ADAPTIVE CONTROL CHARTS FOR MONITORING THE UNIVARIATE AND MULTIVARIATE COEFFICIENT OF VARIATION

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by

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LIST OF ABBREVIATIONS

The abbreviations used in this thesis are presented as follows:

Abbreviations	Details
AATS	Adjusted average time to signal
ARL	Average run length
ARL_0	In-control average run length
ARL_1	Out-of-control average run length
ASI	Average sampling interval
ASI_0	In-control average sampling interval
ASI_1	Out-of-control average sampling interval
ASS	Average sample size
ASS_0	In-control average sample size
ASS_1	Out-of-control average sample size
ATS	Average time to signal
ATS_0	In-control average time to signal
ATS_1	Out-of-control average time to signal
CAT	Computerized axial tomography
CCC	Cumulative count of conforming
cdf	Cumulative distribution function
CRL	Conforming run length
CUSUM	Cumulative sum
CV	Coefficient of variation
CV^2	Squared of the CV statistic
DOE	Design of experiments

EARL	Expected average run length
$EARL_0$	In-control expected average run length
$EARL_1$	Out-of-control expected average run length
EATS	Expected average time to signal
$EATS_0$	In-control expected average time to signal
$EATS_1$	Out-of-control expected average time to signal
EMTS	Expected median time to signal
EWMA	Exponentially weighted moving average
FSI	Fixed sampling interval
LCL	Lower control limit
LWL	Lower warning limit
MCV	Multivariate coefficient of variation
MEWMA	Multivariate exponentially weighted moving average
MRL	Median run length
MTS	Median time to signal
pdf	Probability density function
RR	Runs rule
$RR_{2,3}$	2-out-of-3 runs rule
$RR_{3,4}$	3-out-of-4 runs rule
$RR_{4,5}$	4-out-of-5 runs rule
SDRL	Standard deviation of the run length
$SDRL_0$	In-control standard deviation of the run length
$SDRL_1$	Out-of-control standard deviation of the run length
SDTS	Standard deviation of the time to signal
$SDTS_0$	In-control standard deviation of the time to signal

$SDTS_1$	Out-of-control standard deviation of the time to signal
SH	Shewhart
SPC	Statistical Process Control
SQC	Statistical Quality Control
Syn	Synthetic
tpm	Transition probability matrix
UCL	Upper control limit
UWL	Upper warning limit
VSI	Variable sampling interval
VSS	Variable sample size
VSSI	Variable sample size and sampling interval
VSSIWL	VSSI with warning limits
WLC	Weighted-loss-function based CUSUM scheme

LIST OF NOTATIONS

The notations used in this thesis are presented as follows:

Notations	Details
n	Fixed sample size of the SH CV, Syn CV, RR CV, EWMA CV^2, VSI CV and standard MCV charts or average sample size of the VSS CV, VSSI CV, VSS MCV and VSSI MCV charts
n_1	Small sample size
n_2	Large sample size
t	Fixed sampling interval of the SH CV, Syn CV, RR CV, EWMA CV^2, standard MCV and VSS MCV charts or average sampling interval of the VSI CV, VSSI CV, VSI MCV and VSSI MCV charts
t_1	Short sampling interval
t_2	Long sampling interval
γ	Population CV or population MCV
γ_0	In-control population CV or in-control population MCV
γ_1	Out-of-control population CV or out-of-control population MCV
$\hat{\gamma}$	Sample CV or sample MCV
$\hat{\gamma}_0$	In-control sample CV or in-control sample MCV
μ_0	In-control mean
σ_0	In-control standard deviation
τ	Size of a CV or MCV shift
$\tau_{\min}$	Lower bound of τ
$\tau_{\max}$	Upper bound of τ
α	Type-I error probability
α'	Warning limit's parameter of the VSSI MCV, VSI MCV and VSS MCV charts
m	Number of in-control Phase-I samples
K	Control limit's parameter of the VSSI CV chart
W	Warning limit's parameter of the VSSI CV chart

P_{ij}	Transition probability from state i to state j
b_1	Probability of using the pair (n_1, t_2) as the initial sample size and sampling interval
b_2	Probability of using the pair (n_2, t_1) as the initial sample size and sampling interval
p	Number of quality characteristics monitored simultaneously
δ	Non-centrality parameter for the non-central F distribution
$\boldsymbol{P}$	Transition probability matrix with the transient and absorbing states
$\boldsymbol{b}$	Initial probability vector
$\mathbf{I}$	Identity matrix
$\boldsymbol{Q}$	Transition probability matrix with the transient states
$\boldsymbol{t}$	Vector of sampling intervals
$\mathbf{1}$	Vector with all elements unity
$f_\tau(\tau)$	Probability density function of τ
$F_{\hat{\gamma}}(\cdot)$	Cumulative distribution function of $\hat{\gamma}$
$F_{\hat{\gamma}}^{-1}(\cdot)$	Inverse cumulative distribution function of $\hat{\gamma}$
$F_t\left(\cdot \middle| n-1, \frac{\sqrt{n}}{\gamma}\right)$	Cumulative distribution function of a non-central t random variable with $n-1$ degrees of freedom and non-centrality parameter $\frac{\sqrt{n}}{\gamma}$
$F_t^{-1}\left(\cdot \middle| n-1, \frac{\sqrt{n}}{\gamma}\right)$	Inverse cumulative distribution function of a non-central t random variable with $n-1$ degrees of freedom and non-centrality parameter $\frac{\sqrt{n}}{\gamma}$
$F_F(\cdot \mid p, n-p, \delta)$	Cumulative distribution function of a non-central F distribution with p and $n-p$ degrees of freedom and non-centrality parameter δ

$F_F^{-1}\left(\cdot \mid p, n-p, \delta \right)$	Inverse cumulative distribution function of a non-central F distribution with p and $n - p$ degrees of freedom and non-centrality parameter δ
$\Phi^{-1}\left(\cdot \right)$	Inverse standard normal cumulative distribution function

SH CV Chart

M	Probability of an out-of-control signal on the SH CV chart

Syn CV Chart

L	Lower control limit of the CRL sub-chart of the Syn CV chart
g	Probability of an out-of-control signal on the Syn CV chart

RR CV Chart

$W_{2,3}$	Limit parameter of the 2-out-of-3 RR CV chart
$W_{3,4}$	Limit parameter of the 3-out-of-4 RR CV chart
$W_{4,5}$	Limit parameter of the 4-out-of-5 RR CV chart
P_L	Probability of $\hat{\gamma}$ plotting below LWL_{RR}
P_U	Probability of $\hat{\gamma}$ plotting above UWL_{RR}
P_C	Probability of $\hat{\gamma}$ plotting between LWL_{RR} and UWL_{RR}
$\boldsymbol{Q}_{2,3}$	Transition probability matrix with the transient states for the 2-out-of-3 RR CV chart
$\boldsymbol{Q}_{3,4}$	Transition probability matrix with the transient states for the 3-out-of-4 RR CV chart
$\boldsymbol{Q}_{4,5}$	Transition probability matrix with the transient states for the 4-out-of-5 RR CV chart

$\boldsymbol{G}$	Transition probability matrix with the transient and absorbing states for the 2-out-of-3, 3-out-of-4 and 4-out-of-5 RR CV charts
$\mathbf{q}_{2,3}$	Initial probability vector of the 2-out-of-3 RR CV chart
$\mathbf{q}_{3,4}$	Initial probability vector of the 3-out-of-4 RR CV chart
$\mathbf{q}_{4,5}$	Initial probability vector of the 4-out-of-5 RR CV chart

