BOHR'S INEQUALITY AND ITS EXTENSIONS

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by

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TABLE OF CONTENTS

CHAPTER 6 – BOHR RESEARCH AND CONCLUSION

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

$\Delta_{x,y}$	Laplace operator $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, 7
λ	spherical chordal distance, 18
λ_U	density of hyperbolic metric on U, 13
λ_{U_0}	density of hyperbolic metric on U_0, 56
∂U	unit circle $\{z \in \mathbb{C} : \lvert z\rvert = 1\}$, 15
∂U^+	$\left\{e^{it} : -\frac{\pi}{2} \le t \le \frac{\pi}{2}\right\} \cup \{iy : -1 \le y \le 1\}$, 64
$\frac{\partial}{\partial z}$	$\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, 8
$\frac{\partial}{\partial \bar{z}}$	$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, 8
$\prec$	subordinate to, 6
arg	argument function, 19
$B(G)$	second Bohr radius on domain $G \subset \mathbb{C}^n$, 109
$\mathbb{C}$	complex plane, 1
$\overline{\mathbb{C}}$	extended complex plane $\mathbb{C} \cup \{\infty\}$, 114
$\mathbb{C}^n$	n-fold Cartesian product of $\mathbb{C}$, 107
cos	cosine function, 73
d	Euclidean distance, 15
d_U	hyperbolic distance on U, 13
$d_{U_{1/3}}$	hyperbolic distance on $U_{1/3}$, 58
d_{U_0}	hyperbolic distance on U_0, 57

$\overline{D}_{min}$ smallest closed disk containing the closure of D, 82

exp exponential function, 31

$f_{z\bar{z}}$ $\frac{\partial^2 f}{\partial z \partial \bar{z}}$, 8

$\|f\|_\infty$ $\sup_{|z|<1} |f(z)|$, 16

$\|f\|_G$ $\sup_{z\in G} |f(z)|$, 111

$f * g$ Hadamard product (or convolution) of f and g, 22

$\mathscr{H}$ right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, 6

$H(G)$ class of all analytic functions on some domain G, 113

$H^\infty(G)$ class of all bounded analytic functions on some domain G, 111

$H(U)$ class of all analytic functions on unit disk U, 1

$H(U_r)$ class of all analytic functions on disk U_r, 114

$H(G,D)$ class of of all analytic functions $f : G \to D$, 122

$H(U,U)$ class of of all analytic self-map on unit disk U, 6

$H(U,U_0)$ class of of all analytic functions $f : U \to U_0$, 36

$H(U,U^+)$ class of of all analytic functions $f : U \to U^+$, 61

$H(U,U^h)$ class of of all analytic functions $f : U \to U^h$, 69

$H(U,W_\alpha)$ class of of all analytic functions $f : U \to W_\alpha$, 30

$H(U^h,U^h)$ class of of all analytic self-map on U^h, 61

$H(U^h,U_q)$ class of of all analytic functions $f : U^h \to U_q$, 60

$H^\infty(U,X)$ class of of all bounded analytic functions $f : U \to X$, 116

inf infimum function, 14

J_f Jacobian of f, 9

$\mathcal{K}$ $\{f \in S : f(U)$ is a convex domain$\}$, 5

$K(G)$ first Bohr radius on domain $G \subset \mathbb{C}^n$, 107

$K_n(v,\lambda)$ λ-Bohr radius of v, 117

log logarithmic function, 11

min minimum function, 46

$\mathcal{M}$ majorant function, 16

$\mathbb{N}$ set of natural numbers, 117

$\|\cdot\|$ norm in Banach space, 108

P Poisson kernel, 87

$\mathscr{P}_{LH}$ $\left\{f(z) = h(z)\overline{g(z)} : f \text{ is logharmonic in } U \text{ and } \operatorname{Re} f(z) > 0 \text{ for } z \in U\right\}$, 10

$\mathscr{P}_{LH(M)}$ $\left\{f \in \mathscr{P}_{LH} : \left|\frac{h(z)}{g(z)} - M\right| < M, M \geq 1\right\}$, 10

$p_U(z,w)$ pseudo-hyperbolic distance between z and w, 14

$\mathbb{R}$ set of real numbers, 3

r_h $\tanh(1/2)$, 60

$R(\alpha,\gamma,h)$ $\left\{f \in H(U) : f(z) + \alpha z f'(z) + \gamma z^2 f''(z) \prec h(z), \quad z \in U\right\}$, 22

$R(\alpha,h)$ $\{f \in H(U) : f'(z) + \alpha z f''(z) \prec h(z), \quad z \in U\}$, 22

$s(x)$ $\sin x/x$, 88

sin sine function, 88

$\mathcal{S}$ $\{f \in H(U) : f \text{ is univalent}, f(0) = 0, f'(0) = 1\}$, 3

$\mathcal{S}^*$ $\{f \in S : f(U)$ is a starlike domain with respect to $0\}$, 4

S_{Lh} $\left\{f = zh(z)\overline{g(z)} : f \text{ is univalent logharmonic in } U, h(0) = g(0) = 1\right\}$, 93

ST_{Lh}^0 $\{f \in S_{Lh} : f(U) \text{ is starlike with respect to } 0\}$, 93

$\mathscr{S}$ the strip $\{z \in \mathbb{C} : |\operatorname{Re} z| < \rho\}$, 83

S_W class of all univalent, harmonic, orientation-preserving mappings $f : U \to W$ with normalization $f(0) = 1$, 87

