DYNAMICAL ANALYSIS OF FRACTIONAL-ORDER ROSENZWEIG-MACARTHUR MODELS

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by

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ABSTRAK

Dalam tesis ini, tiga model Rosenzweig-MacArthur (R-M) peringkat pecahan lanjutan dipertimbangkan: i) model R-M dua spesies dengan perlindungan mangsa; ii) model R-M tiga spesies dengan perlindungan mangsa; iii) model R-M tiga spesies dengan struktur berperingkat serta perlindungan mangsa. Model-model ini dibina dan dianalisis secara terperinci. Kewujudan, keunikan, sifat non-negatif dan keterbatasan penyelesaian serta kestabilan tempatan dan asimptotik global bagi titik keseimbangan dikaji. Syarat-syarat yang mencukupi untuk kestabilan dan berlakunya pencabangan Hopf untuk model R-M peringkat pecahan ditunjukkan. Impak peringkat pecahan dan kesan perlindungan mangsa terhadap kestabilan sistem ini juga dikaji secara teori dan dengan menggunakan simulasi berangka. Keputusan menunjukkan bahawa hasil model R-M peringkat pecahan lebih stabil daripada model integer sepadannya kerana domain kestabilan dalam model peringkat pecahan lebih besar daripada domain untuk model sepadan integer yang sama. Rosenzweig dalam makalah yang diterbitkan pada tahun 1971 menekankan bahawa peningkatan kapasiti bawaan mangsa (iaitu sistem diperkayakan) mungkin menyebabkan kepupusan spesies mangsa dalam ekosistem. Ini dikenali sebagai paradoks pengayaan. Dalam kajian ini, didapati bahawa pengenalan peringkat pecahan kepada model R-M mengakibatkan spesies ekosistem menjadi stabil dan dengan itu meleraikan paradoks pengayaan.

DYNAMICAL ANALYSIS OF FRACTIONAL-ORDER ROSENZWEIG-MACARTHUR MODELS

ABSTRACT

In this thesis, three extended fractional order Rosenzweig-MacArthur (R-M) models are considered: i) a two-species R-M model incorporating a prey refuge; ii) a three species R-M model with a prey refuge; iii) a three-species R-M model with stage structure and a prey refuge. The models are constructed and analyzed in detail. The existence, uniqueness, non-negativity and boundedness of the solutions as well as the local and global asymptotic stability of the equilibrium points are studied. Sufficient conditions for the stability and the occurrence of Hopf bifurcation for these fractional order R-M models are demonstrated. The impacts of fractional order and prey refuge on the stability of these systems are also studied both theoretically and by using numerical simulations. The results indicate that the outcomes of R-M fractional order model are more stable than its integer counterpart model because the domain of stability in the fractional order model is larger than the domain for the corresponding integer order model. Rosenzweig in a paper published in 1971 highlighted that increasing the carrying capacity of the prey (i.e. enriching the systems) may lead to destroy the steady state. This is known as the paradox of enrichment. In this study, it was found that the introduction of fractional order to the R-M models can lead to stabilization of the species ecosystems and thus resolve the paradox of enrichment.

CHAPTER 1

INTRODUCTION

1.1 Background

In the natural world, predation describes a biological interaction where a predator feeds on its prey. These behaviors can be modeled mathematically by prey-predator models. The basic model was first proposed by Lotka and Volterra in 1925. The Lotka-Volterra model consists of two coupled non-linear differential equations and illustrates the interactions of one prey and one predator population (Chauvet et al., 2002). The Lotka-Volterra model makes two unrealistic assumptions. First, it assumes that in the absence of predators, the prey population will grow unboundedly (exponential prey growth) therefore, it can be arbitrary large. Second, it implies that individual predators never get satiation (Rocco, 2011). In order to fix this problem, several extensions of the prey-predator models were introduced. These include the Lotka-Volterra model with logistic growth in 1930 and the Rosenzweig-MacArthur model in 1963. The Lotka-Volterra model with logistic growth incorporates logistic growth to fix the unbounded exponential growth problem for the prey. The logistic growth guarantees that the prey can grow only to a certain saturation level. The Rosenzweig-MacArthur (R-M) model (Boccara, 2010; Kot, 2001) is based on the assumption that the eating and digesting process occurs at a non-constant rate. A R-M model normally incorporates the Holling type-II functional response. The Holling type-II functional response is a type of function in which the attack rate of predator increases at a decreasing rate with prey density until it becomes constant due to satiation. The difference between the Lotka-Volterra model with logistic growth and the R-M model is that the predation rate is no longer assumed to be proportional to prey density (Hurkova, 2013). The R-M model is inspired by behavior that can be found in nature and this thesis only focuses on the R-M model. The classical R-M model has been analyzed in Chen et al. (2010); Ivanov and Dimitrova (2017); Kar (2005); Ma et al. (2017).