EWMA CV^2 Chart

K_{EWMA}	Limit constant of the EWMA CV^2 chart
λ	Smoothing constant of the EWMA CV^2 chart
$\boldsymbol{A}$	Transition probability matrix with the transient and absorbing states for the EWMA CV^2 chart
$\hat{\gamma}^2$	Squared of the sample CV statistic
H_i	Midpoint of the i^{th} subinterval
H_j	Midpoint of the j^{th} subinterval
$\mathbf{s}$	Initial probability vector of the EWMA CV^2 chart
$F_{\hat{\gamma}^2}(\cdot)$	Cumulative distribution function of $\hat{\gamma}^2$
$F^{-1}_{\hat{\gamma}^2}(\cdot)$	Inverse cumulative distribution function of $\hat{\gamma}^2$

VSS CV Chart

W_{VSS}	Warning limit’s parameter of the VSS CV chart
K_{VSS}	Control limit’s parameter of the VSS CV chart
$\boldsymbol{B}$	Transition probability matrix with the transients and absorbing states for the VSS CV chart
$\mathbf{d}$	Initial probability vector of the VSS CV chart

VSI CV Chart

W_{VSI}	Warning limit's parameter of the VSI CV chart
K_{VSI}	Control limit's parameter of the VSI CV chart
P_H	Probability of $\hat{\gamma}$ plotting between $\text{LWL}_{\text{VSI CV}}$ and $\text{UWL}_{\text{VSI CV}}$
P_S	Probability of $\hat{\gamma}$ plotting between $\text{LCL}_{\text{VSI CV}}$ and $\text{LWL}_{\text{VSI CV}}$ or $\text{UWL}_{\text{VSI CV}}$ and $\text{UCL}_{\text{VSI CV}}$
q	Probability of $\hat{\gamma}$ plotting above $\text{UCL}_{\text{VSI CV}}$ or below $\text{LCL}_{\text{VSI CV}}$

Standard MCV Chart

R	Probability of an out-of-control signal on the MCV chart

CARTA-CARTA KAWALAN PENYESUAIAN UNTUK PERMANTAUAN PEKALI VARIASI UNIVARIAT DAN MULTIVARIAT

ABSTRAK

Teknik carta kawalan telah digunakan dalam pelbagai bidang. Dalam bidang-bidang tertentu, seperti kewangan dan kesihatan, carta-carta kawalan tradisional untuk memantau min proses dan varians proses tidak dapat berfungsi dengan baik kerana min dan varians bukan tak bersandar antara satu sama lain. Dalam keadaan sedemikian, pekali variasi (CV) yang merupakan nisbah sisihan piawai kepada min perlu dipantau. Objektif pertama kajian ini adalah untuk mencadangkan suatu carta kawalan penyesuaian univariat CV dengan menggunakan pendekatan saiz sampel dan selang pensampelan berubah (VSSI), yang dikenali sebagai carta VSSI CV untuk memantau CV proses. Carta VSSI CV direka bentuk secara optimum, yang mana dua parameter, iaitu saiz sampel dan selang persampelan, dibenarkan untuk berubah. Dalam senario sebenar, terdapat banyak situasi yang mana permantauan serentak dua atau lebih cirian kualiti yang berkorelasi diperlukan. Kesimpulan yang salah akan berlaku jika pengamal kualiti menggunakan carta-carta kawalan univariat untuk memantau suatu proses multivariat. Objektif kedua kajian ini adalah untuk mencadangkan carta-carta CV untuk memantau CV proses multivariat (MCV) dengan menggunakan prosedur penyesuaian. Tiga carta baru bagi permantauan MCV, iaitu carta MCV selang persampelan berubah (dengan mengubah selang persampelan), carta MCV saiz sampel berubah (dengan mengubah saiz sampel) dan carta VSSI MCV (dengan mengubah kedua-dua saiz sampel dan selang pensampelan) dicadangkan untuk meningkatkan prestasi carta MCV yang sedia ada. Kesemua carta yang dicadangkan untuk memantau CV univariat dan multivariat direka bentuk dengan

menggunakan pendekatan rantai Markov. Prosedur pelaksanaan dan reka bentuk pengoptimuman carta-carta yang dicadangkan dibentangkan dalam tesis ini. Carta yang dicadangkan dinilai dengan menggunakan kriteria masa purata untuk memberi isyarat (ATS), sisihan piawai masa untuk memberi isyarat (SDTS) dan jangkaan masa purata untuk memberi isyarat (EATS). Perbandingan prestasi carta-carta CV univariat dan multivariat yang dicadangkan dengan carta-carta sedia ada dijalankan, yang mana carta-carta yang dicadangkan adalah lebih baik daripada carta-carta sedia ada untuk mengesan anjakan CV dan MCV yang kecil dan sederhana. Aplikasi carta-carta optimum VSSI CV dan VSSI MCV ditunjukkan dengan contoh yang menggunakan data sebenar dari industri pembuatan.

ADAPTIVE CONTROL CHARTS FOR MONITORING THE UNIVARIATE AND MULTIVARIATE COEFFICIENT OF VARIATION

ABSTRACT

Control charting techniques have been applied in a wide variety of areas. In certain areas, such as in finance and healthcare, the traditional control charts for monitoring the process mean and process variance are unable to work well, as the mean and variance are not independent of one another. In such circumstances, the coefficient of variation (CV), which is the ratio of the standard deviation to the mean should be monitored. The first objective of this research is to propose a univariate adaptive CV chart using the variable sample size and sampling interval (VSSI) approach, called the VSSI CV chart, to monitor the process CV. The VSSI CV chart will be optimally designed, where two parameters, namely the sample size and sampling intervals are allowed to vary. In real life scenarios, there are many situations in which a simultaneous monitoring of two or more correlated quality characteristics is necessary. Erroneous conclusions will occur if quality practitioners use univariate control charts to monitor a multivariate process. The second objective of this study is to propose CV charts for monitoring the multivariate process CV (MCV) by adopting the adaptive procedures. Three new charts for monitoring the MCV, namely the variable sampling interval MCV (by varying the sampling interval), variable sample size MCV (by varying the sample size) and VSSI MCV (by varying both the sample size and sampling interval) charts are proposed to improve the performance of the existing MCV chart. All the charts proposed for monitoring the univariate and multivariate CVs are designed using the Markov chain approach. The implementation procedures and optimization designs of these proposed charts are enumerated in this thesis. The new proposed charts are evaluated using the average time to signal (ATS), standard deviation of the time to signal (SDTS) and expected average time to signal (EATS) criteria. Performance comparisons of the proposed univariate and multivariate CV charts with the existing charts are conducted, where the proposed charts outperform the existing charts for detecting small and moderate CV and MCV shifts. The application of the optimal VSSI CV and VSSI MCV charts are demonstrated with examples using real data from manufacturing industries.

CHAPTER 1
INTRODUCTION

1.1 Introduction to Statistical Quality Control (SQC)

SQC involves the use of statistical quality control tools for monitoring and evaluating the quality of products and services in a variety of areas. SQC has been commonly recognized after World War II. During World War II, the need for producing reliable electronic equipment and weapons, with low costs, has reinforced the use of SQC. SQC contains three broad categories, namely Statistical Process Control (SPC), Design of Experiments (DOE) and acceptance sampling.

DOE is an approach that varies the input factors in a process systematically, in order to study the impact of the input factors on the output product parameters. DOE is considered as an off-line quality control tool, and it is used in the early stages of manufacturing. SPC refers to a collection of statistical tools that is used to inspect a random sample taken from a process output, for determining whether the process is producing predetermined products. If the process is producing predetermined products, the production process will be continued. Otherwise, the production process may not function properly and it should be stopped or continued after process adjustment is made. Acceptance sampling is an approach of inspecting a certain number of random samples from a batch of goods, and then deciding whether this batch of goods should be accepted or rejected based on the findings of the inspection. Acceptance sampling is a useful tool that provides important information to quality inspectors in accepting or rejecting a particular batch of products. A setback of this method is that the production cost will increase as a new batch of goods need to be reproduced if the current batch of goods fails on quality inspection. SPC can circumvent this problem as it identifies the assignable causes during the production process. An inspector can terminate the production process immediately in order to avoid waste of raw materials and to reduce the unnecessary costs. Thus, an organization usually implements SPC instead of acceptance sampling (Montgomery, 2009).