$\overline{S_W}$ closure of S_W, 87

$S(f)$ $\{g \in H(U) : g \prec f\}$, 82

$S(h)$ $\{f \in H(U) : f \prec h\}$, 23

sup supremum function, 15

tanh hyperbolic tangent, 60

U unit disk $\{z \in \mathbb{C} : |z| < 1\}$, 1

U_0 unit disk punctured at the origin $U \backslash \{0\}$, 36

U^+ unit semi-disk $\{z \in U : \operatorname{Re} z > 0\}$, 61

U^h hyperbolic unit disk $\{z \in U : d_U(0,z) < 1\}$, 60

$\overline{U}$ closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$, 114

U_r $\{z \in \mathbb{C} : |z| < r\}$, 114

$\overline{U}_r$ $\{z \in \mathbb{C} : |z| \leq r\}$, 36

U_{r_h} $\{z \in \mathbb{C} : |z| < r_h\}$, 60

U^n $\{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_k| < 1\}$, 107

W $\left\{w \in \mathbb{C} : |\arg w| < \frac{\pi}{4}\right\}$, 87

W_α $\left\{w \in \mathbb{C} : |\arg w| < \frac{\alpha\pi}{2}\right\}$, 30

KETAKSAMAAN BOHR DAN PELANJUTAN

ABSTRAK

Jika $f(z) = \sum_{n=0}^{\infty} a_n z^n$ merupakan peta diri analisis pada unit cakera U, maka $d(\sum_{n=0}^{\infty} |a_n z^n|, |a_0|) \leq d(a_0, \partial U)$ bagi $|z| \leq 1/3$, dengan d menandakan jarak Euklidan dan ∂U bulatan unit. Pernyataan ini disebut sebagai Teorem Bohr, yang dibuktikan oleh Harald Bohr pada tahun 1914. Tesis ini memberi tumpuan kepada pengitlakan Teorem Bohr. Andaikan h sebagai fungsi univalen yang tertakrif pada U. Andaikan juga $R(\alpha,\gamma,h)$ sebagai kelas fungsi f analisis dalam U dengan $f(z)+\alpha z f'(z)+\gamma z^2 f''(z)$ yang tersubordinasi kepada $h(z)$. Teorem Bohr bagi kelas $R(\alpha,\gamma,h)$ diperoleh untuk h suatu fungsi cembung dan fungsi berbintang terhadap $h(0)$. Teorem Bohr untuk kelas fungsi analisis yang memeta U ke domain cekung dan juga ke domain cakera unit berliang diperoleh dalam bab yang seterusnya. Jejari klasik Bohr $1/3$ ditunjukkan tak berubah apabila jarak Euclidean digantikan sama ada dengan jarak sentuhan sfera atau dengan jarak model cakera Poincaré. Tambahan lagi, teorem Bohr untuk set cembung Euklidan ditunjukkan mempunyai analog dalam model cakera Poincaré. Akhirnya, Teorem Bohr diperoleh untuk beberapa subkelas pemetaan harmonik dan logharmonik yang tertakrif pada unit cakera U.

BOHR'S INEQUALITY AND ITS EXTENSIONS

ABSTRACT

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic self-map defined on the unit disk U, then $d(\sum_{n=0}^{\infty} |a_n z^n|, |a_0|) \leq d(a_0, \partial U)$ for $|z| \leq 1/3$, where d denote the Euclidean distance and ∂U the unit circle. The result is known as the Bohr's theorem which was proved by Harald Bohr in 1914. This thesis focuses on generalizing the Bohr's theorem. Let h be a univalent function defined on U. Also, let $R(\alpha, \gamma, h)$ be the class of functions f analytic in U such that the differential $f(z) + \alpha z f'(z) + \gamma z^2 f''(z)$ is subordinate to $h(z)$. The Bohr's theorems for the class $R(\alpha, \gamma, h)$ are proved for h being a convex function and a starlike function with respect to $h(0)$. The Bohr's theorems for the class of analytic functions mapping U into concave wedges and punctured unit disk are next obtained in the following chapter. The classical Bohr radius $1/3$ is shown to be invariant by replacing the Euclidean distance d with either the spherical chordal distance or the distance in Poincaré disk model. Also, the Bohr's theorem for any Euclidean convex set is shown to have its analogous version in the Poincaré disk model. Finally, the Bohr's theorems are obtained for some subclasses of harmonic and logharmonic mappings defined on the unit disk U.

CHAPTER 1

INTRODUCTION

1.1 Analytic Functions

Let $\mathbb{C}$ be the complex plane and $U := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk. Let f be a function on U and $z_0 \in U$. We say that f is differentiable at z_0 if the derivative of f at z_0 given by

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

exists. If f is differentiable at every point of U, then f is said to be analytic in U since U is an open set. Let $H(U)$ denote the class of all analytic functions defined on U. By using the Cauchy integral formula, it can be shown that if $f \in H(U)$, then f is represented by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in U, \tag{1.1}$$

where

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \oint_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta, \quad n \geq 0,$$

for any fixed r, $0 < r < 1$.

Write $U = \cup_{n=0}^{\infty} K_n$ where $K_0 = \{0\}$ and $K_n = \{z : |z| \leq r_n < 1\}$ for $n \geq 1$ where $(r_n)_{n \geq 1}$ is a strictly increasing sequence of positive real numbers such that $r_n \to 1$ as $n \to \infty$. The space $H(U)$ can be made into a complete metric space by defining the metric on $H(U)$ as

$$\rho(f,g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|f-g\|_n}{1+\|f-g\|_n},\quad f,g\in H(U),$$

where $\|f-g\|_n=\sup_{z\in K_n}|f(z)-g(z)|$. The topology on $H(U)$ given by the metric ρ is then equivalent to the topology of uniform convergence on compact subsets of U (see [14, p. 221]). Finally, it follows from theorems of Weiestrass and Montel that this space is complete [76, p. 38].

