# THE PARTIAL-ISOMETRIC CROSSED PRODUCTS BY SEMIGROUPS OF ENDOMORPHISMS AND THE KHOSHKAM-SKANDALIS ALGEBRA

**by**

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**HASIL DARAB SILANG SEPARA-ISOMETRIK ANTARA SEMIKUMPULAN BAGI ENDOMORFISMA DAN ALJABAR KHOSHKAM-SKANDALIS**

## ABSTRAK

Andaikan $\Gamma^+$ adalah kon positif daripada sebuah kumpulan abelan yang bertertib sepenuhnya $\Gamma$. Biar $(A, \Gamma^+, \alpha)$ sebuah sistem dinamik yang terdiri daripada sebuah aljabar-$C^*$ $A$ dan sebuah tindakan $\alpha$ dari $\Gamma^+$ sebagai endomorfisma yang meluas dari $A$. Dalam tesis ini;

(i) kami menyamaratakan takrifan bagi aljabar Khoshkam-Skandalis kepada konteks endomorphisms yang meluas, iaitu $\mathcal{T}_\alpha$, dan kemudian kami membuktikan bahawa hasil darab silang separa-isometrik bagi $(A, \Gamma^+, \alpha)$ merupakan sudut penuh bagi aljabar $\mathcal{T}_\alpha$;

(ii) kami menunjukkan bahawa apabila $\alpha$ adalah sebuah tindakan automorfisma dari $A$, maka $A \times_\alpha^{\text{piso}} \Gamma^+$ merupakan salah satu dari hasil darab silang isometrik.

Dengan menggunakan (i), kami menunjukkan bahawa jika $\Gamma^+$ $\mathbb{N}$, maka ideal $\ker \phi$ dari $A \times_\alpha^{\text{piso}} \Gamma^+$ dari homomorfisma surjektif semula jadi $\phi : A \times_\alpha^{\text{piso}} \Gamma^+ \to A \times_\alpha^{\text{iso}} \Gamma^+$, adalah merapikan sebuah sudut penuh didalam ideal $\mathcal{K}(\ell^2(\mathbb{N}, A))$ dari $\mathcal{T}_\alpha$. Akibatnya, kami memperoleh dari $A \times_\alpha^{\text{piso}} \mathbb{N}$ oleh dua buah aljabar $A \times_\alpha^{\text{iso}} \mathbb{N}$ dan sudut penuh dan $\mathcal{K}(\ell^2(\mathbb{N}, A))$.

Sebagai aplikasi daripada perluasan tersebut, kami memperoleh perluasan dari Khoshkam-Skandalis dan juga perluasan dari Pimsner-Voiculescu.

Dengan menggunakan (ii), kami menunjukkan bahawa jika $\alpha$ adalah tindakan sebagai automorfisma dari $A$, maka $A \times_\alpha^{\text{piso}} \Gamma^+$ dapat dipandang sebagai sudut penuh didalam hasil darab silang biasa. Tambahan lagi, kami mengidentifikasi ideal $\ker \phi$ dari $A \times_\alpha^{\text{piso}} \Gamma^+$ sebagai sebuah hasil darab silang isometrik.

**THE PARTIAL-ISOMETRIC CROSSED PRODUCTS BY SEMIGROUPS OF ENDOMORPHISMS AND THE KHOSHKAM-SKANDALIS ALGEBRA**

## ABSTRACT


Suppose $\Gamma^+$ is the positive cone of a totally ordered abelian group $\Gamma$. Let $(A, \Gamma^+, \alpha)$ be a dynamical system consisting of a $C^*$-algebra $A$ and an action $\alpha$ of $\Gamma^+$ by extendible endomorphisms of $A$. In this thesis;

(i) we generalize the definition of the algebra of Khoshkam-Skandalis, namely $\mathcal{T}_\alpha$ to the context of extendible endomorphism, and then we prove that the partial-isometric crossed product of $(A, \Gamma^+, \alpha)$ is a full corner in the algebra $\mathcal{T}_\alpha$;

(ii) we show that when $\alpha$ is an action by automorphisms of $A$, then $A \times_\alpha^{\text{piso}} \Gamma^+$ is one of the isometric crossed products.