In recent years, fractional-order differential equations have attracted the attention of researchers due to their ability to provide a good description of certain non-linear phenomena (Kilbas et al., 2006). The fractional order differential equations are generalizations of ordinary differential equations to arbitrary (non-integer) orders. In the last few years, many researchers studied the fractional order differential equations to describe complex systems in different branches of physics, chemistry and engineering (Heymans and Podlubny, 2006). This is because the fractional-order differential equations are naturally related to systems with memory (Hong-Li et al., 2016). Many biological systems possesses memory and the conception of fractional-order system may be closer to real life situations than integer-order systems. The fractional-order systems describe the whole time domain for physical processes, while the integer-order model is related to the local properties of a certain position. Also, the fractional-order allow greater degrees of freedom in the model (Petras, 2011). So, the fractional order model can give a more realistic interpretation of real life phenomena. Further, the fractional order differential equation helps to reduce the errors arising from the neglected parameters in modeling real life phenomena (Podlubny, 1999; Rana et al., 2013). In the last few years, it has been observed in many areas of engineering, physics and life sciences that models based on fractional order derivatives can provide better agreement between measured and simulated data than classical models based on integer order derivatives (Diethelm, 2013). Some papers have studied fractional order prey predator models and have found that the dynamics of fractional order model is more stable than its integer counterpart because the domain of stability in the fractional order model is larger than the domain for the corresponding integer order model (Ahmed et al., 2007; Rana et al., 2013).

Gilpin and Rosenzweig (1972) studied the stability of the positive equilibrium of R-M model by regarding the carrying capacity k as a bifurcation parameter. They found that prey and predator densities tend to a steady state if k is small but oscillate periodically if k is large enough to pass a critical value. Rosenzweig highlighted that increasing the carrying capacity of the prey may lead to destroy the steady state. This is known as the paradox of enrichment (Rana et al., 2013). In this thesis, we show that the introduction of fractional order to the R-M model resolves the paradox of enrichment.

The study of prey refuge on the dynamics of prey-predator systems can be recognized as a major issue in applied mathematics and theoretical ecology. Prey can move to areas called refuges where they are safe from their predators and this behaviour may reduce the prey mortality (González-Olivares and Ramos-Jiliberto, 2003). The use of refuge has been shown to enhance prey-predator coexistence by preventing prey extinction. Thus research on the dynamic behaviors of prey-predator systems incorporating a prey refuges has become a popular topic during the last decade (Chen et al., 2012). Incorporating a refuge is believed to provide a somewhat more realistic prey-predator model i.e. for a number of prey populations some form of refuge in the ecosystem is available.

In this thesis, the dynamical analysis of fractional-order R-M models incorporating a prey refuge is proposed. The focus will be on three populations which are prey, predator and top predator. The top predator (e.g. hawk) feeds on the predator (e.g. snake) only and in turn, the predator feeds on the prey (e.g. frog) only. The qualitative behavior of these models are analyzed. The existence, uniqueness, non-negativity and boundedness of the solutions are studied. The local and global stability of the equilibrium points of the fractional order system are investigated and the emergence of Hopf bifurcation in the fractional order system is illustrated. Moreover, the Adams-Bashforth-Moulton numerical method is applied for the numerical simulation of the fractional-order system to confirm the theoretical results. The numerical simulations focus on the influences of fractional order and prey refuge parameters on the population densities.

The three situations considered in this thesis (a two-species R-M model incorporating a prey refuge, a three species R-M model with a prey refuge, a three-species R-M model with stage structure and a prey refuge) are situations which often arise in studies involving ecosystems. A large body of literature exists for these situations. Hence this is why the situations have been studied.

1.1.1 Motivation

So far as it is known, the dynamical analysis of a fractional-order R-M model incorporating a prey refuge has not been performed before. This research, therefore, seeks to develop a R-M model incorporating fractional order and a prey refuge. This study is focused on the effects of fractional order and prey refuge on the dynamics of the R-M model. The findings of this research are useful for the mathematicians who are interested in ecology because this research allows a better understanding of the R-M model. It is also useful to ecologists who work on prey-predator interactions.

1.1.2 Research questions

The research questions which are relevant to this specific study are:

1. What are the advantages of fractional order models in comparison with classical integer-order models?
2. What are the effects of fractional order on the dynamics of R-M model?
3. What are the effects of refuge on the dynamics of fractional order R-M model?

1.2 Research objectives

The objectives of this study are:

1. To formulate and analyze a fractional-order R-M model incorporating a prey refuge.
2. To formulate and analyze an extended fractional-order R-M model with a prey refuge.
3. To formulate and analyze a fractional-order R-M model with stage structure incorporating a prey refuge.
4. To determine the combined influence of fractional order parameter and prey refuge on the stability of these systems.

5. To resolve the paradox of enrichment.

1.3 Methodology

A fractional-order R-M model incorporating a prey refuge is examined by extending the integer order model.

