motivated by the increasing number of applications of fractional differential equations, growing attention has been given to develop an efficient approximate methods for the solution of this kind of differential equations. In this regard, several methods for the approximate analytical solutions of integer order differential equations have been extended to solve fractional differential equations. These include: Adomian decomposition method (ADM) (Momani and Shawagfeh, 2006; Daftardar and Jafari, 2007), differential transform method (DTM) (Ertürk and Momani, 2008; Al-rabtah et al., 2010), homotopy analysis method (HAM) (Arqub and El-Ajou, 2013; Mishra et al., 2016) and variational iteration method (VIM) (Momani and Odibat, 2007; Abbasbandy, 2007). Also, development of some existing methods have been proposed by numerous authors to solve fractional order equations, these methods include the works of Jang (2014), Cang et al. (2009), and Duan et al. (2013).

Although the approximate analytical methods have been extensively used to solve various kinds of FDEs, some drawbacks of these methods have been frequently reported by numerous authors (we will discuss these drawbacks in detail in the next chapter). Therefore, developing new techniques depending on existing methods to overcome these drawback and also to reduce required computational time and making computations easier is the motivation of this thesis.

1.3 Problem Statement

Most ordinary FDEs do not have exact solution. Thus, the approximate analytical methods such as ADM, FDTM, HAM and VIM have been extensively used to obtain the approximate solution for this type of differential equations. However, these methods have drawbacks in the accuracy and convergence in a wide region and in the required computational time (these drawbacks will be highlighted in Chapter 3). Our aim is to develop new techniques based on these methods to overcome these drawbacks.

1.4 Objectives

The objectives of this study are

- To develop and analyze new classes of Adomian polynomials and employs these classes to solve various kind of initial and boundary value problems as well as systems of initial value problems of nonlinear FDEs.
- To develop a new algorithm of Adomian decomposition method (ADM) to solve boundary value problems of FDEs.
- To apply fractional differential transform method (FDTM) to solve special kinds of fractional initial value problems called Abel differential equations and special kinds of fractional system called stiff systems. In addition, to apply a modification of FDTM to solve boundary value problems of FDEs.
- To apply an existing modification based on a combination of homotopy analysis method (HAM) and Laplace transform, to solve initial and boundary value problems of FDEs. In addition, to develop a new multi-step technique based on this combination to solve systems of initial value problems of FDEs including stiff systems of fractional order.
- To apply a new modification based on a combination of alternative approach of

variational iteration method (VIM) and Laplace transform to solve initial and boundary value problems of FDEs. In addition, to develop a new multi-step technique based on this combination to solve systems of initial value problems of FDEs including stiff systems of fractional order.

1.5 Methodology

The methodology of this study is as follows. The literature on methods for solving initial and boundary value problems as well as systems of initial value problems of ordinary FDEs will be studied. The focus will be on the ADM, FDTM, HAM and VIM. The general structure of these methods will be studied. Then these methods will be constructed and formulated to solve linear and nonlinear problems of initial, boundary and systems of FDEs. This step will provide a basis for the research to follow. New modifications of ADM will be proposed and applied to solve ordinary FDEs. Numerical experiments will be carried out to illustrate the efficiency of these modifications. Existing modifications of HAM, VIM and FDTM for solving initial value problems will be extended to solve boundary value problems of FDEs. New multi-step techniques of HAM and VIM will be proposed and applied to solve systems of ordinary FDEs. The obtained results using the four methods and its modifications will be tabulated and analyzed and comparisons with some obtained results by other methods in the open literature will be made. All the numerical examples in this thesis will be conducted by using Mathematica 10 with HP Laptop (i7-5500U CPU@2.40 GHz, 8.00 GB RAM).

1.6 Thesis Outline

The flow chart of Chapters 2-8 is presented in Figure 1.1. The outline of these chapters is as follows. In Chapter 2, some basic concepts and definitions which are useful in the study of fractional calculus will be reviewed. Also, in this chapter a brief description will be given of the basic principles of the approximate analytical methods that will be studied in this thesis. In Chapter 3, recent studies that have been presented by many authors to find the approximate solution of various kinds of ordinary FDEs are reviewed. Chapter 4 is devoted to study the analytical solution of ordinary FDEs by using ADM. New algorithms are proposed and some numerical examples are tested. At the end of this chapter, the convergence of ADM is discussed. In Chapter 5, we apply the FDTM and its modification to solve special kinds of fractional initial value problems called Abel differential equations and special kinds of fractional system called stiff systems. Comparisons of results by FDTM with other methods in literature are also given. In Chapter 6, the HAM and the homotopy analysis transform method (HATM) are introduced and applied to solve various kind of ordinary FDEs. Also in this chapter, a new multi-step technique based on HATM is proposed to solve systems of FDEs. Comparisons of obtained results by HAM and HATM and other methods are carried out. In Chapter 7, a new combination of Laplace transform with an existing alternative approach of VIM is introduced and used to solve initial and boundary value problems of FDEs. Furthermore, new multi-step technique is proposed in this chapter to solve systems of FDEs. A comparative study is also carried out. Finally, in Chapter 8 we discuss the main results of our study and also we discuss the possibilities for further work.