By using the realization (i), we show that if $\Gamma^+$ is the semigroup $\mathbb{N}$ of the additive group $\mathbb{Z}$, then the ideal $\ker \phi$ of $A \times_\alpha^{\text{piso}} \Gamma^+$ arising by the natural surjective homomorphism $\phi : A \times_\alpha^{\text{piso}} \Gamma^+ \to A \times_\alpha^{\text{iso}} \Gamma^+$ is a full corner in $\mathcal{K}(\ell^2(\mathbb{N}, A))$ of compact operators contained in $\mathcal{T}_\alpha$. Therefore we get the extension of $A \times_\alpha^{\text{piso}} \mathbb{N}$ by the two algebras $A \times_\alpha^{\text{iso}} \mathbb{N}$ and the full corner of $\mathcal{K}(\ell^2(\mathbb{N}, A))$.

As applications of this extension, we obtained the exact sequence of Khoshkam-Skandalis and Pimsner-Voiculescu.

By using the realization (ii), we show that if $\alpha$ is an action by automorphisms of $A$, then $A \times_\alpha^{\text{piso}} \Gamma^+$ can be viewed as a full corner in the usual crossed product of a system by a group of automorphisms. Moreover, we identify the ideal $\ker \phi$ of $A \times_\alpha^{\text{piso}} \Gamma^+$ as one of the ideals of the isometric crossed products.

# CHAPTER 1
# INTRODUCTION

## 1.1 Literature Review

Let $\Gamma$ be a totally ordered abelian group and $\Gamma^+ := \{x \in \Gamma : x \geq 0\}$ the positive cone of $\Gamma$. A dynamical system $(A, \Gamma^+, \alpha)$ is a system consisting of a (not necessarily unital) $C^*$-algebra $A$ and an action $\alpha$ of $\Gamma^+$ by endomorphisms of $A$ such that $\alpha_0 = \mathrm{id}_A$.

Our interest in the present work is to study the representation theory of such dynamical systems. When $\Gamma^+ = \mathbb{N}$ and $1 \in A$, Stacey in [25] defines a covariant representation of $(A, \mathbb{N}, \alpha)$ to be a pair $(\pi, V)$ of a unital representation $\pi : A \to B(H)$ of $A$ on a Hilbert space $H$ and an isometry $V$ on the same Hilbert space $H$ which satisfies the covariance relation:

$$\pi(\alpha(a)) = V\pi(a)V^* \quad \text{for all } a \in A. \tag{1.1}$$

Then Stacey shows in [25] that if $(A, \mathbb{N}, \alpha)$ has a nontrivial covariant representation, then there is a $C^*$-algebra denoted by $A \times_\alpha^{\mathrm{iso}} \mathbb{N}$, generated as a $C^*$-algebra by a universal covariant representation $(i_A, i_{\mathbb{N}})$ such that the unital representations of $A \times_\alpha^{\mathrm{iso}} \mathbb{N}$ are in a bijective correspondence with covariant representations of $(A, \mathbb{N}, \alpha)$. We call the $C^*$-algebra $A \times_\alpha^{\mathrm{iso}} \mathbb{N}$ the isometric crossed product of $(A, \mathbb{N}, \alpha)$. Furthermore, by using the idea of Cuntz [9], Stacey shows that the $C^*$-algebra $A \times_\alpha^{\mathrm{iso}} \mathbb{N}$ is a full corner in the crossed product of a $C^*$-algebra by a group of automorphic actions.

Later on Adji in [1] generalized Stacey's results from the semigroup $\mathbb{N}$ to the positive cone $\Gamma^+$ and to nonunital $C^*$-algebras. Because she studied on nonunital $C^*$-algebras, she had to assume that every endomorphism $\alpha_x : A \to A$ is extendible to a strictly continuous endomorphism $\overline{\alpha}_x$ of the multiplier algebra $M(A)$ of $A$. The assumption on extendible endomorphism is required because the $C^*$-algebra $A$ in the system may not contain an identity. It was shown in [4] that an endomorphism $\phi : A \to A$ is extendible precisely when there is an approximate identity $\{a_\lambda\}$ for $A$ and a projection $P_\phi \in M(A)$ such that $\phi(a_\lambda)$ converges strictly to $P_\phi$ in $M(A)$. We wish to stress that extendibility of endomorphisms $\alpha_x$ does not always imply that $\overline{\alpha}_x$ is unital ($\overline{\alpha}_x(1_{M(A)}) = 1_{M(A)}$), even though $1 \in A$. For example, we see that the endomorphism $\tau(x_1, x_2, ...) = (0, x_1, x_2, ...)$ on the $C^*$-algebra $\mathbf{c}$ of convergent sequences, extends to the endomorphism $\overline{\tau} = \tau$ on $M(\mathbf{c}) = \mathbf{c}$ which satisfies $\overline{\tau}(1, 1, ...) = (0, 1, 1, ...) \neq (1, 1, ...)$.