- The existence and uniqueness of the solutions are studied by using the Lipschitz condition.
- The non-negativity and boundedness of the solutions are studied by using the standard comparison theorem for fractional order and the positivity of Mittag-Leffler function.
- The basic reproduction number of fractional order system is obtained by using the next generation method.
- The local stability of the equilibrium points of the fractional order R-M system is studied by the well-known Matignon's condition.
- The global asymptotic stability of the equilibrium points of the fractional order R-M system is studied by constructing suitable Lyapunov functions.
- Sufficient conditions for the stability of the fractional order R-M model are demonstrated by analyzing the associated characteristic equation of the system at the equilibrium points.
- A Hopf bifurcation is shown to occur as fractional-order α and refuge m passes through critical points, α^* and m^*, respectively.

- These theoretical studies are verified numerically by using MATLAB-R2014a and MATHEMATICA-9.

1.4 Contribution

The main contributions of this study are as follows:

- Dynamical analysis of a fractional-order R-M model incorporating a prey refuge (Chapter 4).
- Dynamical analysis of an extended fractional-order R-M model with a prey refuge (Chapter 5).
- Dynamical analysis of a fractional-order R-M model with stage structure incorporating a prey refuge (Chapter 6).

1.5 Structure of thesis

This thesis consists of seven chapters. Chapter 1 gives an introduction to the study including, the background, motivation, research questions, research objectives, methodology and contribution. Chapter 2 reviews on necessary concepts, definitions, and theorems that will be used throughout this study. Chapter 3 presents the literature on fractional-order prey-predator models. In particular, those models which are related to R-M model. Chapter 4 presents the dynamical analysis of a fractional-order R-M model incorporating a prey refuge while Chapter 5 presents the dynamical systems analysis of an extended fractional-order R-M model with a prey refuge. Chapter 6 presents the dynamical analysis of a fractional-order R-M model with stage structure incorporating a prey refuge. Chapter 7 contains the conclusions and future works.

CHAPTER 2

BASIC CONCEPTS

This chapter is a review of necessary concepts, definitions, and theorems that will be used throughout this thesis.

2.1 Prey-predator model

The dynamics of prey-predator models are active research topics in mathematical ecology. One of the focus areas is the study on the local and global stability of the equilibrium points as well as the occurrence of Hopf bifurcation in the models. In this section, several models and relevant concepts are reviewed.

2.1.1 The classical Lotka-Volterra model

One of the first systems that modelled the interactions between prey and predator is the Lotka-Volterra model. This model, proposed by Alfred Lotka and Vito Volterra in 1925 (Boccara, 2010; Kot, 2001). The classical Lotka-Volterra model is a system of coupled non-linear ordinary differential equation as follows

$$\begin{aligned} \frac{dx}{dt} &= rx - \beta xy, \\ \frac{dy}{dt} &= cxy - \gamma y. \end{aligned} \tag{2.1}$$

All the parameters are non-negative for all time $t \geq 0$. The parameters are described in Table 2.1.

The assumptions of the classical Lotka-Volterra model are

Table 2.1: Parameters table for the R-M model.

Parameter	Description
x	Prey population.
y	Predator population.
r	Natural growth rate of the prey.
k	Carrying capacity of the prey.
γ	Death rate of the predator.
c	Coefficient in converting prey into a new predator.
β	Attack rate of the predator.
a	Half saturation constant.
mx	Refuge protecting of the prey.
$(1-m)x$	Prey available to the predator.
$\frac{\beta x}{1+ax}$	Holling type-II functional response.

- prey population x will grow exponentially in the absence of predators y,
- a constant per capita mortality rate of predators γ.
- a constant conversion rate of eaten prey into new predator abundance c,
- a constant predation rate β.

Unfortunately, this model does not describe actual behavior observed in nature. One of the biggest problems is that the prey population is not self-limiting and, therefore, this species can grow unboundedly. In order to fix this problem, a new version of the Lotka-Volterra model was introduced in 1930 (Hurkova, 2013). This model uses a logistic growth rate instead of exponential growth rate for the prey as follows

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - \beta xy, \\ \frac{dy}{dt} &= cxy - \gamma y, \end{aligned} \tag{2.2}$$

where k is the carrying capacity of the prey and other parameters are described in Table 2.1. The logistic growth guarantees that the prey population is self-limiting, therefore, the prey can grow only to a certain saturation level (Hurkova, 2013).

2.1.2 Holling's type functional responses

In population dynamics, a functional response of the predator to the prey density refers to relationship of an individual predator's rate of food consumption to prey density (Xiao and Ruan, 2001). One of the characteristics of Lotka-Volterra model (2.2) is that the predation term βxy is linear with respect to x and therefore is called a linear functional response of the predator (Britton, 2012). In the linear functional response, the attack rate of the predator rises linearly with prey density but then reaches a constant value when the predator is in satiating (Hurkova, 2013).

After the linear functional response by Lotka and Volterra (i.e. also called Holling type-I functional response), Holling (1959) proposed the well-known Holling type-II functional response as follows

$$f(x) = \frac{\beta x}{1+ax}. \tag{2.3}$$

The Holling type-II functional response is a type of function in which the attack rate of predator increases at a decreasing rate with prey density until it becomes constant due to satiation (Hurkova, 2013). It is a typical response of predators that specialize on one or a few prey. Also, Holling (1959) proposed the Holling type-III functional response as follows

$$f(x) = \frac{\beta x^2}{1+ax^2}. \tag{2.4}$$

The Holling type-III functional response causes prey consumption remains low until a threshold density is reached. The predation rate then increases exponentially until levels out.