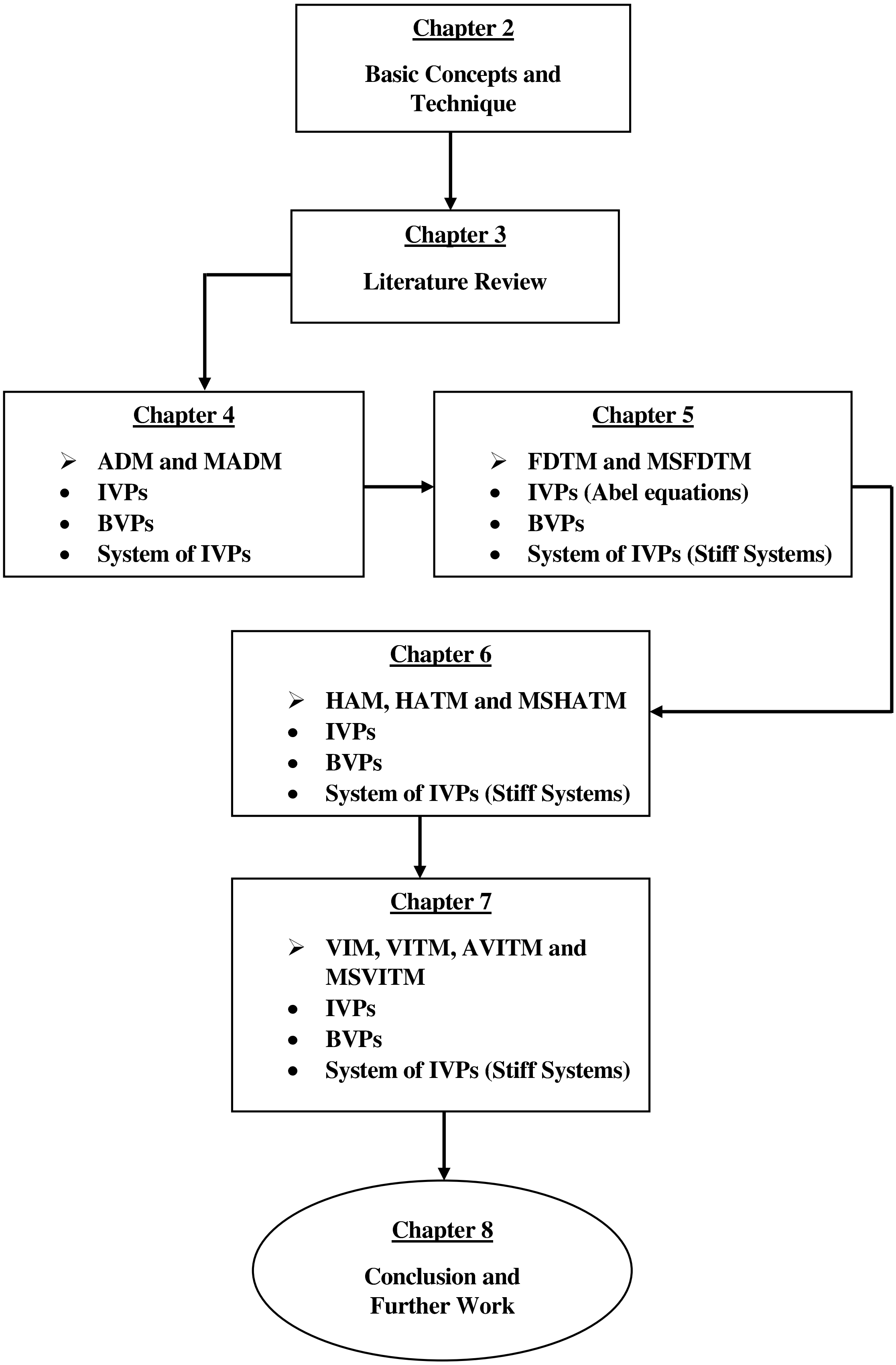

Figure 1.1: Flow chart of Chapters

CHAPTER 2

BASIC CONCEPTS AND TECHNIQUES

2.1 Introduction

In this chapter, we introduce some basic concepts and definitions which play an important role in the theory of fractional calculus and in the theory of fractional differential equations. In addition, a brief description is also given of the basic principles of the analytical methods that will be used in this thesis. These methods are ADM, DTM, HAM and VIM.