A covariant representation of $(A, \Gamma^+, \alpha)$ is given by a nondegenerate representation $\pi : A \to B(H)$ and an isometric representation $V : \Gamma^+ \to B(H)$, where every $V_x$ is an isometry on $H$ and $V_{x+y} = V_x \circ V_y$, and that

$$\pi(\alpha_x(a)) = V_x \pi(a) V_x^* \quad \text{for all } x \in \Gamma^+ \text{ and } a \in A. \tag{1.2}$$

The isometric crossed product $A \times_\alpha \Gamma^+$ has exactly the same property as described by Stacey: it is a full corner in the crossed product of the system by the group of automorphisms.

Adji, Laca, Raeburn and Nilsen described in [6] a condition for a covariant representation $(\pi, V)$ of the system to induce a faithful representation of the isometric crossed product. They considered the system $(B_{\Gamma^+}, \Gamma^+, \tau)$ of the unital $C^*$-algebra $B_{\Gamma^+} := \overline{\text{span}}\{1_s \in \ell^\infty(\Gamma^+) : s \in \Gamma^+\}$ spanned by the characteristics function $1_s$ defined by $1_s(x) = 1$ for $x \geq s$ and $1_s(x) = 0$ for $x < s$, and the action $\tau : \Gamma^+ \to \text{End}(B_{\Gamma^+})$ is given by the right translation on $\ell^\infty(\Gamma^+)$ which satisfies $\tau_t(1_s) = 1_{s+t}$. They show that every isometric representation $V$ of $\Gamma^+$ on a Hilbert space $H$ induces a unital representation $\pi_V : 1_s \mapsto V_s V_s^*$ of $B_{\Gamma^+}$ on $H$ such that $(\pi_V, V)$ is a covariant isometric representation of $(B_{\Gamma^+}, \Gamma^+, \tau)$, and the representation $\pi_V \times V$ of $B_{\Gamma^+} \times_\tau^{\text{iso}} \Gamma^+$ is faithful provided all $V_s$ are nonunitary. This gives us a realization of the *Toeplitz algebra* $\mathcal{T}(\Gamma)$, the $C^*$-subalgebra of $B(\ell^2(\Gamma^+))$ generated by the nonunitary isometries $\{T_s : s \in \Gamma^+\}$ given by $T_s(\varepsilon_r) = \varepsilon_{r+s}$ on the usual orthonormal basis $\{\varepsilon_r : r \in \Gamma^+\}$ of $\ell^2(\Gamma^+)$, as the isometric crossed product $B_{\Gamma^+} \times_\tau^{\text{iso}} \Gamma^+$. More precisely, since each $T_s$ in the isometric representation $T : s \mapsto T_s$ of $\Gamma^+$ on $\ell^2(\Gamma^+)$ is nonunitary, $\pi_T \times T$ is an isomorphism of $B_{\Gamma^+} \times_\tau^{\text{iso}} \Gamma^+$ onto $\mathcal{T}(\Gamma)$.

Since then, several works have been carried out by authors to study the isometric crossed product associated to more general ordered semigroups: Larsen in [16] studied the isometric crossed product of the systems of Ore ordered semigroups, and their applications to the Hecke $C^*$-algebras, Laca and Raeburn in [14] contributed the theory of isometric crossed products of the systems of nonabelian semigroups with quasi-lattice order, and their applications to number theory. Authors in [3, 7, 8] studied the structure of isometric crossed product of a system of positive cones of totally ordered groups, and describe the primitive ideal space of the Toeplitz algebra $\mathcal{T}(\Gamma)$.

These facts confirm that the isometric crossed product theory provides the most effective method to study the Toeplitz algebra (the $C^*$-algebra generated by Toeplitz operators) [3, 6].