The Holling type functional responses are derived from a realistic assumption, Holling improved the linear functional response by incorporating a predator handling time of prey besides attacking. The common feature of the Holling type-II and type-III functional responses lies in that they are both saturating functions when the density of prey becomes large (Holling, 1959; Wang, 2016). In this thesis, the Holling type-II functional response is used so as to represent the prey and predator interactions. This is because the Holling type-II functional response is simpler and derived from a realistic assumption (Freedman, 1980).

2.1.3 Rosenzweig-MacArthur model

The Lotka-Volterra model makes two unrealistic assumptions. First, it assumes that in the absence of predators, the prey population will grow exponentially. Second, it implies that individual predators never get full. The Rosenzweig-MacArthur (R-M) model proposed some corrections to these assumptions.

After killing a prey, a predator typically eats and digests its captured food. Some models assume that this occurs at a constant rate (Boccara, 2010; Kot, 2001). The R-M model is based on the assumption that the eating and digesting process occurs at a non-constant rate. The difference between the Lotka-Volterra model with logistic growth and the R-M model is that the predation rate is no longer assumed to be proportional to prey density. The R-M model is inspired by behavior that can be found in nature (Hurkova, 2013). Studies on the R-M model include Chen et al. (2010); Ivanov and Dimitrova (2017); Javidi and Nyamoradi (2013); Kar (2005); Ma et al. (2017); Moustafa et al. (2018); Nosrati and Shafiee (2017). A R-M model normally incorporates the Holling type-II functional response as follows:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - \frac{\beta xy}{1+ax}, \\ \frac{dy}{dt} &= \frac{c\beta xy}{1+ax} - \gamma y. \end{aligned} \tag{2.5}$$

All the parameters are non-negative for all time $t \geq 0$. The parameters are described in Table 2.1. In this thesis, we will focus on the R-M model.

2.1.4 Paradox of enrichment

Rosenzweig (1971) highlighted that increasing the carrying capacity of the prey (i.e. enriching the systems) may lead to destroy the steady state as shown in Fig. 2.1. This is known as the paradox of enrichment. Gilpin and Rosenzweig (1972) studied the stability of the positive equilibrium of the R-M model (2.5) by regarding the carrying capacity k as a bifurcation parameter, they find that prey and predator densities tend to a steady state if k is small but oscillate periodically if k is large enough to pass a critical value.

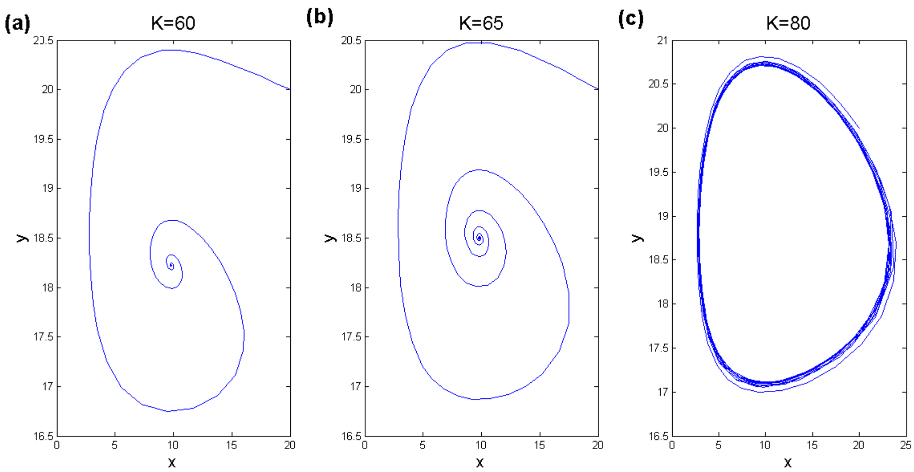

Figure 2.1: Phase portraits of Rosenzweig-MacArthur model with various values of k.

2.2 Fractional calculus

Fractional calculus was originated at the end of the seventeenth century, since the letter by Leibniz to L'Hopital in 1695, in which a half order derivative was mentioned (Podlubny, 1999). In recent years, fractional calculus has attracted much attention among researchers. It is the area of mathematics that extends derivatives and integrals to an arbitrary order. Fractional calculus as an important tool for mathematical modeling has been applied in different fields of sciences such as biological systems, economics and engineering (Podlubny, 1999). In this section, some basic definitions and preliminary concepts on fractional calculus used in this thesis are discussed.