1.2 Univalent Functions

An analytic function f is said to be univalent in a domain D if $f(z)\neq f(w)$ whenever $z\neq w$ for all $z,w\in D$. In particular, f is locally univalent at a point $z_0\in D$ if it is univalent in some neighborhood of z_0. The existence of a unique analytic function which maps U conformally onto any simply connected domain strictly contained in $\mathbb{C}$ follows from the Riemann Mapping Theorem:

Theorem 1.1. *[14, p. 230] (see also [69, p. 11]) Given any simply connected domain D which is not the whole plane, and a point $z_0\in D$, there exists a unique analytic function f in D, normalized by the conditions $f(z_0)=0$ and $f'(z_0)>0$, such that f defines a one-to-one mapping of D onto the unit disk U.*

As a consequence of this theorem, the study of analytic univalent functions on a simply connected domain D can now be reduced to the study of analytic univalent functions on the unit disk U.

The post-composition of a univalent function with the affine map $\alpha z+\beta$ defined on $\mathbb{C}$, $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$, is again a univalent function. Thus, the study of analytic univalent functions can be further restricted to the class $\mathcal{S}$ which consists of all analytic univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$. The *Koebe function*

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n, \quad z \in U$$

is a function in $\mathcal{S}$ which maps U conformally onto $\mathbb{C} \backslash (-\infty, -1/4]$. Indeed, the Koebe function and its rotations $e^{-it} k(e^{it} z)$, $t \in \mathbb{R}$, appear as extremal functions for various research problems arisen in exploring the class $\mathcal{S}$.

One such problem is to determine the maximum value of $|a_n|$ in $\mathcal{S}$ for $n \geq 2$. This is a well-defined problem as $\mathcal{S}$ is a compact subset of $H(U)$ (see [76, Theorem 4.1]) and the function $J(f) = a_n$ defined on $\mathcal{S}$ has a maximum modulus, that is, there exists a $f_0 \in \mathcal{S}$ such that $|J(f)| \leq |J(f_0)|$ for all f (see [76, Theorem 4.2]). In 1916, Bieberbarch[33] obtained the estimate for a_2:

Theorem 1.2. *(Bieberbarch Theorem)[69, Theorem 2.2] If $f \in \mathcal{S}$, then $|a_2| \leq 2$, with equality if and only if f is a rotation of the Koebe function.*

In the same paper, Bieberbarch made a conjecture:

Theorem 1.3. *(Bieberbarch Conjecture)[69, p. 37] If $f \in \mathcal{S}$, then $|a_n| \leq n$, with equality if and only if f is a rotation of the Koebe function.*

The Bieberbarch theorem is applied to prove theorems regarding the class $\mathcal{S}$ such as the Koebe one-quater theorem [69, Theorem 2.3], the distortion theorem [69, Theorem 2.5] and the growth theorem [69, Theorem 2.6]. Consequently, the researchers reckoned that the Bieberbarch conjecture is true because of the extremal role played by Koebe function (and its rotations) in those theorems. A proof of Bieberbarch conjecture was eventually given by Louis de Branges [48] in 1985.

1.2.1 Starlike and Convex Functions

In the effort of validating the Bieberbarch Conjecture, researchers considered certain subclasses of $\mathcal{S}$ which are determined by natural geometric conditions.

A domain D is called *a starlike domain with respect to* $w_0 \in D$ if $tw+(1-t)w_0 \in D$ whenever $w \in D$ for all $0 \leq t \leq 1$. A univalent function f in U is called *a starlike function with respect to* $w_0 \in f(U)$ if $f(U)$ is a starlike domain with respect to w_0. In particular, if $w_0 = 0$, then f is known as a starlike function. Let $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ which consists of starlike functions. An analytic characterization of $\mathcal{S}^*$ is given as follows.

Theorem 1.4. *[76, Theorem 2.2] A function $f \in \mathcal{S}^*$ if and only if $f \in \mathcal{S}$ and*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in U.$$

Since $\mathcal{S}^*$ contains the Koebe function and it is a compact subset of $H(U)$ (see [76, Theorem 4.1]), it can be proved that the Bieberbarch's Conjecture is true for the subclass $\mathcal{S}^*$ (see [76, Theorem 2.4]).

Another kind of function which is closely related to the starlike function is the convex function. A univalent function f in U is called *a convex function* if $f(U)$ is a *convex domain*, that is, $tw_1 + (1-t)w_2 \in f(U)$ for all $w_1, w_2 \in f(U)$ and $0 \leq t \leq 1$. Let $\mathcal{K}$ denote the subclass of $\mathcal{S}$ which consists of convex functions. Similarly, an analytic characterization of $\mathcal{K}$ is given by

Theorem 1.5. *[76, Theorem 2.6] A function $f \in \mathcal{K}$ if and only if $f \in \mathcal{S}$ and*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in U.$$

A close connection between classes $\mathcal{S}^*$ and $\mathcal{K}$ is shown in Alexander's theorem [69, Theorem 2.12] which states that $f \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$. The relation is then applied to deduce the coefficient bounds from the previously known coefficient bounds of $\mathcal{S}^*$ giving $|a_n| \leq 1$, $n \geq 2$ for all $f \in \mathcal{K}$.

1.3 Differential Subordinations

The famous Noshiro-Warschawski theorem states that if f is analytic in a convex domain D and

$$\operatorname{Re} f'(z) > 0, \quad z \in U,$$

then f is univalent in D (see [69, Theorem 2.16]). This theorem suggests the characterization of an analytic function through its derivative which is a type of the differential implications [95, p. 1]. Another example is the lemma proved by Miller, Mocanu and Reade [96]: if α is real and $p \in H(U)$ such that

$$\operatorname{Re}\left[p(z) + \alpha\frac{zp'(z)}{p(z)}\right] > 0 \quad \text{for all } z \in U,$$

then $\operatorname{Re} p(z) > 0$. Let $\mathscr{H} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ denote the right half-plane. In other words, if

$$p(z) + \alpha \frac{zp'(z)}{p(z)} \in \mathscr{H} \quad \text{for all } z \in U,$$

then $p(U) \subseteq \mathscr{H}$.