The success of isometric crossed product attracts the authors in [17] to study a more general representation theory of a system $(A, \Gamma^+, \alpha)$ in which the semigroup $\Gamma^+$ is represented by partial isometries (instead of isometries). They define a covariant representation of $(A, \Gamma^+, \alpha)$ as a pair $(\pi, W)$ which consists of a nondegenerate representation $\pi$ of $A$ and a partial-isometric representation $W$ of $\Gamma^+$ on $H$ such that

$$\pi(\alpha_s(a)) = W_s \pi(a) W_s^* \quad \text{and} \quad W_s^* W_s \pi(a) = \pi(a) W_s^* W_s \quad \text{for } s \in \Gamma^+, a \in A. \tag{1.3}$$

.

Unlike the isometric covariant representation theory of $(A, \Gamma^+, \alpha)$, every system $(A, \Gamma^+, \alpha)$ admits a nontrivial covariant partial-isometric representation $(\pi, W)$ with $\pi$ faithful [17, Example 4.6]. Then the crossed product $A \times_\alpha^{\text{piso}} \Gamma^+$ (we call the partial-isometric crossed product) of $(A, \Gamma^+, \alpha)$ which by definition is the Toeplitz $C^*$-algebra of Hilbert bimodules, which was studied enormously by Fowler [10]. It is the universal $C^*$-algebra generated by the canonical partial-isometric covariant representation $(i_A, i_{\Gamma^+})$ such that $A \times_\alpha^{\text{piso}} \Gamma^+$ has one-to-one correspondence with the partial-isometric covariant representations of $(A, \Gamma^+, \alpha)$ (see Definition 2.5). The main result of [17] shows that the crossed product of $(B_{\Gamma^+}, \Gamma^+, \tau)$ has a similar property to the isometric crossed products, and is universal for partial isometric representation of the semigroup $\Gamma^+$. So every partial isometric representation of $\Gamma^+$ gives a representation $\pi_V$ of $B_{\Gamma^+}$ defined by $\pi_V(1_x) = V_x V_x^*$ such that $(\pi_V, V)$ is the partial-isometric covariant representation of the Proposition 5.1 [17]. Moreover, the authors described the structure of $B_{\Gamma^+} \times_\tau^{\text{piso}} \Gamma^+$ in the large commutative diagram [17, Theorem 5.6 ], in which this diagram determines completely the structure of the crossed product of the system when $\Gamma^+ = \mathbb{N}$ [17, Theorem 6.1].

The question that naturally arises here is whether there is a realization of partial-isometric crossed products as full corners similar to the one of isometric crossed product. The work of this thesis is dedicated towards answering the question: Can $A \times_\alpha^{\text{piso}} \Gamma^+$ be viewed as a corner in a particular $C^*$-algebra? We want to point out that the method we use here is adapted from Khoshkam-Skandalis [12], which is different from the one of isometric crossed products: it does not involve the dilation process. In fact we construct a partial isometric representation of the system $(A, \Gamma^+, \alpha)$ such that the associated representation of the crossed product is an isomorphism onto the full corner.

To view the construction, we set up the subalgebra $\mathcal{T}_\alpha$ of $\mathcal{L}(\ell^2(\Gamma^+, A))$ associated to a system $(A, \Gamma^+, \alpha)$ using a pair $(\pi_\alpha, S)$ in which $\pi_\alpha : A \to \mathcal{L}(\ell^2(\Gamma^+, A))$ is a faithful representation defined by $(\pi_\alpha(a)f)(x) = \alpha_x(a)f(x)$ and $S : \Gamma^+ \to \mathcal{L}(\ell^2(\Gamma^+, A))$ is an isometric representation given by $(S_y f)(x) = f(x-y)$ for $x \geq y$ and $(S_y f)(x) = 0$ for $x < y$ where $a \in A$ and $x, y \in \Gamma^+$. The pair $(\pi_\alpha, S)$ satisfies the equation