Statistical tools are often used on two types of data, namely attribute and variable data. Attribute data are data having quality characteristics with outcomes that can be grouped into two categories, such as absent or present, conforming or non-conforming, defective or non-defective, etc. The quality characteristics of variable data are variables that can be measured, for example, weight and diameter. Attribute data only provide information on whether a product is good or bad, but the data do not show how good or bad a product is (Lind et al., 2011).

In a real manufacturing process, the characteristics, such as the dimension, shape and central tendency of the distribution of two identical products will not be exactly similar. There will be some variation between them. Two general causes of variation in a process are the common and assignable causes of variation. Common causes of variation are random in nature and cannot be totally omitted. For example, a rapid change of the humidity and temperature, due to the atmospheric conditions in a factory is beyond control. The second type of variation is due to assignable causes. This type of variation is non-random, correctable and can be precisely identified and eliminated by finding the assignable causes.

According to Besterfield (2004), SPC improves the quality of products and increases profit. SPC is useful in reducing errors and rework in a process and it also helps in the decision-making process. Through the use of SPC, a process is managed by facts and statistics (and not based on subjective opinions). This will enhance an organization’s efficiency and effectiveness in producing quality products (Garrity, 1993). In SPC, there are seven common diagnostic tools to investigate quality problems. These tools, often called “the magnificent seven”, comprise the Pareto chart, histogram, fishbone diagram, check sheet, scatter diagram, run chart and control chart (Montgomery, 2009).

The Pareto chart, often called the 80-20 rule, emphasizes that roughly 80 percent of the effects on the output come from 20 percent of the causes. The Pareto chart identifies the factors that should be prioritized. A histogram is commonly used to study the frequency distribution and it is a graphical method that is used as a first assessment in statistical measurements. Fishbone diagram, also known as the cause-and-effect diagram, is used to identify a set of possible causes that produce a particular effect. Check sheet is a technique of collecting and analyzing data using structured and prepared forms. It is a useful technique in observing and collecting data from instruments or processes that are repeated daily, monthly or annually. A scatter diagram is a graphical display that investigates the relationship between two variables. Run chart is an approach that separates the variety of sources lumped together, in order to let the analyst analyze the data independently. A control chart is a graph that shows the measured quality characteristic from a sample versus the sample number or time.

1.2 Control Charting Technique

The control charting technique was developed by Dr. Walter A. Shewhart, who is also known as the father of quality control. A control chart signals the presence of an assignable cause of variation in the process when a sample point falls in the out-of-control region. The in-control and out-of-control regions are separated by a set of borders, called the upper and lower control limits. A control chart contains a center line, which represents the average value of the quality characteristic corresponding to the in-control process. A control chart is a useful tool in SPC as it is a proven technique for improving productivity, reducing defective products and unnecessary process adjustments, and providing key information on the process (Montgomery, 2009).

A control chart is often classified based on the type of quality characteristic, which comprises either variables or attribute data. Variable control charts are used when quality practitioners wish to monitor data that can be measured on a continuous scale, while attribute control charts are applied when monitoring the data that have a discrete value. The common charts for monitoring variable data are the Shewhart $\bar{X}$ chart, which monitors the process mean and the range chart (or R chart) that monitors the process variability. Some organizations prefer to apply both the $\bar{X}$ and R charts at the same time, for monitoring the process mean and variance, respectively. Control charts commonly used to control attribute data are the c chart and p chart. Both the c and p charts are constructed based on the Poisson and binomial distributions, respectively. The c chart plots the number of defects or failures per unit while the p chart plots the percentage of defective items in a sample. For the charts mentioned above, when a sample point plotted within the control limits, the process is in-control. Otherwise, the process is out-of-control and corrective actions need to be taken (Reid and Sanders, 2012).

Control charting techniques have gained increasing importance recently due to the rapid advancement in technology. Since control charting techniques are easy to apply, many industries tend to use control charts to monitor the quality of their products or services. This is evident from a wide variety of publications. For instance, Ghute and Shirke (2008) reported the spring manufacturing industry in their research on the multivariate synthetic control chart for monitoring the process mean vector while Castagliola et al. (2011) considered an example based on real data from the sintering process in manufacturing mechanical parts. The data from the die casting hot chamber process in manufacturing zinc alloy (ZAMAK) parts for the sanitary sector was studied by Castagliola et al. (2013a) in their construction of the variable sampling interval (VSI) chart for monitoring the coefficient of variation (CV).

Scordaki and Psarakis (2005), and Woodall (2006) showed the effectiveness of using control charts in healthcare industry. The variations in antibiotic prescriptions among different doctors were examined by Marshall and Mohammed (2003) using control charts. Rodriquez and Ransdell (2010) applied a control chart to analyze data from computerized axial tomography (CAT) scans. Control charting techniques are also applied to monitor financial data. Dull and Tegarden (2004) noted that control charts can be used to detect irregular patterns in financial data.

More recently, research works on control charting techniques have been shown through various publications, such as those by, Haq and Khoo (2016), Haridy et al. (2016), Tran et al. (2016), Celano et al. (2016), Rakitzis et al. (2016), Maleki et al. (2017), Gunaratne et al. (2017), Nenes et al. (2017), Teoh et al. (2017) and Bersimis et al. (2017).

1.3 Problem Statement

Control charting techniques have been applied to a wide variety of fields. Unfortunately, in certain industries, such as in some cases in finance and healthcare, traditional control charts for monitoring the process mean and variance may not work well, as the process mean and the process variance in those cases are not independent of one another, where the variance is a function of the mean.

To circumvent this problem, Kang et al. (2007) recommended the use of the Shewhart (SH) CV chart. However, the SH CV chart is insensitive to small and moderate CV shifts. Thus, the variable sample size and sampling interval (VSSI) control chart for monitoring the CV is suggested in this thesis to overcome this problem.

In real life, there are many situations in which a simultaneous monitoring of two or more correlated quality characteristics is necessary. Erroneous conclusions will occur if practitioners use univariate control charts to monitor a multivariate process. Hence, the use of multivariate control charts to monitor a multivariate process is inevitable. Yeong et al. (2016) proposed a multivariate chart to monitor the multivariate CV (MCV). The proposed chart is called the MCV chart. Past research works have shown that the performance of charts in detecting small and moderate shifts can be significantly improved by adopting the adaptive approach (see Aparisi and Haro, 2003; Zhang et al., 2014). In this study, the VSI, variable sample size (VSS) and VSSI approaches are adopted on the MCV chart so that the new charts proposed, namely the VSI MCV, VSS MCV and VSSI MCV charts will outperform the existing MCV chart of Yeong et al. (2016).

1.4 Research Objectives

The objectives of this thesis are as follows:

(i) To develop a univariate control chart for monitoring the CV by adopting the VSSI strategy, in order to improve the sensitivity of the SH CV chart in detecting small and moderate CV shifts.

(ii) To develop multivariate adaptive control charts, i.e. the VSSI MCV, VSI MCV and VSS MCV charts, for monitoring the MCV, in order to enhance the performance of the standard MCV chart, in detecting small and moderate MCV shifts.

1.5 Organization of the Thesis

Chapter 1 gives an overview on SQC and the related tools. Some preliminaries on control charting techniques in process monitoring and their usefulness in a variety of fields are enumerated. The objectives of this thesis are provided after the problem statement is discussed. Then the organization of the thesis is presented in the last section of Chapter 1.