Definition 2.1. *(Kilbas et al., 2006) The Riemann-Liouville fractional integral operator of order* $\alpha > 0$*, of function* $f : [0,\infty) \to \mathbb{R}$ *is defined as*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds,$$

where $\Gamma(.)$ *is the Gamma function.*

Definition 2.2. *(Kilbas et al., 2006) The Riemann-Liouville fractional derivative of order* $\alpha > 0$*, of a continuous function* $f : [0,\infty) \to \mathbb{R}$ *is defined as*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds, \quad n-1 < \alpha < n, \quad n \in \mathbb{N},$$

From Definition 2.2, one can observe that the definition of fractional derivative involves integration. Since integration is a non-local operator (as it is defined on an interval), fractional derivative is also a non-local operator. Calculating time-fractional derivative of a function $f(t)$ at some $t = t_1$ requires all the past history, i.e. all $f(t)$

from $t = 0$ to $t = t_1$ (Srivastava et al., 2015).

Definition 2.3. *(Diethelm and Ford, 2002; Kilbas et al., 2006) Suppose that* $\alpha > 0$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$, *the fractional operator*

$$^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds\,,$$

is called the Caputo fractional derivative of order α, *where* $f^{(n)}(s) = \frac{d^n}{ds^n} f(s)$ *and* $\Gamma(.)$ *is the Gamma function.*

The first main advantage of Caputo's derivatives is that the initial conditions of fractional differential equations take on the same form as for integer-order ones, which have more applications in modelling and analysis. The second advantage is that Caputo's derivative for a constant is zero, while the Riemann-Liouville fractional-order derivative for a constant is not zero (Podlubny, 1999). Therefore, Caputo's definition of fractional derivatives is used throughout this thesis.

Definition 2.4. *(Kilbas et al., 2006) The Mittag-Leffler function is defined by*

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)},$$

where $\alpha > 0$, $z \in \mathbb{R}$. *The two-parameter Mittag-Leffler function is defined by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)},$$

where $\alpha > 0$ *and* $\beta \in \mathbb{R}$, $z \in \mathbb{R}$.

Lemma 2.1. *(Li et al., 2010) Consider the following fractional-order system:*

$$ {}^{c}D_t^{\alpha}x(t) = f(t,x), \quad t > 0, \tag{2.6}$$

where $0 < \alpha < 1$*,* $f : [0,\infty) \times \Psi \to \mathbb{R}^n$*,* $\Psi \in \mathbb{R}^n$*, if* $f(t,x)$ *satisfies the locally Lipschitz condition with respect to x, then there exists a unique solution of* (2.6) *on* $[0,\infty) \times \Psi$.

Lemma 2.2. *(Wei et al., 2010) (The positivity of Mittag-Leffler function) For any* $\alpha \in (0,1)$ *and* $t \in \mathbb{R}$*, we have*

$$E_{\alpha}(t) = E_{\alpha,1}(t) > 0,\ \ E_{\alpha,\alpha}(t) > 0 \ \ \text{and} \ \ \frac{d}{dt}E_{\alpha,\alpha}(t) > 0.$$

Lemma 2.3. *(Choi et al., 2014) Let* $0 < \alpha < 1$ *and* $\lambda < 0$*. Then,* $E_{\alpha,\alpha}(\lambda t^{\alpha})$ *tend monotonically to zero as* $t \to \infty$.

Corollary 2.1. *(Choi et al., 2014) Let* $0 < \alpha < 1$ *and* $|\arg(\lambda)| > \frac{\alpha\pi}{2}$*. Then, one has*

$$t^{\alpha}E_{\alpha,\alpha+1}(\lambda t^{\alpha}) = -\frac{1}{\lambda} - \frac{1}{\Gamma(1-\alpha)\lambda^2 t^{\alpha}} + O\left(\frac{1}{\lambda^3 t^{2\alpha}}\right) \ \text{as}\ t \to \infty.$$

Lemma 2.4. *(Choi et al., 2014) (Standard comparison theorem in fractional order) Suppose that* $m \in C_p(\mathbb{R}^+,\mathbb{R})$ *satisfies*

$$ {}_{0}^{c}D_t^{\alpha}m(t) \leq \lambda m(t) + d, \quad m(t_0) = m_0,\ 0 \leq t_0 \leq t,$$

where λ*,* $d \in \mathbb{R}$*. Then one has*

$$m(t) \leq m(t_0)E_{\alpha}\left(\lambda(t-t_0)^{\alpha}\right) + d(t-t_0)^{\alpha}E_{\alpha,\alpha+1}(\lambda(t-t_0)^{\alpha}).$$

Lemma 2.5. *(Vargas-De-León, 2015) Let* $x(t) : \mathbb{R} \to \mathbb{R}$ *be a continuous and differentiable function. Then the following relationship holds:*

$$ {}_{0}^{c}D_t^{\alpha}\left(x(t) - x^* - x^* ln\frac{x(t)}{x^*}\right) \leq \left(1 - \frac{x^*}{x(t)}\right) {}_{t_0}^{c}D_t^{\alpha} x(t), \quad x^* \in \mathbb{R}^+, \forall \alpha \in (0,1), $$

for any time instant $t \geq t_0$.