$$\pi_\alpha(a)S_y = S_y\pi_\alpha(\alpha_y(a)) \tag{1.4}$$

for all $a \in A$ and $y \in \Gamma^+$. The algebra $\mathcal{T}_\alpha$ is then defined as the $C^*$-subalgebra of $\mathcal{L}(\ell^2(\Gamma^+, A))$ spanned by $\{S_x\pi_\alpha(a)S_y^* : a \in A, x, y \in \Gamma^+\}$. The pair $(\pi_\alpha, S)$ is neither the partial-isometric covariant nor isometric-covariant representation, however we prove in Proposition 4.1 that there is a right covariant partial-isometric representation $(k_A, w) : (A, \Gamma^+, \alpha) \to \mathcal{L}(\ell^2(\Gamma^+, A))$ such that the representation $k_A \times w$ of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ is an isomorphism onto the full corner of $\mathcal{T}_\alpha$ .

Furthermore by using this full corner identification, we obtain in Theorem 5.1 the extension of $A \times_\alpha^{\mathrm{piso}} \mathbb{N}$, from which we recover the exact sequences of [12] and [22].

Finally, we want to mention that when dealing with unital endomorphisms, $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ is exactly the Koshkham-Skandalis algebra $\mathcal{T}_\alpha$ described in [12]. Next, if $\alpha : \Gamma^+ \to \mathrm{Aut}(A)$ is an action by automorphisms of $A$, then we realize that $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ is the isometric crossed product of the system $(B_{\Gamma^+} \otimes A, \tau \otimes \alpha^{-1}, \Gamma^+)$ (see Proposition 6.1).

### 1.2 Overview of the Thesis

We begin with a preliminary chapter (Chapter 2) containing background materials about Hilbert $C^*$-modules and the extendibility of homomorphisms and ideals of $C^*$-algebras. Moreover, we briefly recall the theories of isometric crossed product and partial-isometric crossed product.

In Chapter 3 we identify the spanning elements of the kernel of the natural homomorphism from partial-isometric crossed product onto the isometric crossed product.

In Chapter 4, we construct a covariant partial-isometric representation of $(A, \Gamma^+, \alpha)$ in $\mathcal{L}(\ell^2(\Gamma^+, A))$ and show that this gives an isomorphism of $A \times_\alpha^{\text{piso}} \Gamma^+$ onto a full corner of the subalgebra $\mathcal{T}_\alpha$ of $\mathcal{L}(\ell^2(\Gamma^+, A))$.

In Chapter 5, we show that when the semigroup $\Gamma^+$ is $\mathbb{N} = \mathbb{Z}^+$ then the kernel of natural homomorphism is a full corner in the compact operators of $\ell^2(\mathbb{N}, A)$.

In Chapter 6, we discuss the theory of partial-isometric crossed products for systems by automorphic actions of the semigroups $\Gamma^+$. We show that $A \times_\alpha^{\text{piso}} \Gamma^+$ is a full corner in the usual crossed product $(B_\Gamma \otimes A) \times \Gamma$ by the group of automorphisms.

Finally, Chapter 7 discusses conclusions and proposes an open question.

# CHAPTER 2
# PRELIMINARIES

This chapter contains the required background theory of $C^*$-algebras which supports the main results of the thesis. However it does not contain the basic theory of $C^*$-algebras on a graduate course level. The readers who wish to know about basic terminologies in $C^*$-algebras can refer to the textbook [20], and the theory of crossed products can be founded in [26]. We wish to mention that we specifically elaborate Proposition 2.1 and Proposition 2.2 on Hilbert $C^*$-modules although they are known since we frequently use them considering that their proofs are existing in other works or mathematics textbooks. We will use these in Chapters 4 and 5.

## 2.1 Hilbert $C^*$-Modules

### 2.1.1 Basic Theories of Hilbert $C^*$-Modules

**Definition 2.1** Let $A$ be a $C^*$-algebra and $E$ be a right $A$-module with compatible scalar multiplication, that is $\lambda(x \cdot a) = (\lambda x) \cdot a = x \cdot (\lambda a)$ for every $x \in E$, $a \in A$, and $\lambda \in \mathbb{C}$. We say $E$ is a (right) *inner product $A$-module*, if there is a map

$$E \times E \to A$$

$$(x, y) \mapsto \langle x, y \rangle_A$$

such that

(i) $\langle x, \lambda y + \mu z \rangle_A = \lambda \langle x, y \rangle_A + \mu \langle x, z \rangle_A$;

(ii) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$;

(iii) $\langle x, y \rangle_A^* = \langle y, x \rangle_A$;

(iv) $\langle x, x \rangle_A$ is a positive element of $A$;

(v) If $\langle x, x\rangle_A = 0$ then $x = 0$,

for every $x, y, z \in E$, $\lambda, \mu \in \mathbb{C}$, and $a \in A$.