2.2 Fractional Calculus

2.2.1 Brief history

Fractional calculus is a branch of applied mathematics that deal with integrals and derivatives of arbitrary order (i.e. non-integer). The original ideas of fractional calculus can be traced back to the end of 17th century, the time when the classical differential and integral calculus theories were developed by Newton and Leibniz (Diethelm, 2010). In a letter in 1695, L'Hôpital wrote to Leibniz asking him about the symbol $\frac{d^n}{dt^n}f(t)$ which he had used in his publications to denoted the n-th derivative of the function $f(t)$ (apparently with the assumption $n \in \mathbb{N}$). L'Hopital posed the question to Leibniz, what would the result when $n = \frac{1}{2}$. Leibniz replied "it will lead to a paradox, which one day useful consequences will be drawn" (Kilbas et al., 2006). This letter from L'Hôpital is nowadays commonly accepted as the first occurrence of what we today call a fractional derivative.

Since then many famous mathematicians have concerned in the subject of fractional calculus and they have provided important contributions up to the middle of the twentieth century. In 1812 Laplace defined a fractional derivative by means of an integral, and in 1819 the first mention of a derivative of arbitrary order appears in a text. In 1819 Lacroix developed a formula for fractional differentiation for the nth derivative of the function $u = x^m$ by induction. Then, he replaced n with the fraction $\frac{1}{2}$, and together with the fact that $\Gamma(\frac{1}{2} = \sqrt{\pi})$, he obtained when $u = x$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}u = \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

In 1822 Fourier suggested an integral representation in order to define the derivative and his version can be considered the first definition for derivative of arbitrary order. In 1826 Abel solved the integral equation of the tautochrone problem, and this problem is considered as the first application of fractional calculus. In 1849 Lagrange contributed to fractional calculus indirectly. He developed the law of exponents for differential operators of integer order

$$\frac{d^m}{dx^m}.\frac{d^n}{dx^n}y = \frac{d^{m+n}}{dx^{m+n}}y.$$

Later, when the theory of fractional calculus started, it became important to know whether it held true if n and m were fractions.

Here we give a list of the most important mathematicians who have contributed on the subject of fractional calculus. This list includes: Liouville (1832-1837), De Morgan (1840), Riemann (1847), Grunwald (1867-1872), Letnikove (1868-1872), Laurent (1884), Hadamard (1892), Heaviside (1892-1912), Littlewood (1917-1928), Weyl

(1917), Levy (1923), Zygmund (1935-1945), Erdelyi (1939-1965), Kober (1940), Riesz (1949) and Feller (1952) (Oldham and Spanier, 1974; Kilbas et al., 1993; Ross, 1977). For the last three centuries or so the theory of fractional derivatives developed mainly as an abstract theoretical field of mathematics of use only for pure mathematicians. However in the last few decades, the theory of fractional derivatives has been object of specialized conferences and treatises. The first conference of note was the "The First Conference on Fractional Calculus and Its Applications" at the University of New Haven in 1979 (Yang, 2010). For the first in-depth study, Oldham and Spanier (1974) published the first book devoted to fractional calculus in 1974.

2.2.2 Special functions

In this section, some special functions which are useful in the theory of fractional calculus are given. These functions are Euler's gamma function, beta function and Mittag-Leffler function.

2.2.2(a) Euler's gamma function

Undoubtedly, gamma function $\Gamma(z)$ is one of the fundamental functions of the fractional calculus. It generalizes the factorial $n!$ and allows n to take non-integer or complex values. Gamma function is defined as follows:

Definition 2.1 *(Podlubny, 1998) The gamma function* $\Gamma(z)$ *is defined by the integral*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad Re(z) > 0. \tag{2.1}$$

Euler's gamma function satisfies the following properties (Podlubny, 1998; Kilbas et al., 1993)

1. Gamma function is analytic for all $z \in \mathbb{C} \backslash 0, -1, -2, \cdots$
2. $\Gamma(z) = \frac{\Gamma(z+1)}{z}, \quad z \neq 0, -1, -2, \cdots$
3. $\Gamma(n+1) = n\Gamma(n) = n!, \quad n \in \{0, 1, 2, \cdots\}$
4. $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(z\pi)} \quad z \neq 0, -1, -2, \cdots$
5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
6. $\Gamma(nz) = \frac{n^{(nz-\frac{1}{2})}}{(2\pi)^{\frac{(n-1)}{2}}} \prod_{k=0}^{n-1} \Gamma(z + \frac{k}{n}), \quad n = 2, 3, \cdots.$

The gamma function also can be represented by the limit

$$\Gamma(z) \quad = \quad \lim_{n\to\infty} \frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)}. \tag{2.2}$$

Some values for $\Gamma(\alpha)$ which often occur in classical application are given in Table 2.1.