As the proposed charts are compared with existing charts in Chapters 3 and 4, Chapter 2 reviews the existing CV and MCV charts considered in Chapters 3 and 4, respectively. The existing CV and MCV charts considered are the SH CV, exponentially weighted moving average (EWMA) CV^2, synthetic (Syn) CV, runs rules (RR) CV, VSI CV, VSS CV and MCV charts. Note that the EWMA CV^2 chart comprises two one-sided EWMA charts for monitoring the squared of the CV statistics.

In Chapter 3, a new chart, namely the VSSI CV chart, is proposed to provide a quick detection of small and moderate CV shifts. The basic properties of the VSSI CV chart are discussed and an optimization procedure for the chart, in computing the optimal parameters is presented. Subsequently, the performance of the proposed chart is compared with its existing counterparts, in terms of the average time to signal (ATS), standard deviation of the time to signal (SDTS) and expected average time to signal (EATS) criteria. An example of application is then illustrated using a real dataset from a car radio manufacturing industry.

In Chapter 4, the adaptive approach is adopted to propose three new charts for monitoring the MCV, namely the VSSI MCV, VSI MCV and VSS MCV charts. The optimal design parameters of these MCV charts, obtained through optimization programs, are discussed. The performance of the proposed charts are compared with the existing standard MCV chart, in terms of the ATS, SDTS and EATS criteria, in this chapter. This is followed by a real life example to illustrate the implementation of the charts using the spring manufacturing data.

Lastly, Chapter 5 summarizes the findings in this thesis, where the main contributions are highlighted. Potential topics for further research are also mentioned in this chapter.

1.6 Conclusion

This chapter explains the importance of applying SQC in real life applications. The use of control charting techniques is enumerated with their applications in a variety of fields, such as in manufacturing, service industries, finance and healthcare. The problem statement and research objectives are also highlighted in this chapter. The organization of the thesis is provided to facilitate readers in comprehending the various chapters in the thesis. In the next chapter, a literature review on existing univariate and multivariate CV charts is given.

CHAPTER 2
LITERATURE REVIEW ON EXISTING UNIVARIATE AND MULTIVARIATE COEFFICIENT OF VARIATION CONTROL CHARTS

2.1 Introduction

The CV is a useful tool in measuring an investor's risk, by determining the volatility of the return on an asset to the expected value of the return (Sharpe, 1994). According to Castagliola et al. (2011), the CV is utilized in renewal, queuing and reliability theory. Additionally, it is commonly needed in manufacturing and materials engineering.

Numerous research works on univariate CV charts have been made over the years. Kang et al. (2007) were the pioneers who introduced the SH CV chart by using rational subgroups. The SH CV chart performs well in detecting large shifts but it is less effective in detecting small and moderate shifts. To circumvent this problem, Hong et al. (2008) presented an EWMA CV chart. The EWMA CV chart surpasses the SH CV chart. Castagliola et al. (2011) proposed two one-sided EWMA chart, called the EWMA CV^2 chart, to monitor the squared of the CV statistics. The EWMA CV^2 chart yields smaller out-of-control average run length (ARL_1) values than the EWMA CV chart. Calzada and Scariano (2013) suggested the monitoring of the CV using a Syn CV chart. The Syn CV chart surpasses the SH CV chart but loses out to the EWMA CV^2 chart in detecting small and moderate CV shifts. The use of selected runs rules, such as the 2-out-of-3, 3-out-of-4 and 4-out-of-5 rules, on the SH CV chart, to construct the RR CV chart, to monitor the CV was recommended by Castagliola et al. (2013b). The RR CV chart outperforms the SH CV chart and it is also easier to implement compared with the Syn CV, EWMA CV and EWMA CV^2 charts. Castagliola et al. (2013a) used the VSI CV chart to monitor the CV. A control chart for monitoring the CV by means of the VSS strategy was suggested by Castagliola et al. (2015a). Castagliola et al. (2015b) presented the one-sided Shewhart-type charts for monitoring the CV in a finite horizon production while Amdouni et al. (2015; 2016) proposed the VSS chart and the one-sided runs rules charts for monitoring the CV in short production runs, respectively.

In many real life applications, a simultaneous monitoring of at least two related quality characteristics is necessary, hence the need for multivariate charts arises. The first chart for monitoring the MCV was suggested by Yeong et al. (2016).

2.2 Control Charts for Univariate Coefficient of Variation (CV)

This section explains the existing univariate CV charts, which are the SH CV, Syn CV, RR CV, EWMA CV^2, VSS CV and VSI CV charts. The properties of these charts are enumerated. These charts are compared with the proposed chart in Chapter 3.

2.2.1 Shewhart (SH) CV Chart

According to Kang et al. (2007), CV is the ratio of the standard deviation to the mean. Let μ and σ be the mean and standard deviation of a positive random variable, *X*, respectively. Then the CV of *X* is

$$\gamma = \frac{\sigma}{\mu}. \tag{2.1}$$

Assume that $\{X_1, X_2, ..., X_n\}$ is a random sample of size *n* from the normal distribution and $\bar{X}$ and *S* are the sample mean and sample standard deviation, respectively, i.e.

$$\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k \tag{2.2}$$

and

$$S = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}\left(X_k - \bar{X}\right)^2}\,. \tag{2.3}$$

The sample CV is computed as

$$\hat{\gamma} = \frac{S}{\bar{X}}\,. \tag{2.4}$$

Note that $\sqrt{n}/\hat{\gamma}$ follows a noncentral *t* distribution with *n* – 1 degrees of freedom and noncentrality parameter $\sqrt{n}/\gamma$ (Iglewicz et al., 1968). If γ (> 0) is not too large, say $\gamma \in (0,\ 0.5]$, the cumulative distribution function (cdf) of $\hat{\gamma}$ can be accurately approximated as (Iglewicz et al., 1968)

$$F_{\hat{\gamma}}(x|n,\gamma) \approx 1 - F_t\left(\frac{\sqrt{n}}{x}\middle| n-1, \frac{\sqrt{n}}{\gamma}\right), \tag{2.5}$$

where $F_t\left(\ \cdot\ \middle| n-1, \frac{\sqrt{n}}{\gamma}\right)$ denotes the cdf of a noncentral *t* random variable with $n-1$ degrees of freedom and noncentrality parameter $\frac{\sqrt{n}}{\gamma}$. The inverse cdf of $\hat{\gamma}$ is accurately approximated as

$$F_{\hat{\gamma}}^{-1}(\alpha|n,\gamma) \approx \frac{\sqrt{n}}{F_t^{-1}\left(1-\alpha\middle| n-1, \frac{\sqrt{n}}{\gamma}\right)}, \tag{2.6}$$

where $F_t^{-1}\left(\ \cdot\ \middle| n-1, \frac{\sqrt{n}}{\gamma}\right)$ represents the inverse cdf of the noncentral *t* random variable with $n-1$ degrees of freedom and noncentrality parameter $\frac{\sqrt{n}}{\gamma}$.

The center line of the SH CV chart is set as the in-control CV value, γ_0, while the chart's limits are obtained, based on probability limits. The lower and upper control limits, $\mathrm{LCL}_{\mathrm{CV}}$ and $\mathrm{UCL}_{\mathrm{CV}}$, respectively, of the SH CV chart with a Type-I error probability of α are equal to

$$\mathrm{LCL}_{\mathrm{CV}} = F_{\hat{\gamma}}^{-1}\left(\frac{\alpha}{2}\middle|n,\gamma_0\right), \quad (2.7\text{a})$$

and

$$\mathrm{UCL}_{\mathrm{CV}} = F_{\hat{\gamma}}^{-1}\left(1-\frac{\alpha}{2}\middle|n,\gamma_0\right). \quad (2.7\text{b})$$

The SH CV chart plots the sample CV statistic, $\hat{\gamma}$. The probability of an out-of-control signal on the SH CV chart is obtained as

$$M = 1-\Pr\left(\mathrm{LCL}_{\mathrm{CV}} < \hat{\gamma} < \mathrm{UCL}_{\mathrm{CV}}\right). \quad (2.8)$$

Consequently, the average run length (ARL) and standard deviation of the run length (SDRL) of the SH CV chart are

$$\mathrm{ARL}_{\mathrm{CV}} = \frac{1}{M}, \quad (2.9\text{a})$$

and

$$\mathrm{SDRL}_{\mathrm{CV}} = \frac{\sqrt{1-M}}{M}. \quad (2.9\text{b})$$

respectively (Kang et al., 2007). Note that the ARL measures the average number of sample CVs that need to be plotted on the SH CV chart until the first out-of-control signal is detected by the chart. Therefore, for an out-of-control process, the smaller the ARL value, the better the chart is, when the in-control ARL (ARL_0) is specified at a desired value. On the other hand, the SDRL measures the spread of the run length distribution, hence, the smaller the SDRL value, the better is the chart's performance.