Let $H(U,U)$ denote the class of of all analytic self-map on U. Before making any further progress, recall that for functions $f, g \in H(U)$, g is said to be subordinate to f, written $g \prec f$, if $g = f \circ \phi$ for some $\phi \in H(U,U)$ with $\phi(0) = 0$. Further, if f is univalent in U, then $g \prec f$ if $g(0) = f(0)$ and $g(U) \subseteq f(U)$. Miller and Mocanu [95, p. 3] introduced the notion of differential subordination, which is the complex analogue of differential inequality by replacing the real variable concept with the theory of subordination.

Let Ω and Δ be sets in $\mathbb{C}$, let $p \in H(U)$ with $p(0) = a$ for some constant $a \in \mathbb{C}$ and let $\psi(r,s,t;z) : \mathbb{C}^3 \times U \to \mathbb{C}$. Then the following relation

$$\left\{\psi(p(z), zp'(z), z^2 p''(z); z) : z \in U\right\} \subset \Omega \quad \Rightarrow \quad p(U) \subset \Delta, \tag{1.2}$$

is a general formulation of function characterization. There are three problems that can be stated based on the inclusion (1.2).

(i) Given Ω and Δ, find the condition on ψ so that (1.2) holds. Such a ψ is called an admissible function.

(ii) Given ψ and Ω, find the smallest Δ so that (1.2) holds.

(iii) Given ψ and Δ, find the largest Ω so that (1.2) holds.

If Ω is a simply connected domain and $\Omega \neq \mathbb{C}$, then the Riemann mapping theorem ensures the existence of a unique conformal mapping h of U onto Ω such that $h(0) = \psi(a,0,0;0)$. Further, if $\psi(p(z),zp'(z),z^2p''(z);z) \in H(U)$, then in terms of subordination, (1.2) can be rewritten as

$$\psi(p(z),zp'(z),z^2p''(z);z) \prec h(z) \Rightarrow p(U) \subset \Delta.$$

If p is analytic in U, then p is called a *solution* of the (second-order) differential subordination. Further, if q is conformal mapping of U onto Δ such that $q(0) = a$, then (1.2) becomes

$$\psi(p(z),zp'(z),z^2p''(z);z) \prec h(z) \Rightarrow p(z) \prec q(z),$$

and the univalent function q is called a *dominant* if $p \prec q$ for all solutions p. Also, the *best dominant* $\tilde{q}$ is the dominant such that $\tilde{q} \prec q$ for all dominants q (see [95, p. 16]). The monograph [61] by Milller and Mocanu and references therein are excellent resources for the study on differential subordination.

1.4 Harmonic Mappings

Recall that a real-valued function $u(x,y) : \mathbb{R}^2 \to \mathbb{R}$, with continuous second partial derivatives, is (real) harmonic if it satisfies Laplace's equation:

$$\Delta_{x,y}\, u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A complex-valued function $f(x,y) = u(x,y) + iv(x,y)$ is harmonic if both u and v are (real) harmonic. Write $z = x + iy$. The Wirtinger derivatives (differential operators) are defined as follows:

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

Then for a complex-valued function f, f is harmonic if

$$f_{z\bar{z}}=\frac{\partial^2 f}{\partial z\partial \bar{z}}=\frac{1}{4}\Delta_{x,y}f=0.$$

If f is a complex-valued harmonic function defined on a simply connected domain $D\subset\mathbb{C}$, then f can be expressed as

$$f(z)=h(z)+\overline{g(z)}=a_0+\sum_{n=1}^{\infty}a_nz^n+\overline{\sum_{m=1}^{\infty}b_mz^m},$$

where h and g are analytic in D. If D is the unit disk U and $h(0)=f(0)$, then the representation is unique and is called the *canonical representation* of f (see [66, p. 7]). The Jacobian of f is given by

$$J_f(z)=|h'(z)|^2-|g'(z)|^2.$$

It is well known that (see [66, p. 2] or [92]) a complex-valued harmonic function f is locally one-to-one in D if and only if J_f is nonvanishing in D. Further, if $J_f>0$ in D then f is said to be *locally univalent* in D, that is, locally one-to-one and sense preserving in D. A complex-valued harmonic function f is said to be *univalent* in D if f is one-to-one and sense preserving in D.

A complex-valued harmonic function can also be viewed as a solution to a partial differential equation as stated in the following result:

Theorem 1.6. *([81, Lemma 2.1]) A complex valued function f defined in a domain D is open, harmonic and sense preserving in D if and only if there is an $a \in H(U,U)$ such that f is a non-constant solution of*

$$\overline{\left(\frac{\partial f}{\partial \bar{z}}\right)} = a\frac{\partial f}{\partial z}.$$

The theory of complex-valued harmonic functions serves as an active research area which can be seen from [11, 46, 66, 67, 68, 80, 81] as such mappings are closely related to the theory of minimal surfaces (see [99, 100]).

Throughout this thesis, we shall use the term harmonic function to indicate a complex-valued harmonic function.

1.5 Logharmonic Mappings

A logharmonic mapping defined in U is a solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{f_{\bar{z}}}}{\overline{f}} = a\frac{f_z}{f},$$

where $a \in H(U,U)$ is called the second dilatation function. Thus the Jacobian

$$J_f = |f_z|^2 (1-|a|^2)$$

is positive and all non-constant logharmonic mappings are therefore sense-preserving and open in U. In [44], the class of locally univalent logharmonic mappings is shown to play an instrumental role in validating the Iwaniec conjecture involving the Beurling-Ahlfors operator.