Table 2.1: $\Gamma(\alpha)$ for some selected values

α	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$\Gamma(\alpha)$	$\frac{4}{3}\sqrt{\pi}$	$\pm\infty$	$-2\sqrt{\pi}$	$\pm\infty$	$\sqrt{\pi}$	$\frac{1}{2}\sqrt{\pi}$	$\frac{3}{4}\sqrt{\pi}$

2.2.2(b) Beta function

A special function that is closely related to the gamma function in a direct way is the beta function $B(p,q)$, which defined as follows:

Definition 2.2 *(Oldham and Spanier, 1974) If* $p > 0$ *and* $q > 0$ *then*

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt. \tag{2.3}$$

If either p or q is non-positive, the integral diverges and then beta function is defined by the relationship

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{2.4}$$

which is valid for all values of p and q.

Beta function satisfies the following identities (Weilbeer, 2005):

1. $B(p,q) = B(q,p)$

2. $B(p,q) = B(p+1,q) + B(p,q+1)$

3. $B(p,q+1) = \frac{q}{p}B(p+1,q) = \frac{q}{p+q}B(p,q)$

2.2.2(c) Mittag-Leffler function

Mittag-Leffler function plays an important role in the theory of fractional equations. The two parameters Mittag-Leffler function which denoted by $E_{\alpha,\beta}(z)$ is defined as follows:

Definition 2.3 *(Podlubny, 1998) If $z \in \mathbb{C}$ and $\alpha, \beta > 0$ then*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{2.5}$$

The Mittag-Leffler function satisfies the following identities (Podlubny, 1998):

1. For $\beta = 1$, we obtain the one parameter Mittag-Leffler Function

$$E_{\alpha,1}(z) = E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{2.6}$$

2. $E_{1,1}(z) = e^z$, $E_{1,2}(z) = \frac{e^z - 1}{z}$ and in general $E_{1,m}(z) = \frac{1}{z^{m-1}}[e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!}]$.

3. $E_{2,1}(z^2) = \cosh(z)$, $E_{2,2}(z^2) = \frac{\sinh(z)}{z}$ and in general
   $E_{n,r}(z^n) = \frac{1}{z^{1-r}}[\sum_{k=0}^{\infty} \frac{z^{nk+r-1}}{(nk+r-1)!}], \ \ r = 1, 2, \cdots, n.$

2.3 Riemann-Liouville Integrals

The Riemann-Liouville fractional integral is a direct generalization of Cauchy formula for the n-fold integral J^n. If $f(t)$ is Riemann integrable on $[a, b]$, then for $a \leq t \leq b$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
J^1 f(t) &= \int_a^t f(s_1) ds_1, \\
J^2 f(t) &= \int_a^t \left(\int_a^{s_1} f(s_2) ds_2 \right) ds_1, \\
&\vdots \\
J^n f(t) &= \underbrace{\int_a^t \int_a^{s_1} \cdots \int_a^{s_{n-1}}}_{n\text{-integrals}} f(s_n) ds_n \cdots ds_1.
\end{aligned}$$

Cauchy formula is that (Diethelm, 2010)

$$J^n f(t) \quad = \quad \frac{1}{(n-1)!}\int_a^t (t-x)^{n-1} f(x)dx, \quad n \in \mathbb{N}, \ a \geq 0. \tag{2.7}$$

Now, to get an integral for any $\alpha \in \mathbb{R}^+$, we simply generalize the Cauchy formula Eq.(2.7) by replacing the positive integer n by $\alpha \in \mathbb{R}^+$ and using the gamma function instead of the factorial. So it is natural to define the integration of the arbitrary order α as follows.

Definition 2.4 *(Diethelm, 2010) Let $\alpha \in \mathbb{R}^+$ and $f(t) \in L_1[a,b]$. Then the integral*

$$J_a^\alpha f(t) \quad = \quad \frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1} f(x)dx, \quad a \leq t \leq b, \tag{2.8}$$

is called the Riemann-Liouville fractional Integral of order α . And for $\alpha = 0$, we set $J_a^0 = I$, the identity operator.