In some scenarios, the shift size, τ is not deterministic. For this case, the expected average run length (EARL) can be used as a measure of performance when the shift τ is unknown. The in-control EARL (EARL_0) is set equal to the ARL_0. The out-of-control EARL (EARL_1) of the SH CV chart is computed as

$$\text{EARL}_1 = \int_{\tau_{\min}}^{\tau_{\max}} \text{ARL}_1\left(\text{LCL}_{\text{SH CV}}, \text{UCL}_{\text{SH CV}}, M, n, \gamma_0, \tau\right) f_\tau\left(\tau\right) d\tau, \tag{2.10}$$

where ARL_1 is computed using Equation (2.9a) and $f_\tau\left(\tau\right)$ defines the probability density function (pdf) of τ. If no information on $f_\tau\left(\tau\right)$ is available, it is reasonable to assume that τ follows a uniform distribution over the interval $\left(\tau_{\min}, \tau_{\max}\right)$. Here, $\tau_{\min}$ and $\tau_{\max}$ refer to the lower bound and upper bound of the shift, τ, respectively, i.e. $\tau_{\min} < \tau < \tau_{\max}$.

2.2.2 Synthetic (Syn) CV Chart

The Syn CV chart, proposed by Caldaza and Scariano (2013) is an integration of the SH CV and CRL sub-charts. The Syn CV chart functions on the basis that when a non-conforming sample exists, a count of the number of samples between the present and the previous non-conforming samples is made. The process is said to be out-of-control only when this count is less than or equal to a threshold value, L. As soon as an out-of-control signal is detected and corrective actions are taken, the count is reset to zero and the same process monitoring procedure continues.

The lower and upper control limits of the Syn CV chart are

$$\text{LCL}_{\text{Syn CV}} = F_{\hat{\gamma}}^{-1}\left(\frac{g}{2} \middle| n, \gamma_0\right), \tag{2.11a}$$

and

$$\mathrm{UCL}_{\mathrm{Syn\ CV}} = F_{\hat{\gamma}}^{-1}\left(1-\frac{g}{2}\middle|n,\gamma_0\right). \tag{2.11b}$$

The ARL and SDRL of the Syn CV chart are obtained as (Caldaza and Scariano, 2013)

$$\mathrm{ARL}_{\mathrm{Syn\ CV}} = \left(\frac{1}{1-(1-g)^L}\right)\times\left(\frac{1}{g}\right) \tag{2.12a}$$

and

$$\mathrm{SDRL}_{\mathrm{Syn\ CV}} = \left(\frac{2-g}{(1-(1-g)^L)\,\mathrm{g}^2}+\left(\frac{1}{(1-(1-g)^L)^2}\right)\left[\frac{1}{g^2}-2\sum_{t=1}^{L}t(1-g)^{t-1}\right]\right)^{\frac{1}{2}}, \tag{2.12b}$$

where $g = 1-\Pr\left(\mathrm{LCL}_{\mathrm{Syn\ CV}} < \hat{\gamma} < \mathrm{UCL}_{\mathrm{Syn\ CV}}\right)$. The limits of the SH CV sub-chart, i.e. $\mathrm{LCL}_{\mathrm{Syn\ CV}}$ and $\mathrm{UCL}_{\mathrm{Syn\ CV}}$ are obtained using the procedure explained below. Note that *L* denotes the lower limit of the CRL sub-chart. The parameters $\mathrm{LCL}_{\mathrm{Syn\ CV}}$, $\mathrm{UCL}_{\mathrm{Syn\ CV}}$ and *L* can be optimally designed to minimize the ARL_1, based on a specified ARL_0 value using the following optimization procedure:

Step 1: Specify the sample size (*n*), in-control CV (γ_0), ARL_0 and τ, where $\tau = \frac{\gamma_1}{\gamma_0}$, and γ_1 is the out-of-control CV. Here, τ is the magnitude of shift in the CV, where a quick detection is important.

Step 2: Initialize $L = 1$.

Step 3: Solve Equation (2.12a) for *g*, when the process is in-control, by letting $\mathrm{ARL}_{\mathrm{Syn\ CV}} = \mathrm{ARL}_0$. Consequently, substitute *g* into Equations (2.11a) and (2.11b), to compute the limits $\mathrm{LCL}_{\mathrm{Syn\ CV}}$ and $\mathrm{UCL}_{\mathrm{Syn\ CV}}$, respectively.

Step 4: After obtaining the limits L, $\text{LCL}_{\text{Syn CV}}$ and $\text{UCL}_{\text{Syn CV}}$ in Step 3, compute ARL_1 for the shift τ specified in Step 1, using Equation (2.12a).

Step 5: Repeat Steps 3 and 4 by letting $L = L + 1$ until the largest L value set by the user is reached.

Step 6: Choose the limits L, $\text{LCL}_{\text{Syn CV}}$ and $\text{UCL}_{\text{Syn CV}}$ that minimize the ARL_1, for the shift τ.

Meanwhile, the EARL_1 of the Syn CV chart is computed as

$$\text{EARL}_1 = \int_{\tau_{\min}}^{\tau_{\max}} \text{ARL}_1\left(\text{LCL}_{\text{Syn CV}}, \text{UCL}_{\text{Syn CV}}, L, n, \gamma_0, \tau\right) f_\tau\left(\tau\right) d\tau, \qquad (2.13)$$

where EARL_0 is set as ARL_0 and $f_\tau\left(\tau\right)$ defines the pdf of τ. Here, it is assumed that τ follows a uniform $U\left(\tau_{\min}, \tau_{\max}\right)$ distribution, where $\tau_{\min}$ and $\tau_{\max}$ denotes the minimum and maximum shifts, respectively.

2.2.3 Runs Rules (RR) CV Charts

The RR CV chart, suggested by Castagliola et al. (2013b) is different from other CV charts as it only considers the lower and upper warning limits, i.e. $\text{LWL}_{\text{RR CV}}$ and $\text{UWL}_{\text{RR CV}}$, instead of using the control limits. An out-of-control signal is detected by the RR CV chart if the selected runs rules pattern has occurred. Three runs rules (RR) strategies were considered by Castagliola et al. (2013b), namely the 2-out-of-3 RR ($\text{RR}_{2,3}$ CV), 3-out-of-4 RR ($\text{RR}_{3,4}$ CV) and 4-out-of-5 RR ($\text{RR}_{4,5}$ CV) charts.