When f is a nonvanishing logharmonic mapping in U, it is known that f can be expressed as

$$f(z) = h(z)\overline{g(z)}, \tag{1.3}$$

where h and g are in $H(U)$. In [94], Mao *et al.* introduced the Schwarzian derivative for these nonvanishing logharmonic mappings. They established the Schwarz lemma for this class and obtained two versions of Landau's theorem. Denote by $\mathscr{P}_{LH}$ the class consisting of logharmonic mappings f in U of the form (1.3) satisfying $\operatorname{Re} f(z) > 0$ for all $z \in U$. The subclass $\mathscr{P}_{LH(M)}$ defined by

$$\mathscr{P}_{LH(M)} = \left\{ f : f = h(z)\overline{g(z)} \in \mathscr{P}_{LH}, \left| \frac{h(z)}{g(z)} - M \right| < M, M \geq 1 \right\}$$

was recently investigated in [101].

If f is a non-constant logharmonic mapping of U which vanishes only at $z = 0$, then [2] f admits the representation

$$f(z) = z^m |z|^{2\beta m} h(z)\overline{g(z)}, \tag{1.4}$$

where m is a nonnegative integer, $\operatorname{Re}\beta > -1/2$, and h and g are analytic functions on U satisfying $g(0) = 1$ and $h(0) \neq 0$. The exponent β in (1.4) depends only on $a(0)$ and can be expressed by

$$\beta = \overline{a(0)} \frac{1 + a(0)}{1 - |a(0)|^2}.$$

Note that $f(0) \neq 0$ if and only if $m = 0$, and that a univalent logharmonic mapping in U vanishes at the origin if and only if $m = 1$, that is, f has the form

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)}, \quad z \in U,$$

where $\operatorname{Re}\beta > -1/2$, $0 \notin (hg)(U)$ and $g(0) = 1$. This class has been widely studied in the works of [1, 2, 3, 4, 5]. In this case, it follows that $F(\zeta) = \log f(e^{\zeta})$ are univalent harmonic mappings of the half-plane $\{\zeta : \operatorname{Re}\zeta < 0\}$.

1.6 Spherical Chordal Distance

Let $\mathbb{S}$ denote the unit sphere $\{Z = (Z_1, Z_2, Z_3) \in \mathbb{R}^3 : |Z|^2 = 1\}$ and $N = (0,0,1)$ be its north pole. Then every point in the complex plane $\mathbb{C}$ corresponds to an unique point on $\mathbb{S}\backslash\{N\}$ via stereographic projection from N. Let $L_z(t) = (tx, ty, 1-t)$ be the line segment connecting N and $z = x + iy \in \mathbb{C}$ with coordinate (x, y) in the xy-plane. Note that L_z intersects $\mathbb{S}$ at a unique point Z indicating t satisfies the equation

$$(tx)^2 + (ty)^2 + (1-t)^2 = 1.$$

Thus $t = 2/(1 + |z|^2)$ giving

$$Z = \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|^2 - 1}{1+|z|^2} \right) = \left(\frac{z + \bar{z}}{1+|z|^2}, \frac{z - \bar{z}}{i(1+|z|^2)}, \frac{|z|^2 - 1}{1+|z|^2} \right).$$

Discussions on conformality and circles preserving properties of the stereographic projection can be found in [64, Problem 75]. The Euclidean distance between points Z and W on $\mathbb{S}$ is known as the *spherical chordal distance* between z and w, denoted by $\lambda(w,z)$, where

$$\lambda^2(Z,W) = (Z_1 - W_1)^2 + (Z_2 - W_2)^2 + (Z_3 - W_3)^2 = 2 - 2(Z_1W_1 + Z_2W_2 + Z_3W_3).$$

If Z and W are the stereographic projections of z and w in $\mathbb{C}$ respectively, then

$$\begin{aligned} Z_1W_1 + Z_2W_2 + Z_3W_3 =& \frac{(z+\bar{z})(w+\overline{w}) - (z-\bar{z})(w-\overline{w}) + (|z|^2-1)(|w|^2-1)}{(1+|z|^2)(1+|w|^2)} \\ =& \frac{2(z\overline{w}+\bar{z}w) + |zw|^2 - |z|^2 - |w|^2 + 1}{(1+|z|^2)(1+|w|^2)} \\ =& \frac{2(z\overline{w}+\bar{z}w) + (1+|z|^2)(1+|w|^2) - 2|z|^2 - 2|w|^2}{(1+|z|^2)(1+|w|^2)} \\ =& \frac{(1+|z|^2)(1+|w|^2) - 2|z-w|^2}{(1+|z|^2)(1+|w|^2)}. \end{aligned}$$

Thus

$$\lambda(z,w) = \frac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}.$$

1.7 Poincaré Disk Model

Recall the classical Schwarz's Lemma:

Theorem 1.7. *[60, p. 4](see also [85, Theorem 2.1]) Let f be an analytic self-map of U. If $f(0)=0$, then $|f(z)| \le |z|$ for all $z \in U$ and $|f'(0)| \le 1$. Further, if $|f(z_0)| = |z_0|$ for some $z_0 \in U \backslash \{0\}$, or if $|f'(0)| = 1$, then $f(z) = e^{i\theta} z$ for some constant $\theta \in \mathbb{R}$.*

A generalization of Schwarz's Lemma was presented by Pick [106], which is known as the Schwarz-Pick Lemma:

Theorem 1.8. *[60, p. 5](see also [85, Theorem 2.3]) If f is an analytic self-map of U, then*

(i)

$$\left|\frac{f(z)-f(w)}{1-f(z)\overline{f(w)}}\right| \leq \left|\frac{z-w}{1-z\overline{w}}\right| \quad \textit{for all } z,w \in U;$$

(ii)

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2} \quad \textit{for all } z,w \in U.$$

Equality occurs in both (i) and (ii) if f is an conformal automorphism of U. If equality holds in (i) for one pair of points $z \neq w$ or if equality holds in (ii) at one point z, then f is a conformal automorphism of U.