Note 2.1 (Diethelm, 2010) $L_p[a,b] := \{f : [a,b] \to \mathbb{R};\ f$ is measurable on $[a,b]$ *and* $\int_a^b |f(t)|^p dt < \infty\}$.

Some important properties of Riemann-Liouville fractional integral are as follows.

Property 2.1 (Diethelm, 2010) Let $\alpha, \beta \geq 0$ and $f(t) \in L_1[a,b]$, then

i) $J_a^\alpha J_a^\beta f(t) = J_a^{\alpha+\beta} f(t)$ hold almost every where on $[a,b]$. If additionally $f(t) \in C[a,b]$ or $\alpha + \beta \geq 1$, then the identity holds everywhere on $[a,b]$.

ii) $J_a^\alpha J_a^\beta f(t) = J_a^\beta J_a^\alpha f(t)$.

Property 2.2 (Gorenflo and Mainardi, 2000) If $f(t) = t^\beta$ for some $\beta > -1$ and $\alpha > 0$, then

$J_0^{\alpha}(f(t)) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}, \quad t > 0.$

2.4 Riemann-Liouville Derivatives

Definition 2.5 *(Diethelm, 2010) Let $\alpha \in \mathbb{R}^+$ and $m = \lceil \alpha \rceil$. Then the Riemann-Liouville fractional derivative of order α is defined by*

$$D_a^{\alpha} f(t) = D^m J_a^{m-\alpha} f(t) \quad = \quad \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau & , m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t) & , \alpha = m. \end{cases}$$

For $\alpha = 0$, we set $D_a^0 = I$, the identity operator.

Definition 2.5 is valid for any integer m provided $m > \alpha$. For example, if $\alpha = 0.8$ then the equation $D_a^{\alpha} f(t) = D^m J_a^{m-\alpha} f(t)$ remains valid for any integer $m > 0.8$. There is no loss of the generality while considering narrow condition $m-1 < \alpha < m$ or $m = \lceil \alpha \rceil$. Some important properties of Riemann-Liouville fractional derivatives are as follows.

Property 2.3 (Diethelm, 2010) Let f_1 and f_2 are two functions defined on $[a,b]$ such that both $D_a^{\alpha} f_1$ and $D_a^{\alpha} f_2$ exist, then

$D_a^{\alpha}(c_1 f_1 + c_2 f_2) = c_1 D_a^{\alpha} f_1 + c_2 D_a^{\alpha} f_2$, where $c_1, c_2 \in \mathbb{R}$.

Property 2.4 (Weilbeer, 2005) If $\alpha > 0$, then for every $f(t) \in L_1[a,b]$ we have

$D_a^{\alpha} J_a^{\alpha} f(t) = f(t)$,

almost everywhere. If furthermore there exists a function $h(t) \in L_1[a,b]$ such that $f(t) = J_a^{\alpha} h$, then

$J_a^{\alpha} D_a^{\alpha} f(t) = f(t)$

holds almost everywhere.

Property 2.5 (Weilbeer, 2005) Let $\alpha > 0$ and $m = \lceil \alpha \rceil$, also assume that f is such that $J^{m-\alpha} f \in A^m[a,b]$, then

$$J_a^\alpha D_a^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} \frac{1}{\Gamma(\alpha-k)}(t-a)^{\alpha-k-1} . \lim_{z\to a^+} D^{m-k-1} J_a^{m-\alpha} f(z),$$

and for the special case $0 < \alpha < 1$ we have

$$J_a^\alpha D_a^\alpha f(t) = f(t) - \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1} \lim_{z\to a^+} J_a^{1-\alpha} f(z).$$

Note 2.2 (Diethelm, 2010) $A^n[a,b]$ denotes to the set of functions with an absolutely continuous $(n-1)$st derivative on $[a,b]$.

Property 2.6 (Gorenflo and Mainardi, 2000) If $f(t) = t^\beta$ for some $\beta > -1$ and $\alpha > 0$, then

$$D_0^\alpha f(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t)^{\beta-\alpha}, \quad t > 0.$$

Property 2.6 is straightforward generalization of what we know for integer order derivatives. We can note that the Riemann-Liouville fractional derivative for the constant function is not zero if $\alpha \notin \mathbb{N}$. In fact if $\beta = 0$, we have

$$D_a^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha > 0, \ t > 0.$$

This, of course equal zero for $\alpha \in \mathbb{N}$, due to the poles of the gamma function at the points $0, -1, -2, \cdots$.