Lemma 2.6. *(Wang and Li, 2014) If* ${}_{0}^{c}D_t^{\alpha} x(t) \geq 0$ *and* $x(0) \geq 0,\ 0 < \alpha < 1$, *then* $x(t) \geq 0$.

Lemma 2.7. *(Wang and Li, 2014) (Comparison theorem)*

Let $0 < \alpha < 1$ *and let* $x(0) = y(0)$; *then* $x(t) \geq y(t)$, *if* ${}_{0}^{c}D_t^{\alpha} x(t) \geq {}_{0}^{c}D_t^{\alpha} y(t)$.

2.3 Important dynamical concepts

The dynamical system approach is used to explore the population dynamics of the prey-predator system presented in this thesis. The main concepts are as follows.

2.3.1 Equilibrium Points and stability

Now we introduce important concepts pertaining to equilibrium points and stability of the fractional order system.

Studying equilibrium solutions is important in mathematical ecology because it predicts long-term behaviors of a system (Wang, 2016). Consider the fractional order autonomous system

$$ {}^{c}D^{\alpha} x(t) = f(x), \quad x(0) = x_0 \in \mathbb{R}^n. \tag{2.7} $$

Definition 2.5. *(Li and Zhang, 2011) A point* $E \in \mathbb{R}^n$ *is called as an equilibrium point* *(steady states) of* (2.7) *if* $f(E) = 0$.

An equilibrium point E (steady states) of system (2.7) implies that the system (2.7) at this point remain unchanged with time.

Matignon (1996) studied the following autonomous fractional differential system involving Caputo derivative

$$^{c}D^{\alpha}x(t) = Ax(t), \tag{2.8}$$

with initial value $x(0) = x_0$, $\alpha \in (0,1)$. Sayevand (2016) stated that the qualitative behavior of the solution set of a nonlinear system of fractional differential equations near an equilibrium point is typically the same as the qualitative behavior of the solution set of the corresponding linearized system near the equilibrium point.

The stability of the equilibrium of system (2.8) was defined and established by Matignon as follows.

Definition 2.6. *(Qian et al., 2010) The autonomous system* (2.8) *is said to be*

(i) stable iff for any x_0*, there exists* $\varepsilon > 0$ *such that* $\|x(t)\| \leq \varepsilon$ *for* $t \geq 0$;

(ii) asymptotically stable iff $\lim_{t \to +\infty} \|x(t)\| = 0$.

In Definition 2.6, the global stability include the local stability and the globally asymptotically stable include the locally asymptotically stable. In general, the system is stable if it always returns to and stays near a steady state, and is unstable if it goes farther away from any state, without being bounded (Franklin et al., 1994).

2.3.2 Matignon's conditions

Consider the following non-linear fractional-order system:

$$^{c}D_t^{\alpha}x(t) = f(x), \tag{2.9}$$

where $0 < \alpha < 1$ and $x \in \mathbb{R}^n$. The equilibrium points of the system (2.9) are solutions to the following equation

$$f(x) = 0,$$

an equilibrium is locally asymptotically stable if all eigenvalues $\mu_i,\ (i = 1,2,\cdots,n)$ of the Jacobian matrix $J = \frac{\partial f}{\partial x}$ evaluated at the equilibrium satisfy the following condition (Petras, 2011),

$$|\arg(\mu_i)| > \frac{\alpha\pi}{2}, \quad i = 1,2,\cdots,n.$$

For $\alpha = 1$ and $0 < \alpha < 1$, Figure 2.2 shows the stability and instability regions of the

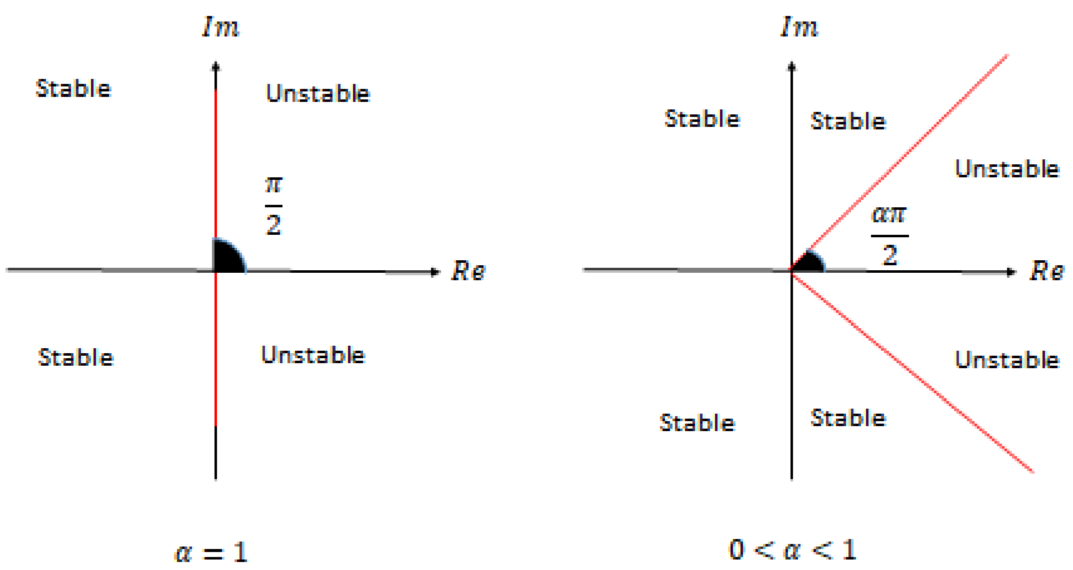

Figure 2.2: Stability and instability region of the fractional-order system, when $0 < \alpha < 1$ and $\alpha = 1$.