2.2.3 (a) $\text{RR}_{2,3}$ CV Chart

For the $\text{RR}_{2,3}$ CV chart, the lower and upper warning limits of the chart are (Castagliola et al., 2013b)

$$\text{LWL}_{\text{RR}_{2,3}\ \text{CV}} = \mu_0\left(\gamma\right) - W_{2,3}\sigma_0\left(\gamma\right) \tag{2.14a}$$

and

$$\text{UWL}_{\text{RR}_{2,3}\ \text{CV}} = \mu_0\left(\gamma\right) + W_{2,3}\sigma_0\left(\gamma\right), \tag{2.14b}$$

where $\mu_0\left(\gamma\right)$ and $\sigma_0\left(\gamma\right)$ denote the mean and standard deviation of the sample CV, γ when the process is in-control. Here, $W_{2,3}$ is the warning limits' parameter. Since there is no closed form for $\mu_0\left(\gamma\right)$ and $\sigma_0\left(\gamma\right)$, the following approximations suggested by Reh and Scheffler (1996) can be used (Castagliola et al., 2013b):

$$\mu_0\left(\hat{\gamma}\right) \approx \gamma_0\left(1 + \frac{1}{n}\left(\gamma_0^2 - \frac{1}{4}\right) + \frac{1}{n^2}\left(3\gamma_0^4 - \frac{\gamma_0^2}{4} - \frac{7}{32}\right) + \frac{1}{n^3}\left(15\gamma_0^6 - \frac{3\gamma_0^4}{4} - \frac{7\gamma_0^2}{32} - \frac{19}{128}\right)\right) \tag{2.15a}$$

and

$$\sigma_0\left(\hat{\gamma}\right) \approx \gamma_0\left(\frac{1}{n}\left(\gamma_0^2 + \frac{1}{2}\right) + \frac{1}{n^2}\left(8\gamma_0^4 + \gamma_0^2 + \frac{3}{8}\right) + \frac{1}{n^3}\left(69\gamma_0^6 + \frac{7\gamma_0^4}{2} + \frac{3\gamma_0^2}{4} + \frac{3}{16}\right)\right)^{1/2}, \tag{2.15b}$$

where γ_0 and n represent the in-control CV and sample size, respectively. An out-of-control signal is issued by the $\text{RR}_{2,3}$ CV chart when two out of three sample CVs, γ plot above $\text{UWL}_{\text{RR}_{2,3}\ \text{CV}}$ or below $\text{LWL}_{\text{RR}_{2,3}\ \text{CV}}$.

Consequently, there are seven possible in-control (transient) Markov chain states, based on the position of the last two sample CV points plotted on the chart, i.e. (Castagliola et al., 2013b)

(1) $\hat{\gamma}_{i-2} > \text{UWL}_{\text{RR}_{2,3}\text{ CV}}$ and $\text{LWL}_{\text{RR}_{2,3}\text{ CV}} \le \hat{\gamma}_{i-1} \le \text{UWL}_{\text{RR}_{2,3}\text{ CV}}$

(2) $\hat{\gamma}_{i-2} > \text{UWL}_{\text{RR}_{2,3}\text{ CV}}$ and $\hat{\gamma}_{i-1} < \text{LWL}_{\text{RR}_{2,3}\text{ CV}}$

(3) $\text{LWL}_{\text{RR}_{2,3}\text{ CV}} \le \hat{\gamma}_{i-2} \le \text{UWL}_{\text{RR}_{2,3}\text{ CV}}$ and $\hat{\gamma}_{i-1} > \text{UWL}_{\text{RR}_{2,3}\text{ CV}}$

(4) $\text{LWL}_{\text{RR}_{2,3}\text{ CV}} \le \hat{\gamma}_{i-2} \le \text{UWL}_{\text{RR}_{2,3}\text{ CV}}$ and $\text{LWL}_{\text{RR}_{2,3}\text{ CV}} \le \hat{\gamma}_{i-1} \le \text{UWL}_{\text{RR}_{2,3}\text{ CV}}$

(5) $\text{LWL}_{\text{RR}_{2,3}\text{ CV}} \le \hat{\gamma}_{i-2} \le \text{UWL}_{\text{RR}_{2,3}\text{ CV}}$ and $\hat{\gamma}_{i-1} < \text{LWL}_{\text{RR}_{2,3}\text{ CV}}$

(6) $\hat{\gamma}_{i-2} < \text{LWL}_{\text{RR}_{2,3}\text{ CV}}$ and $\hat{\gamma}_{i-1} > \text{UWL}_{\text{RR}_{2,3}\text{ CV}}$

(7) $\hat{\gamma}_{i-2} < \text{LWL}_{\text{RR}_{2,3}\text{ CV}}$ and $\text{LWL}_{\text{RR}_{2,3}\text{ CV}} \le \hat{\gamma}_{i-1} \le \text{UWL}_{\text{RR}_{2,3}\text{ CV}}$

Based on the above seven transient states, the location of the current sample CV, $\hat{\gamma}_i$, (also referred to as the third point of the run) will decide whether the process is in-control or out-of-control. If there are two successive sample CVs, i.e. $\hat{\gamma}_{i-1}$ and $\hat{\gamma}_i$ plotted beyond the same warning limit, an out-of-control signal is detected. The transition probability matrix (tpm) with the absorbing state for the $\text{RR}_{2,3}$ CV chart is constructed as follows (Castagliola et al., 2013b):

$$\boldsymbol{G} = \begin{pmatrix} \boldsymbol{Q}_{2,3} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{pmatrix} = \left(\begin{array}{ccccccc|c} 0 & 0 & 0 & p_C & p_L & 0 & 0 & p_U \\ 0 & 0 & 0 & 0 & 0 & 0 & p_C & p_L + p_U \\ p_C & p_L & 0 & 0 & 0 & 0 & 0 & p_U \\ 0 & 0 & p_U & p_C & p_L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_U & p_C & p_L \\ p_C & 0 & 0 & 0 & 0 & 0 & 0 & p_L + p_U \\ 0 & 0 & p_U & p_C & 0 & 0 & 0 & p_L \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \qquad (2.16)$$

Here, $\boldsymbol{Q}_{2,3\ (7\text{x}7)}$ is the tpm of transient probabilities while the vector $\mathbf{r}$ satisfies $\mathbf{r} = \mathbf{1} - \boldsymbol{Q}_{2,3}\mathbf{1}$, i.e. the sum of row probabilities equal to 1. Note also that $\mathbf{0}^T = (0,0,0,0,0,0,0)$ and $\mathbf{1} = (1,1,1,1,1,1,1)^T$. The starting probabilities of the $\text{RR}_{2,3}$ CV chart is $\mathbf{q}_{2,3} = (0,0,0,1,0,0,0)^T$ and P_L, P_U and P_C are computed as

$$P_L = \Pr\left(\gamma \le \text{LWL}_{\text{RR}_{2,3}\ \text{CV}}\right) = F_\gamma\left(\text{LWL}_{\text{RR}_{2,3}\ \text{CV}} \middle| n, \gamma_1\right),$$

$$P_U = \Pr\left(\gamma \ge \text{UWL}_{\text{RR}_{2,3}\ \text{CV}}\right) = 1 - F_\gamma\left(\text{UWL}_{\text{RR}_{2,3}\ \text{CV}} \middle| n, \gamma_1\right),$$

$$P_C = \Pr\left(\text{LWL}_{\text{RR}_{2,3}\ \text{CV}} \le \gamma \le \text{UWL}_{\text{RR}_{2,3}\ \text{CV}}\right) = 1 - P_L - P_U, \qquad (2.17)$$

where $F_\gamma\left(\cdot \middle| n, \gamma_1\right)$ is the cdf of γ defined in Equation (2.5), $\gamma_1 = \tau\gamma_0$ and τ is the shift size in the CV.