2.5 Caputo's Fractional Derivatives

Real-world phenomena require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $u(a),\ u'(a),\ u''(a)$, etc. (Podlubny, 1998). Unfortunately, the Riemann-Liouville approach D_a^α leads to initial conditions in the form of limit values of the Riemann-Liouville fractional derivatives, for example

$$\begin{aligned}
\lim_{t\to a} D_a^{\alpha-1}u(t) &= c_1, \\
\lim_{t\to a} D_a^{\alpha-2}u(t) &= c_2, \\
&\vdots \\
\lim_{t\to a} D_a^{\alpha-n}u(t) &= c_n.
\end{aligned} \tag{2.9}$$

In fact, the initial value problems with such initial conditions Eq.(2.9) can be successfully solved, but their solutions are of little use because there is no clear physical interpretation for such types of initial conditions.

To overcome this disadvantage of Riemann-Liouville approach, we will discuss below a modified version of Riemann-Liouville operator called Caputo's approach. This approach was proposed by Caputo first in 1967 in his paper (Bella et al., 1990). In this thesis, to distinguish between the two approaches we will use the symbol D_{*a}^{α} to denote the Caputo approach.

Definition 2.6 *(Gorenflo and Mainardi, 2000) Let $\alpha \in \mathbb{R}^+$ and $m = \lceil \alpha \rceil$. Then the Caputo fractional derivative of order α is defined by*

$$D_{*a}^{\alpha} f(t) = J_a^{m-\alpha} D^m f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} f(\tau) d\tau & , m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t) & , \alpha = m. \end{cases}$$

Some important properties of Caputo fractional derivatives (Weilbeer, 2005)

Property 2.7 Let f_1 and f_2 are two functions defined on $[a,b]$ such that the fractional derivatives $D^{\alpha}_{*a}f_1$ and $D^{\alpha}_{*a}f_2$ exist, then

$D^{\alpha}_{*a}(c_1f_1+c_2f_2)=c_1D^{\alpha}_{*a}f_1+c_2D^{\alpha}_{*a}f_2$, where $c_1,c_2\in\mathbb{R}$.

Property 2.8 If $\alpha>0,\ \ m=\lceil\alpha\rceil$ and $f\in A^m[a,b]$, then

$$J^{\alpha}_{a}D^{\alpha}_{*a}f(t) \;=\; f(t)-\sum_{k=0}^{m-1}\frac{D^kf(a)}{k!}(t-a)^k. \tag{2.10}$$

Property 2.9 If $f(t)$ is continuous and $\alpha\geq 0$, then

$$D^{\alpha}_{*a}J^{\alpha}_{a}f(t) \;=\; f(t). \tag{2.11}$$

We note that from the Property 2.8 and the Property 2.9, the Caputo derivative is the left inverse of Riemann-Liouville integral but it is not right inverse.

Property 2.10 Let $\alpha>0,\ \ m=\lceil\alpha\rceil$ and assume that both $D^{\alpha}_{*a}f(t)$ and $D^{\alpha}_{a}f(t)$ exist, then

i) $$D^{\alpha}_{*a}f(t)=D^{\alpha}_{a}f(t)-\sum_{k=0}^{m-1}\frac{D^kf(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}. \tag{2.12}$$

ii) If $D^kf(a)=0,\ \ k=0,1,2,\cdots,m-1,$ then

$$D^{\alpha}_{*a}f(t)=D^{\alpha}_{a}f(t). \tag{2.13}$$

From Property 2.10, we can state that if the initial conditions are homogeneous in a fractional differential equations, then the two approaches Riemann-Liouville and Caputo coincide.

In view of the Caputo approach definition, we note that as in the integer order derivative, the Caputo fractional derivative of a constant c is zero, whereas the Riemann-Liouville derivative is not equal zero since if we substituted $f(t)=c$ in the Definition

2.5 we have

$$D_a^{\alpha} c \;=\; \frac{c\, t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{2.14}$$

On the other hand, the Caputo fractional derivatives are more restrictive than the fractional derivatives of Riemann-Liouville, that is because the Caputo approach requires the existence of the m derivative of the function. Fortunately most function that appear in real-life applications fulfill this requirement (Podlubny, 1998). In this thesis, whenever the Caputo approach is used, we assume the m derivative of the function $f(t)$ exists.

2.6 Laplace Transforms

The Laplace transform plays an important role in the methods to find the exact or approximate solutions for many problems of integer or fractional order differential equations that arise in various field of science and engineering. In this section, we briefly give some important definitions and properties of Laplace transform that will be useful in thesis.