The unit disk U with the hyperbolic metric (see [31])

$$\lambda_U(z)|dz| = \frac{2|dz|}{1-|z|^2},$$

is known as the Poincaré disk model. By (ii), the metric $\lambda_U(z)|dz|$ is invariant under conformal automorphism of U and induces a distance function d_U on U by

$$d_U(z,w) = \inf_{\gamma} \int_{\gamma} \lambda_U(z)\,|dz|$$

over all smooth curves γ in U joining z to w. Similar to the invariance of Euclidean distance under rotation and translation in $\mathbb{C}$, d_U is invariant under conformal automorphism of U. It was shown in [31, Theorem 2.2] that

$$d_U(z,w) = \log \frac{1+p_U(z,w)}{1-p_U(z,w)} = 2\tanh^{-1} p_U(z,w),$$

where the pseudo-hyperbolic distance $p_U(z,w)$ is given by

$$p_U(z,w) = \left| \frac{z-w}{1-z\overline{w}} \right|.$$

1.8 Bohr's inequality

A series of the form $\sum_{n=1}^{\infty} a_n n^{-s}$ is an ordinary Dirichlet series, where $a_n, s \in \mathbb{C}$. Now, if the series converges for some $s_0 = \sigma_0 + it_0$, then it is convergent for all $s = \sigma + it$ with $\sigma > \sigma_0$ (see [79, Theorem 1]). Thus, the maximal domain of convergence is exactly the half-plane $\{s \in \mathbb{C} : \mathrm{Re}\, s > \sigma_c\}$ where

$$\sigma_c = \inf_{s\in\mathbb{C}} \left\{ \mathrm{Re}\, s : \sum_{n=1}^{\infty} \frac{a_n}{n^s} < \infty \right\}.$$

The term σ_c is then known as the *the abscissa of convergence* for $\sum_{n=1}^{\infty} a_n n^{-s}$. Similarly, the quantity

$$\sigma_a = \inf_{\sigma} \left\{ \sigma \text{ is real} : \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}} < \infty \right\},$$

is called the *the abscissa of absolute convergence* for $\sum_{n=1}^{\infty} a_n n^{-s}$. Finally, *the abscissa of uniform convergence* for $\sum_{n=1}^{\infty} a_n n^{-s}$ is defined to be the unique real number σ_u such that the Dirichlet series converges uniformly in the half-plane $\{s \in \mathbb{C} : \mathrm{Re}\, s > \sigma_u\}$.

In 1913, Harald Bohr published the *absolute convergence problem* [39] which asked for the value of

$$S_0 := \sup(\sigma_a - \sigma_u),$$

where the supremum is taken over all ordinary Dirichlet series. In fact, this problem can be reduced to a problem on power series in an infinite number of complex variables [39, 38], which allowed Bohr to obtain the inequality $S_0 \leq 1/2$ [39, Satz X]. While attempting the absolute convergence problem, Bohr returned to the one dimensional case and proved the Bohr's inequality (or Bohr's theorem):

Theorem 1.9. *([40]) If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(U,U)$, then*

$$\sum_{n=0}^{\infty} |a_n z^n| \leq 1 \tag{1.5}$$

for $|z| \leq 1/6$.

The value $1/6$ is further improved independently by Riesz, Schur and Wiener to $1/3$ which is optimal. Other proofs can also be found in [102, 112, 115]. Thus $1/3$ is then known as the *Bohr radius* of $H(U,U)$, and the class $H(U,U)$ is said to have Bohr phenomenon. The notion of the Bohr phenomenon was first introduced by Bénéteau, Dahlner and Khavinson [32] for a Banach space X of analytic functions on the disk U. The Bohr's inequality (1.5) can also be put in the form

$$d\left(\sum_{n=0}^{\infty} |a_n z^n|, |f(0)|\right) \leq d(f(0), \partial U), \tag{1.6}$$

where d is the Euclidean distance and ∂U the unit circle. Further, the Bohr's inequality can be paraphrased in terms of the supremum norm, $\|f\|_\infty$:

$$\sum_{n=0}^{\infty} |a_n z^n| \le \|f\|_\infty = \sup_{|z|<1} |f(z)|. \tag{1.7}$$

1.9 About the thesis

1.9.1 Background - Bohr and distances

For an analytic function f defined in U of the form (1.1), define its associated *majorant function* [36] by

$$\mathcal{M}f(z) := \sum_{n=0}^{\infty} |a_n| z^n.$$

If $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is another analytic function on U, then

$$\begin{aligned} \mathcal{M}(f+g)(|z|) \le& \mathcal{M}f(|z|) + \mathcal{M}g(|z|); \\ \mathcal{M}(fg)(|z|) \le& \mathcal{M}f(|z|)\mathcal{M}g(|z|). \end{aligned} \tag{1.8}$$

Recall the classical Bohr's theorem with Bohr's inequality of the form in (1.6):

Theorem 1.10. *If $f \in H(U,U)$, then*

$$d\left(\mathcal{M}f(|z|), |f(0)|\right) \le d(f(0), \partial U)$$

for $|z| \le 1/3$, where d is the Euclidean distance, and ∂U is the boundary of U. The radius $1/3$ is sharp.

The research on investigating the Bohr's theorem in distance form was initiated by Aizenberg. He proved that

Theorem 1.11. *[17, Theorem 2.1] Let f be an analytic function from U into a domain $G \subset \mathbb{C}$. Further suppose the convex hull $\tilde{G}$ of G satisfies $\tilde{G} \neq \mathbb{C}$. Then*

$$d(\mathcal{M}f(|z|), |f(0)|) \leq d(f(0), \partial\tilde{G})$$

for $|z| \leq 1/3$. The value $1/3$ is the best, provided there exists a point $p \in \mathbb{C}$ satisfying $p \in \partial\tilde{G} \cap \partial G \cap \partial D$ for some disk $D \subset G$.