The ARL and SDRL of the $\text{RR}_{2,3}$ CV chart are computed as (Castagliola et al., 2013b)

$$\text{ARL}_{\text{RR}_{2,3}\ \text{CV}} = \mathbf{q}_{2,3}^T(\mathbf{I} - \boldsymbol{Q}_{2,3})^{-1}\mathbf{1}, \qquad (2.18\text{a})$$

and

$$\text{SDRL}_{\text{RR}_{2,3}\ \text{CV}} = \sqrt{2\mathbf{q}_{2,3}^T(\mathbf{I} - \boldsymbol{Q}_{2,3})^{-2}\boldsymbol{Q}_{2,3}\mathbf{1} - \text{ARL}^2_{\text{RR}_{2,3}\ \text{CV}} + \text{ARL}_{\text{RR}_{2,3}\ \text{CV}}}. \qquad (2.18\text{b})$$

The parameter $W_{2,3}$ in Equations (2.14a) and (2.14b) is obtained by solving Equation (2.19), based on a specified ARL_0 value.

$$\text{ARL}_{\text{RR}_{2,3}\ \text{CV}}\left(W_{2,3}, n, \gamma_0, \tau(=1)\right) = \text{ARL}_0. \qquad (2.19)$$

Note that $\text{ARL}_{\text{RR}_{2,3}\ \text{CV}}\left(W_{2,3}, n, \gamma_0, \tau(=1)\right)$, which is a function of $W_{2,3}$, n, γ_0 and τ is computed using Equation (2.18a). When the shift size τ cannot be specified, it is natural to use EARL as a performance measure of the chart. The EARL_0 is set equal to ARL_0 and the EARL_1 of the $\text{RR}_{2,3}$ CV chart is obtained as

$$\mathrm{EARL}_1 = \int_{\tau_{\min}}^{\tau_{\max}} \mathrm{ARL}_1\left(\mathrm{LWL}_{\mathrm{RR}_{2,3}\ \mathrm{CV}}, \mathrm{UWL}_{\mathrm{RR}_{2,3}\ \mathrm{CV}}, W_{2,3}, n, \gamma_0, \tau\right) f_\tau\left(\tau\right) d\tau, \quad (2.20)$$

where $f_\tau\left(\tau\right)$ defines the pdf of the shift size τ, while $\tau_{\min}$ and $\tau_{\max}$ denote the lower and upper bound for the shift size τ.

2.2.3 (b) $\mathbf{RR_{3,4}}$ CV Chart

For the $\mathrm{RR}_{3,4}$ CV chart, there are twenty-five in-control states, based on the position of the last three sample points plotted on the chart. These states are (Castagliola et al., 2013b)

(1) $\hat{\gamma}_{i-3} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$, $\hat{\gamma}_{i-2} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$ and $\mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}} \le \hat{\gamma}_{i-1} \le \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$

(2) $\hat{\gamma}_{i-3} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$, $\hat{\gamma}_{i-2} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$ and $\hat{\gamma}_{i-1} < \mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$

(3) $\hat{\gamma}_{i-3} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$, $\mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}} \le \hat{\gamma}_{i-2} \le \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$ and $\hat{\gamma}_{i-1} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$

(4) $\hat{\gamma}_{i-3} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$, $\mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}} \le \hat{\gamma}_{i-2} \le \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$ and $\mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}} \le \hat{\gamma}_{i-1} \le \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$

(5) $\hat{\gamma}_{i-3} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$, $\mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}} \le \hat{\gamma}_{i-2} \le \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$ and $\hat{\gamma}_{i-1} < \mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$

(6) $\hat{\gamma}_{i-3} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$, $\hat{\gamma}_{i-2} < \mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$ and $\hat{\gamma}_{i-1} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$

(7) $\hat{\gamma}_{i-3} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$, $\hat{\gamma}_{i-2} < \mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$ and $\mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}} \le \hat{\gamma}_{i-1} \le \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$

(8) $\hat{\gamma}_{i-3} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$, $\hat{\gamma}_{i-2} < \mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$ and $\hat{\gamma}_{i-1} < \mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$

(9) $\mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}} \le \hat{\gamma}_{i-3} \le \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$, $\hat{\gamma}_{i-2} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$ and $\hat{\gamma}_{i-1} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$

(10) $\mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}} \le \hat{\gamma}_{i-3} \le \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$, $\hat{\gamma}_{i-2} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$ and $\mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}} \le \hat{\gamma}_{i-1} \le \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$

(11) $\mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}} \le \hat{\gamma}_{i-3} \le \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$, $\hat{\gamma}_{i-2} > \mathrm{UWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$ and $\hat{\gamma}_{i-1} < \mathrm{LWL}_{\mathrm{RR}_{3,4}\ \mathrm{CV}}$

(12) $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-3} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-2} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\hat{\gamma}_{i-1} > \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$

(13) $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-3} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-2} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-1} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$

(14) $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-3} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-2} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\hat{\gamma}_{i-1} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$

(15) $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-3} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\hat{\gamma}_{i-2} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\hat{\gamma}_{i-1} > \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$

(16) $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-3} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\hat{\gamma}_{i-2} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-1} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$

(17) $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-3} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\hat{\gamma}_{i-2} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\hat{\gamma}_{i-1} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$

(18) $\hat{\gamma}_{i-3} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\hat{\gamma}_{i-2} > \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\hat{\gamma}_{i-1} > \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$

(19) $\hat{\gamma}_{i-3} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\hat{\gamma}_{i-2} > \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-1} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$

(20) $\hat{\gamma}_{i-3} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\hat{\gamma}_{i-2} > \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\hat{\gamma}_{i-1} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$

(21) $\hat{\gamma}_{i-3} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-2} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\hat{\gamma}_{i-1} > \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$

(22) $\hat{\gamma}_{i-3} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-2} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-1} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$

(23) $\hat{\gamma}_{i-3} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-2} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\hat{\gamma}_{i-1} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$

(24) $\hat{\gamma}_{i-3} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\hat{\gamma}_{i-2} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\hat{\gamma}_{i-1} > \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$

(25) $\hat{\gamma}_{i-3} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$, $\hat{\gamma}_{i-2} < \text{LWL}_{\text{RR}_{3,4}\ \text{CV}}$ and $\text{LWL}_{\text{RR}_{3,4}\ \text{CV}} \le \hat{\gamma}_{i-1} \le \text{UWL}_{\text{RR}_{3,4}\ \text{CV}}$

Consequently, the location of where the current sample CV, $\hat{\gamma}_i$, plots on the chart will decide whether the process is in-control or out-of-control. The $\text{RR}_{3,4}$ CV chart signals an out-of-control if three out of four successive sample CVs plot above $\text{UWL}_{\text{RR}_{3,4}\text{ CV}}$ or below $\text{LWL}_{\text{RR}_{3,4}\text{ CV}}$. The chart also issues an out-of-control signal when three successive sample CVs, i.e. $\hat{\gamma}_{i-2}$, $\hat{\gamma}_{i-1}$ and $\hat{\gamma}_i$ plot beyond the same warning limit. The upper and lower warning limits of the $\text{RR}_{3,4}$ CV chart can be obtained using Equations (2.14a) and (2.14b) by substituting $W_{2,3}$ with $W_{3,4}$. The parameter $W_{3,4}$ can be obtained by solving Equation (2.21), based on a specified ARL_0 value.

$$\text{ARL}_{\text{RR}_{3,4}\text{ CV}}\left(W_{3,4}, n, \gamma_0, \tau\left(=1\right)\right) = \text{ARL}_0. \qquad (2.21)$$

Note that $\text{ARL}_{\text{RR}_{3,4}\text{ CV}}\left(W_{3,4}, n, \gamma_0, \tau\left(=1\right)\right)$, SDRL and EARL of the $\text{RR}_{3,4}$ CV chart can be computed using Equations (2.18a), (2.18b) and (2.20), respectively. The only difference is by replacing $\mathbf{q}_{2,3}$ and $\boldsymbol{Q}_{2,3}$ with $\mathbf{q}_{3,4}$ and $\boldsymbol{Q}_{3,4}$, respectively. Here, $\mathbf{q}_{3,4}=\left(0,...,0,1,0,...,0\right)^T$, where the entry containing unity is the 13th entry, while $\boldsymbol{Q}_{3,4}$ is a 25×25 tpm for the transient states given in Castagliola et al. (2013b). The tpm $\boldsymbol{Q}_{3,4}$ is not presented here in order not to lengthen the discussion in this section, as the $\text{RR}_{3,4}$ CV chart is not the proposed chart in this thesis.