Definition 2.7 *(Spiegel, 1965) If the function*

$$F(s) = \int_0^{\infty} e^{-st} f(t)dt, \quad s \in \mathbb{C} \tag{2.15}$$

exist, it is called the Laplace transform of $f(t)$, *and we denote it by* $\mathscr{L}\{f(t), s\}$.

Therefore some useful properties of Laplace transform (Spiegel, 1965):

1. $\mathscr{L}\{\frac{d^m f(t)}{dt^m}, s\} = s^m F(s) - \sum_{k=1}^{m} s^{m-k} f^{(k-1)}(0)$

2. $\mathscr{L}\{\int_0^t f(\tau) d\tau, s\} = \frac{F(s)}{s}$

3. $\mathscr{L}\{f(t) * g(t), s\} = F(s) * G(s)$, where the convolution is defined by $f(t) * g(t) = \int_0^t f(t-\tau).g(\tau) d\tau = \int_0^t g(t-\tau).f(\tau) d\tau$.

4. $\mathscr{L}\{t^\alpha, s\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha > -1$

5. $\mathscr{L}\{c_1 f_1(t) + c_2 f_2(t), s\} = c_1 \mathscr{L}\{f_1, s\} + c_2 \mathscr{L}\{f_2, s\} = c_1 F_1(s) + c_2 F_2(s)$.

Definition 2.8 *(Podlubny, 1998) The Laplace transforms of $J^\alpha f(t)$ and $D^\alpha f(t)$ where $n-1 < \alpha \leq n$, are given by*

$$i) \qquad \mathscr{L}\{J^\alpha f(t), s\} = \frac{F(s)}{s^\alpha} \tag{2.16}$$

$$ii) \qquad \mathscr{L}\{D^\alpha f(t), s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} \left[D^{\alpha-k-1} f(t) \right]_{t=0} s^k \tag{2.17}$$

where $J^\alpha f(t)$ and $D^\alpha f(t)$ are respectively the Riemann-Liouville integral and the Riemann-Liouville fractional derivative at $a = 0$.

Definition 2.9 *(Podlubny, 1998) If $n-1 < \alpha \leq n$, Then the Laplace transforms of the Caputo fractional derivative is given by*

$$\mathscr{L}\{D_*^\alpha f(t), s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-k-1} \tag{2.18}$$

where $D_*^\alpha f(t)$ is the Caputo fractional derivative at $a = 0$.

From Definitions 2.8 and 2.9, it can be seen that the Riemann-Liouville fractional derivative requires initial conditions with non-integer order derivatives $D^{\alpha-k-1}f(t)$ at $t = 0$, while the Caputo derivatives requires the values of the integer order derivatives $f^{(k)}(0^+)$. However, the practical applicability of Riemann-Liouville derivative is limited due to the absence of the physical interpretation of the initial conditions with fractional derivatives. Since the formula of the Laplace transform of the Caputo derivative uses the values of the integer order derivatives at $t = 0$, for which a certain physical interpretation exists, for example ($f(0)$ for the initial position, $f'(0)$ for the initial velocity) (Podlubny, 1998), we can expect that it can be useful for solving many problems as we will see in Chapters 5-7.

2.7 Ordinary Differential Equations of Fractional Order

Equations in which an unknown function $u(t)$ is contained under the sign of a derivative of fractional order D_a^α, this means equations in the following general form

$F(t, u(t), D_{*a_1}^{\alpha_1}\omega_1(t)u(t), D_{*a_2}^{\alpha_2}\omega_2(t)u(t), \cdots, D_{*a_n}^{\alpha_n}\omega_n(t)u(t)) = g(t)$

are called ordinary differential equations of fractional order. By analogy with the classical differential equations, differential equations of fractional order are divided into linear, nonlinear, homogeneous and inhomogeneous equations with constant and variable coefficients (Kilbas et al., 1993).

This thesis is devoted to study the analytical approximate solution for initial, boundary and system of ordinary fractional differential equations in the following general forms

1- Initial value problems (IVPs)

$$\begin{aligned} D_*^{\alpha}u(t) &= f(t,u(t),D_*^{\alpha_1}u,D_*^{\alpha_2}u,\cdots,D_*^{\alpha_n}u), \qquad (2.19)\\ u^{(k)}(0) &= c_k,\ \ k=0,1,\ldots,m-1 \end{aligned}$$

where $m-1<\alpha\leq m,\ \ m\in\mathbb{N},\ \ 0\leq\alpha_i<\alpha,\ \ i=1,2,\cdots,n,\ \ t\in[0,T]$ and D_*^{α} denotes the Caputo fractional derivative of order α (Daftardar and Jafari, 2007).