The result covered the case where G is a convex domain and so extended the classical Bohr's theorem where $G = U$. The domain G was further extended by Abu-Muhanna [6] by using the technique of subordination. He applied both the Koebe one-quarter theorem and de Branges's theorem, or the Bieberbarch's conjecture, to prove

Theorem 1.12. *[6, Theorem 1] Let f be a univalent (analytic and injective) function on U. If $g \prec f$, then*

$$d\left(\mathcal{M}g(|z|), |g(0)|\right) \leq d(f(0), \partial f(U))$$

for $|z| \leq 3 - 2\sqrt{2} \approx 0.17157$. The sharp radius $3 - 2\sqrt{2}$ is attained by the Koebe function $z/(1-z)^2$.

Recently, Abu-Muhanna and Ali [7] studied the class $H(U, \Omega)$ where Ω is a domain exterior to a compact convex set and proved

Theorem 1.13. *Suppose that the universal covering map from U into Ω has a univalent logarithmic branch that maps U into the complement of a convex set. If $0 \notin \Omega$, $1 \in \partial\Omega$*

and $f \in H(U,\Omega)$ *with* $f(0) > 1$, *then for* $|z| < 3-2\sqrt{2} \approx 0.17157$,

$$\lambda\left(\mathcal{M}f(|z|), |f(0)|\right) \leq \lambda\left(f(0), \partial\Omega\right),$$

where λ *is the spherical chordal distance. In particular, if* $\overline{G}$ *is the closed unit disk, then the sharp radius* $1/3$ *is obtained.*

Meanwhile, a link was established between the Bohr's inequality for classes of analytic functions $H(U,G)$ and the hyperbolic metric done by Abu-Muhanna and Ali [8] in the following year. That paper discussed the case where G is the right half-plane, the slit region and the exterior of U.

We end this subsection by stating the Bohr's inequality for bounded harmonic mappings as proved by Abu-Muhanna [6].

Theorem 1.14. *[6, Theorem 2] Let* $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ *be a complex-valued harmonic function on* U. *If* $|f(z)| < 1$ *for all* $z \in U$, *then*

$$\sum_{n=1}^{\infty} |e^{i\mu} a_n + e^{-i\mu} b_n||z|^n \leq d(|\mathrm{Re}\, e^{i\mu} a_0|, \partial U), \quad \textit{for any } \mu \in \mathbb{R},$$

for $|z| \leq 1/3$. *The radius* $1/3$ *is sharp.*

1.9.2 Scope of thesis

The aim of the research work is to extend Theorem 1.10 by

(a) establishing the Bohr's theorem for the class of analytic functions mapping U into some non-convex domain D,

(b) replacing the Euclidean distance with other distances, and

(c) extending the Bohr's theorem to some subclasses of analytic functions as well as classes of non-analytic functions.

The thesis is divided into six chapters. Briefly, Chapter 2 discusses the Bohr's theorem for the class $R(\alpha,\gamma,h)$ consisting of functions f which are analytic in U and satisfying the differential subordination relation

$$f(z)+\alpha z f'(z)+\gamma z^2 f''(z) \prec h(z), \quad z\in U, \alpha\geq\gamma\geq 0.$$

The Bohr's theorems are developed for the case when h is a convex function in U as well as the case when h is starlike with respect to $h(0)$. The results are proved by applying the Koebe one-quarter theorem and the theory of differential subordination. Simply note that if $\alpha=\gamma=0$, then the Bohr radii $1/3$ (convex h) and $3-\sqrt{2}$ (starlike h) are the known radii in Theorem 1.11 and Theorem 1.12, respectively.

Chapter 3 consists of two sections. The first section studies the Bohr's theorem for the class of analytic functions mapping the unit disk U to concave-wedge domains

$$W_\alpha=\left\{w\in\mathbb{C}:|\arg w|<\frac{\alpha\pi}{2}\right\}, \quad 1\leq\alpha\leq 2.$$

The Bohr radius is obtained by using the technique of subordination and has the value $(2^{\frac{1}{\alpha}}-1)/(2^{\frac{1}{\alpha}}+1)$. In particular, if $\alpha=1$, then the Bohr radius is $1/3$ as stated in Theorem 1.11 and $3-2\sqrt{2}$ for $\alpha=2$ as stated in Theorem 1.12. The next section focuses on the class of analytic functions f that maps the unit disk U to the punctured unit disk $U_0 = U \backslash \{0\}$. The development of the Bohr's theorem depends heavily on the coefficient estimate obtained by Koepf and Schmersau [86, p. 248] as well as the Herglotz representation theorem for analytic functions [76, Corollary 3.6].

Chapter 4 focuses on developing the Bohr's theorem in non-Euclidean geometry. The classical Bohr's theorem with respect to the spherical chordal distance λ defined by

$$\lambda(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}, \quad z_1, z_2 \in U,$$

is shown to have value $1/3$. The first section also shows that by replacing the Euclidean distance d with λ, it is possible to slightly improve the constraint in a Bohr's theorem obtained in earlier chapter. The hyperbolic Bohr's theorem is presented in the following section. By defining the hyperbolic unit disk U^h in the Poincaré disk model, an analogous Bohr's theorem for the class of analytic self-maps of U^h is obtained and the (hyperbolic) Bohr radius has the value $\tanh(1/2)/3$. Further, Theorem 1.11 has its hyperbolic version in the Poincaré disk model and the Bohr radius is shown to be $\tanh(1/2)/3$, implying the invariance of Bohr radius in hyperbolic geometry. Additionally, the main theorem is applied to obtain the Bohr-type theorem for other hyperbolic regions.