fractional-order system (2.9). It is interesting to note that the fractional order system is more stable than its integer counterpart because the domain of stability of eigenvalues for fractional order system is larger than the domain for the corresponding integer order system as shown in Fig. 2.2.

2.3.3 Fractional order Routh–Hurwitz conditions

Let us consider the following three-dimensional fractional-order commensurate system:

$$D^{\alpha}x(t) = f(x),$$

where $\alpha \in (0,1)$, $x \in \mathbb{R}^3$, and suppose that E is an equilibrium point of this system, then its characteristic equation is given as

$$F(\mu) = \mu^3 + B_1\mu^2 + B_2\mu + B_3 = 0. \tag{2.10}$$

The discriminant $D(F)$ of the polynomial $F(\mu)$ is

$$D(F) = 18B_1B_2B_3 + (B_1B_2)^2 - 4B_3B_1^3 - 4B_2^3 - 27B_3^2.$$

According to Abdelouahab et al. (2012); Ahmed et al. (2006), one obtains the following proposition.

Proposition 2.1.

(i) If $D(F) > 0$, $B_1 > 0$, $B_3 > 0$ and $B_1B_2 > B_3$, then the equilibrium point E is locally asymptotically stable for $0 < \alpha < 1$.

(ii) *If* $D(F) < 0$, $B_1 \geq 0$, $B_2 \geq 0$, $B_3 > 0$ *and* $0 < \alpha < \frac{2}{3}$, *then the equilibrium point E is locally asymptotically stable.*

(iii) *If* $D(F) < 0$, $B_1 < 0$, $B_2 < 0$ *and* $\alpha > \frac{2}{3}$, *then the equilibrium point E is unstable.*

(iv) *If* $D(F) < 0$, $B_1 > 0$, $B_2 > 0$, $B_1 B_2 = B_3$ *and* $0 < \alpha < 1$, *then the equilibrium point E is locally asymptotically stable.*

2.3.4 Volterra Lyapunov function

Throughout the study of global stability in next chapters, the following function will be considered;

$$\begin{aligned} f(x) =& x - x^* - x^* \ln\left(\frac{x}{x^*}\right), \\ f'(x) =& 1 - \frac{x^*}{x},\ f''(x) = \frac{x^*}{x^2}, \end{aligned}$$

therefore, $f'(x) < 0$ for $0 < x < x^*$, $f'(x) > 0$ for $x > x^*$ and $f''(x) > 0$. Hence $f(x)$ has global minimum at x^* for $x > 0$. Thus, the function $f(x) = x - x^* - x^* \ln\left(\frac{x}{x^*}\right)$ is positive definite Lyapunov function (Korobeinikov, 2001).

2.3.5 Hopf bifurcation

Bifurcation describes an abrupt change from one state to the other when some parameters pass the critical values. Bifurcation study is a powerful tool in understanding an ecological community because bifurcation implies an abrupt change from one state to the other (Wang, 2016). Hopf bifurcation of fractional-order systems can be analyzed through stability theory of equilibrium points and numerical simulations (Li and Wu, 2014).

Let us consider the following two-dimensional fractional-order commensurate system:

$$^{c}D^{\alpha}x = f(\varphi, x), \tag{2.11}$$

where $0 < \alpha < 1$, $x \in \mathbb{R}^2$ and suppose that E is an equilibrium point of system (2.11).

The stability of equilibrium point E is related to the sign of

$$\theta_i(\alpha, \varphi) = \frac{\alpha\pi}{2} - |\arg(\mu_i(\varphi))|, \quad i = 1,2.$$

If $\theta_i(\alpha, \varphi) < 0$ for all $i = 1,2$, then E is locally asymptotically stable. If there exist i such that $\theta_i(\alpha, \varphi) > 0$, then, the equilibrium point E is unstable (Abdelouahab et al., 2012).

In Abdelouahab et al. (2012), a fractional order Hopf bifurcation is proposed which states that system (2.11) undergoes a Hopf bifurcation through the equilibrium E at the value φ^* of φ if:

(i) The Jacobian matrix has two complex-conjugate eigenvalues $\mu_{1,2}$,

(ii) $\theta_{1,2}(\alpha, \varphi^*) = 0$,

(iii) $\frac{\partial \theta_{1,2}}{\partial \varphi}|_{\varphi=\varphi^*} \neq 0$, where

$$\theta(\alpha, \varphi) = \frac{\alpha\pi}{2} - |\arg(\mu_i(\varphi))|, \quad i = 1,2.$$

In next chapters, we will investigate the occurrence of Hopf bifurcation in the fractional-order R-M model using above conditions.