2.2.3 (c) $\mathbf{RR_{4,5}}$ CV Chart

The $\text{RR}_{4,5}$ CV chart contains seventy-nine transient states, based on the position of the last four sample CVs plotted on the chart. These states are described in Castagliola et al. (2013b).

The location of the current sample CV, $\hat{\gamma}_i$, on the chart determines whether the process is in-control or out-of-control. An out-of-control is signalled by the chart when four out of five consecutive sample CVs fall above $\text{UWL}_{\text{RR}_{4,5}\text{ CV}}$ or below $\text{LWL}_{\text{RR}_{4,5}\text{ CV}}$. Additionally, an out-of-control is also signalled if four successive sample CVs, i.e. $\hat{\gamma}_{i-3}$, $\hat{\gamma}_{i-2}$, $\hat{\gamma}_{i-1}$ and $\hat{\gamma}_i$ plot beyond the same warning limit. The upper and lower warning limits of the $\text{RR}_{4,5}$ CV chart are obtained using Equations (2.14a) and (2.14b) by substituting the parameter $W_{2,3}$ with $W_{4,5}$. The parameter $W_{4,5}$ can be obtained by solving Equation (2.22), based on a specified ARL_0 value.

$$\text{ARL}_{\text{RR}_{4,5}\text{ CV}}\left(W_{4,5}, n, \gamma_0, \tau(=1)\right) = \text{ARL}_0. \tag{2.22}$$

Equations (2.18a), (2.18b) and (2.20) are used to compute $\text{ARL}_{\text{RR}_{4,5}\text{ CV}}\left(W_{4,5}, n, \gamma_0, \tau(=1)\right)$, SDRL and EARL, respectively, of the $\text{RR}_{4,5}$ CV chart but by replacing $\mathbf{q}_{2,3}$ and $\boldsymbol{Q}_{2,3}$ with $\mathbf{q}_{4,5}$ and $\boldsymbol{Q}_{4,5}$, respectively. Here, $\mathbf{q}_{4,5} = (0,\ldots,0,1,0,\ldots,0)^T$ (the entry with unity is the 40th entry) while $\boldsymbol{Q}_{4,5}$ is a 79×79 tpm for the transient states discussed in Castagliola et al. (2013b).

2.2.4 Exponentially Weighted Moving Average (EWMA) CV^2 Charts

The EWMA CV^2 charts, presented by Castagliola et al. (2011) are superior to the EWMA CV chart proposed by Hong et al. (2008). Castagliola et al. (2011) suggested using two one-sided (upward and downward) EWMA CV^2 charts, instead of using a two-sided chart.

The cdf of the sample CV^2, $\hat{\gamma}^2$ is (Castagliola et al., 2011)

$$F_{\hat{\gamma}^2}(x|n,\gamma) \approx 1 - F_F\left(\frac{n}{x}\middle|1, n-1, \frac{n}{\gamma^2}\right), \tag{2.23}$$

where $F_F\left(\ \cdot\ \middle|1, n-1, \frac{n}{\gamma^2}\right)$ is the cdf of the noncentral *F* distribution with degrees of freedom 1 and $n-1$, and noncentrality parameter $\frac{n}{\gamma^2}$. The inverse cdf of $\hat{\gamma}^2$ can be accurately approximated as (Castagliola et al., 2011)

$$F_{\hat{\gamma}^2}^{-1}(\alpha|n,\gamma) \approx \frac{n}{F_F^{-1}\left(1-\alpha\middle|1, n-1, \frac{n}{\gamma^2}\right)}, \tag{2.24}$$

where $F_F^{-1}\left(\ \cdot\ \middle|1, n-1, \frac{n}{\gamma^2}\right)$ refers to the inverse cdf of the noncentral *F* distribution with degrees of freedom 1 and $n-1$, and noncentrality parameter $\frac{n}{\gamma^2}$.

The upward and downward EWMA CV^2 charts are used to monitor increasing and decreasing CV shifts, respectively. The upper (of the upward EWMA CV^2 chart) and lower (of the downward EWMA CV^2 chart) control limits are computed as (Castagliola et al., 2011)

$$\text{UCL}_{\text{EWMA CV}^2} = \mu_0\left(\hat{\gamma}^2\right) + K_{\text{EWMA}}^{+}\sqrt{\frac{\lambda^+}{2-\lambda^+}}\sigma_0\left(\hat{\gamma}^2\right), \tag{2.25a}$$

and

$$\text{LCL}_{\text{EWMA CV}^2} = \mu_0\left(\hat{\gamma}^2\right) - K_{\text{EWMA}}^{-}\sqrt{\frac{\lambda^-}{2-\lambda^-}}\sigma_0\left(\hat{\gamma}^2\right), \tag{2.25b}$$

respectively, where $K_{\text{EWMA}}^{+}, K_{\text{EWMA}}^{-}$, λ^+ and λ^- are the optimal charts' parameters. Here, $\mu_0\left(\hat{\gamma}^2\right)$ and $\sigma_0\left(\hat{\gamma}^2\right)$ refer to the mean and standard deviation of $\hat{\gamma}^2$ when the process is in-control. Since there is no closed form for $\mu_0\left(\hat{\gamma}^2\right)$ and $\sigma_0\left(\hat{\gamma}^2\right)$, Castagliola et al. (2011) adopted the following approximations due to Breunig (2001):

$$\mu_0\left(\hat{\gamma}^2\right) = \gamma_0^2\left(1-\frac{3\gamma_0^2}{n}\right) \tag{2.26a}$$

and

$$\sigma_0\left(\hat{\gamma}^2\right) = \left(\gamma_0^4\left(\frac{2}{n-1}+\gamma_0^2\left(\frac{4}{n}+\frac{20}{n(n-1)}+\frac{75\gamma_0^2}{n^2}\right)\right)-\left(\mu_0\left(\hat{\gamma}^2\right)-\gamma_0^2\right)^2\right)^{\frac{1}{2}}. \tag{2.26b}$$

The Markov chain approach is employed to obtain the run length properties of the EWMA CV^2 chart. The Markov chain for the EWMA CV^2 chart comprises $h+2$ states, where the $(h+2)^{\text{th}}$ state is absorbing. The tpm $\boldsymbol{A}$ (with the absorbing state) for the EWMA CV^2 chart is (Castagliola et al., 2011)

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{Q} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} Q_{0,0} & Q_{0,1} & \cdots & Q_{0,h} & r_0 \\ Q_{1,0} & Q_{1,1} & \cdots & Q_{1,h} & r_1 \\ \vdots & \vdots & & \vdots & \vdots \\ Q_{h,0} & Q_{h,1} & \cdots & Q_{h,h} & r_h \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{2.27}$$

where $\boldsymbol{Q}$ is the $(h+1)\times(h+1)$ tpm of transient probabilities and the $(h+1)\times 1$ vector $\mathbf{r}$ satisfies $\mathbf{r} = \mathbf{1} - \boldsymbol{Q}\mathbf{1}$, i.e. the sum of row probabilities in the tpm $\boldsymbol{A}$ equals to 1, with $\mathbf{0}^T = (0, 0, ..., 0)$ and $\mathbf{1} = (1, 1, ..., 1)^T$. Consequently, the ARL and SDRL of the upward and downward EWMA CV^2 charts are computed as

$$\text{ARL}_{\text{EWMA CV}^2} = \mathbf{s}^{\boldsymbol{T}}(\mathbf{I}-\boldsymbol{Q})^{-1}\mathbf{1} \tag{2.28a}$$

and