Chapter 5 is devoted to studying the Bohr's theorem in the class of non-analytic functions. The Bohr's theorem for the class of harmonic functions mapping U into a bounded domain in $\mathbb{C}$ can be found in the first section. In particular, if the bounded domain is taken to be U itself, then the Bohr's theorem is reduced to Theorem 2 in [6]. The Bohr's theorem for the class of univalent, harmonic, orientation-preserving mappings of U into the convex wedge

$$W = \{w \in \mathbb{C} : |\arg w| < \pi/4\}.$$

is established as well. Both the Bohr's theorems are shown to have the same Bohr radius $1/3$. The final section deals with the construction of Bohr-type inequality for the class of univalent logharmonic functions f of the form $f(z) = zh(z)\overline{g(z)}$ mapping U onto a domain which is starlike with respect to the origin. The distortion theorem for this class of functions can also be found in this section.

Chapter 6 serves as a survey of the work on developing Bohr's theorem. There are several directions in extending the classical Bohr's theorem. Among those researches, the n-dimensional Bohr radii study is very much well developed and the *first Bohr radius* (see Chapter 6, Section 6.1) has its asymptotic value proved to be $\sqrt{\log n/n}$ in [30] recently.

CHAPTER 2

BOHR AND DIFFERENTIAL SUBORDINATIONS

In this chapter, we shall investigate a special class of differential subordination $R(\alpha,\gamma,h)$. For $\alpha \geq \gamma \geq 0$, and for a given univalent function $h \in H(U)$, let

$$R(\alpha,\gamma,h) := \left\{ f \in H(U) : f(z) + \alpha z f'(z) + \gamma z^2 f''(z) \prec h(z), \quad z \in U \right\}.$$

The investigation of such functions f can be seen as an extension to the study of the class

$$R(\alpha,h) = \left\{ f \in H(U) : f'(z) + \alpha z f''(z) \prec h(z), \quad z \in U \right\}$$

or its variations for an appropriate function h. This class has been investigated in several works, and more recently in [114, 116]. It was shown in Ali *et. al* [23] that $f(z) \prec h(z)$ whenever $f \in R(\alpha,\gamma,h)$. The notion of convolution will be needed to deduce the latter assertion.

For two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $H(U)$, the Hadamard product (or convolution) of f and g is the function $f * g$ defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

The following auxiliary function will be useful: let

$$\phi_\lambda(z) = \int_0^1 \frac{dt}{1 - zt^\lambda} = \sum_{n=0}^{\infty} \frac{z^n}{1 + \lambda n}.$$

From [110] it is known that ϕ_λ is convex in U provided $\operatorname{Re}\lambda \geq 0$.

Now for $\alpha \geq \gamma \geq 0$, let

$$\nu + \mu = \alpha - \gamma, \quad \mu\nu = \gamma,$$

and

$$q(z) = \int_0^1 \int_0^1 h(zt^\mu s^\nu) dt ds = (\phi_\nu * \phi_\mu) * h(z) \in R(\alpha, \gamma, h). \tag{2.1}$$

Let $S(h) := \{f \in H(U) : f \prec h\}$ denote the class of analytic functions on U subordinate to h. In [23], Ali *et. al* showed that

$$f(z) \prec q(z) \prec h(z)$$

for every $f \in R(\alpha, \gamma, h)$. Thus $R(\alpha, \gamma, h) \subset S(h)$.

2.1 $R(\alpha, \gamma, h)$ with convex h

The following result gives the Bohr radius for $R(\alpha, \gamma, h)$ with convex function h.

Theorem 2.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in R(\alpha, \gamma, h)$, and $h \in \mathcal{S}$ be convex. Then*

$$d(\mathcal{M}f(|z|), |f(0)|) = \sum_{n=1}^{\infty} |a_n z^n| \leq d(h(0), \partial h(U))$$

for all $|z| \le r_{CV}(\alpha,\gamma)$*, where* $r_{CV}(\alpha,\gamma)$ *is the smallest positive root of the equation*

$$(\phi_\mu * \phi_\nu)(r) - 1 = \sum_{n=1}^{\infty} \frac{1}{(1+\mu n)(1+\nu n)} r^n = \frac{1}{2}.$$

Further, this bound is sharp. An extremal case occurs when $f(z) := q(z)$ *as defined in* (2.1) *and* $h(z) := z/(1-z)$*.*

Proof. Let $F(z) = f(z) + \alpha z f'(z) + \gamma z^2 f''(z) \prec h(z)$. Then

$$F(z) = \sum_{n=0}^{\infty} [1 + \alpha n + \gamma n(n-1)] a_n z^n,$$

and

$$\frac{1}{h'(0)} \sum_{n=1}^{\infty} [1 + \alpha n + \gamma n(n-1)] a_n z^n = \frac{F(z) - F(0)}{h'(0)} \prec \frac{h(z) - h(0)}{h'(0)}.$$

It follows from [69, Theorem 6.4(i)] that

$$\left| \frac{1 + \alpha n + \gamma n(n-1)}{h'(0)} \right| |a_n| \le 1, \quad n \ge 1.$$

Hence

$$|a_n| \le \frac{|h'(0)|}{1 + (\mu+\nu)n + \mu\nu n^2}, \quad n \ge 1,$$

which readily yields

$$\sum_{n=1}^{\infty} |a_n| r^n \le \sum_{n=1}^{\infty} \frac{|h'(0)|}{1 + (\mu+\nu)n + \mu\nu n^2} r^n.$$

Since $H(z) = \frac{h(z)-h(0)}{h'(0)}$ is a normalized convex function on U, it follows from [69,