2.4 Summary

In this chapter, some basic concepts and theorems required for the mathematical models and their analysis in this thesis are discussed. The ideas presented include, Lotka-Volterra model, Holling's type functional responses, R-M model, paradox of enrichment, fractional calculus and important dynamical concepts. These theorems the positivity of Mittag-Leffler function, standard comparison theorem in fractional order, Matignon's conditions and several others are used in Chapters 4, 5, and 6.

CHAPTER 3

LITERATURE REVIEW

In this chapter, we review the literature on integer and fractional prey-predator models in the recent literatures. In particular, those models related to prey-predator system with different functional responses incorporating prey refuge, three species food chain models and prey-predator models with stage structure.

3.1 Prey-predator model with prey refuge

The study of prey refuge on the dynamics of prey-predator systems can be recognized as a major issue in applied mathematics and theoretical ecology. The use of refuge has been shown to enhance prey-predator coexistence by preventing prey extinction. Thus research on the dynamical behaviors of prey-predator systems incorporating a prey refuges has become a popular topic during the last decade (Chen et al., 2012). Prey can move to areas called refuges where they are safe from their predators and this behaviour may reduce the prey mortality (González-Olivares and Ramos-Jiliberto, 2003). Incorporating a refuge is believed to provide a somewhat more realistic prey-predator model i.e. for a number of prey populations some form of refuge in the ecosystem is available. In this thesis, the constant proportion of prey refuge is used. This is because the presence of a constant proportion of prey refuge does not change the nature of the dynamical stability of the model (Sarwardi et al., 2013). Some studies of the dynamical behaviour of prey-predator models incorporating refuge include Ali and Chakravarty (2016); Das et al. (2013); González-Olivares and Ramos-Jiliberto

(2003); Hong-Li et al. (2016); Naji and Majeed (2016); Samanta et al. (2016); Sarwardi et al. (2012); Tripathi et al. (2015); Verma and Misra (2018); Wei and Fu (2016); Yue (2016); Zhang et al. (2017); Chen et al. (2010); Chen et al. (2012); Ma et al. (2009); Sarwardi et al. (2013).

Ma et al. (2017) presented a prey-predator system with Holling type function response incorporating prey refuge as follows

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{\beta(1-m)^n x^n y}{1+a(1-m)^n x^n},\\ \frac{dy}{dt} &= \left(\frac{c\beta(1-m)^n x^n}{1+a(1-m)^n x^n}-\gamma\right)y. \end{aligned} \tag{3.1}$$

All the parameters are non-negative for all time $t \geq 0$. The parameters are described in Table 2.1. The exponent n describes the shape of the functional response. When $n = 1$, the system (3.1) reduces to a R-M model incorporating a prey refuge which was investigated by Chen et al. (2010); Kar (2005). The model is as follows

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{\beta(1-m)xy}{1+a(1-m)x},\\ \frac{dy}{dt} &= \frac{c\beta(1-m)xy}{1+a(1-m)x}-\gamma y. \end{aligned} \tag{3.2}$$

Where mx is a refuge protecting of the prey and $m \in [0,1)$. Note that if $m = 1$ there is no predation.

When $n = 2$, the system (3.1) reduces to a prey-predator model with Holling type-III functional response incorporating a prey refuge which was investigated by Huang et al. (2006). The model is

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{\beta(1-m)^2x^2y}{1+a(1-m)^2x^2},\\ \frac{dy}{dt} &= \frac{c\beta(1-m)^2x^2y}{1+a(1-m)^2x^2}-\gamma y.\end{aligned} \tag{3.3}$$

Ma et al. (2017) investigated the dynamical behaviours of the system (3.1), including stability, limit cycle and bifurcation. They did not present any numerical simulations of the system (3.1) to clarify their results. There is also a major error in section 4.3 (Page 8). The authors stated that the positive equilibrium point of the system (3.1) is globally asymptotic stable in the article written by Ma et al. (2017). Using the documented data in Kar (2005), for $n=1,\ r=10,\ K=100,\ a=0.02,\ \gamma=0.09,\ \beta=0.6$ and $c=0.02$, the eigenvalues of the Jacobian matrix of system (3.1) are

$$\mu_1 = 0.00389273+0.815917i \text{ and } \mu_2 = 0.00389273-0.815917i.$$

This means that the system (3.1) is unstable and there is a periodic solution around the positive equilibrium point $(12.9758, 25.0935)$.

3.2 Three species food chain model

The dynamics of food chain model are active research topics in mathematical ecology. This mechanism helps us to understand the predation process as well as the stable and unstable dynamics of the ecosystem in the long run (Ali and Chakravarty, 2016). The food chain model contains several layers such that the consumers which eat from the bottom resource layer become the prey of another predator. The standard prey-