2- Boundary value problems (BVPs)

$$D_*^{\alpha}u(t) = f(t,u(t),D_*^{\alpha_1}u,D_*^{\alpha_2}u,\cdots,D_*^{\alpha_n}u), \qquad (2.20)$$

where $t\in[0,L],\ \ m-1<\alpha\leq m,\ \ m\in\mathbb{N},\ \ 0\leq\alpha_i<\alpha,\ \ i=1,2,\cdots,n$, subject to k mixed set of Dirichlet and Neumann boundary conditions

$$\begin{aligned} &u^{(q_0)}(t_0)=c_0,\ \ u^{(q_j)}(t_r)=c_j,\ \ u^{(q_{m-1})}(t_{k-1})=c_{m-1},\ \ r=0,1,2,\cdots,k-1\\ &j=0,1,2,\cdots,m-1,\ \ 0\leq q_0,q_1,\cdots\leq m-1,\ \ t_r\in[0,L],\ \text{such that}\ \ t_0=0,\ \ t_{k-1}=L,\\ &q_i\neq q_j\ \ \text{if}\ \ t_i=t_j \end{aligned}$$

and the total number of boundary conditions is equal to m.

3- System of Initial value problems

$$\begin{aligned} D_*^{\alpha_i}u_i &= \sum_{j=1}^{n}B_jD_*^{\alpha_{ij}}u_j+N_i(t,u_1,u_2,\ldots,u_n)+f_i(t), \qquad (2.21)\\ & u_i^{(r)}(0)=c_r^i \end{aligned}$$

$n_i - 1 < \alpha_i \leq n_i,\ n_i = \lceil \alpha_i \rceil,\ n_{ij} - 1 < \alpha_{ij} \leq n_{ij} - 1,\ \ i = 1, 2, \cdots, n,\ \ r = 0, 1, \cdots, n_i - 1,\ \ \alpha_{ij} \leq \alpha_i$, where $\alpha_i, \alpha_{ij} \in \mathbb{R}^+$, N_i's are nonlinear functions of $t, u_1, u_2, \ldots, u_n$, $f_i(t) \in C[0, T]$ and B_j are constant (Jafari and Daftardar, 2006b).

2.8 Definition of the Homotopy

A homotopy between two continuous functions $f(x)$ and $g(x)$ from a topological space $\mathbb{X}$ to topological space $\mathbb{Y}$ is formally defined to be a continuous function (Liao, 2012)

$H : \mathbb{X} \times [0, 1] \rightarrow \mathbb{Y}$ such that if $x \in \mathbb{X}$ then

$$H(x, 0) = f(x) \ \text{ and } \ H(x, 1) = g(x).$$

2.9 Approximate Analytical Methods

Approximate analytical methods such as ADM, DTM, HAM, VIM, homotopy perturbation method (HPM) (He, 2003) have been widely used in recent years to solve various types of differential equations. These methods provide the solution of differential equations in the form of infinite series. An advantage of the approximate analytical methods is that it is capable of providing us a continuous representation of the approximate solution, which gives better information of the solution over the time interval (Ghorbani et al., 2011). In contrast to the implicit finite difference method (which have better stability characteristic than explicit methods), the approximate analytical method do not require the numerical solution of system of equations (Ghoreishi et al., 2013). In this section, a brief description of the basic principles of ADM, DTM, HAM and VIM are presented. In addition some important advantages and disadvantages for these methods are discussed.

2.9.1 Adomian decomposition method

Since the beginning of 1980's, Adomian (1988a, 1991, 1994, 2013) has presented and developed a so-called ADM to solve linear and nonlinear problems such as ordinary differential equations, partial differential equations and integral equations. ADM provides an easily computable and rapidly convergent series of analytical approximate solution. It is free of rounding errors since it does not involve any discretization, and also it is computationally inexpensive.

To clarify the basic principle of ADM, consider the following general form of differential equation

$$Lu+Ru+Nu = g(t), \tag{2.22}$$

where L is easily invertible operator, R is the remainder of the linear operator less than L, N represents the nonlinear terms and g is known function.

The ADM assumes that the unknown function $u(t)$ can be decomposed by the infinite sum

$$u(t) = \sum_{k=0}^{\infty} u_k. \tag{2.23}$$

Further, the nonlinear term Nu is usually represented by the infinite sum

$$Nu = \sum_{k=0}^{\infty} A_k. \tag{2.24}$$