**Remark 2.1** In the definition above, condition (i) implies that the inner product $\langle \cdot, \cdot \rangle_A$ is linear in the second variable, and from conditions (i) and (iii), it follows that it is conjugate linear in the first variable:

$$\begin{aligned}\langle \lambda x + \mu y, z\rangle_A = \langle z, \lambda x + \mu y\rangle_A^* &= (\lambda \langle z, x\rangle_A + \mu \langle z, y\rangle_A)^* \\ &= \overline{\lambda}\langle z, x\rangle_A^* + \overline{\mu}\langle z, y\rangle_A^* \\ &= \overline{\lambda}\langle x, z\rangle_A + \overline{\mu}\langle y, z\rangle_A\end{aligned}$$

Moreover, conditions (ii) and (iii) imply that

$$a\langle x, y\rangle_A = (\langle y, x\rangle_A a^*)^* = \langle y, x \cdot a^*\rangle_A^* = \langle x \cdot a^*, y\rangle_A.$$

Therefore this together with condition (ii) show that

$$\mathrm{span}\{\langle x, y\rangle_A : x, y \in E\}$$

is a two-sided ideal of $A$.

**Example 2.1** Every $C^*$-algebra $A$ can be viewed as an inner product $A$-module in which the action of $A$ is given by the right multiplication $a \cdot b = ab$ and the inner product is given by $\langle a, b\rangle_A = a^* b$ for every $a, b \in A$.

**Lemma 2.1** (**The Cauchy-Schwarz inequality**) [24, Lemma 2.5] *If $E$ is an inner product $A$-module, and $x, y \in E$, then*

$$\langle x, y\rangle_A^* \langle x, y\rangle_A \leq \|\langle x, x\rangle_A\| \langle y, y\rangle_A. \tag{2.1}$$

Next we define

$$\|x\|_A := \|\langle x, x\rangle_A\|^{1/2} \text{ for every } x \in E. \tag{2.2}$$

By using the Cauchy-Schwarz inequality, we can show that $\|\cdot\|_A$ is a norm on $E$. More precisely, we have:

**Corollary 2.1** [24, Corollary 2.7] *If $E$ is an inner product $A$-module, then*

$$\|x\|_A := \|\langle x, x\rangle_A\|^{1/2}$$

*defines a norm on $E$ such that $\|x \cdot a\|_A \leq \|x\|_A \|a\|$. The normed module $(E, \|\cdot\|_A)$ is non-degenerate in the sense that the elements of the form $x \cdot a$ span a dense subspace of $E$. Indeed,*

$$E \cdot \langle E, E\rangle_A := \operatorname{span}\{x \cdot \langle y, z\rangle_A : x, y, z \in E\}$$

*is $\|\cdot\|_A$-dense in $E$.*

**Definition 2.2** An inner product $A$-module $E$ which is complete with respect to the norm $\|\cdot\|_A$ is called a *Hilbert $A$-module.* Moreover, if the ideal

$$I = \operatorname{span}\{\langle x, y\rangle_A : x, y \in E\}$$

is dense in $A$, then we say $E$ is a *full* Hilbert $A$-module.

**Example 2.2** Every Hilbert space $H$ with the usual scalar multiplication and the inner product is a Hilbert $\mathbb{C}$-module, meaning that the action of $\mathbb{C}$ on $H$ and the $\mathbb{C}$-valued inner product are given by

$$h \cdot \lambda := \lambda h \text{ and } \langle h, k\rangle_{\mathbb{C}} := \langle k|h\rangle$$

where $h, k \in H$ and $\lambda \in \mathbb{C}$.

**Example 2.3** Every $C^*$-algebra $A$ is a full Hilbert $A$-module with $a \cdot b = ab$ and $\langle a, b\rangle_A = a^*b$ for every $a, b \in A$. In this case, the norm $\|\cdot\|_A$ on $A$ equals to the usual norm ($C^*$-norm) of $A$, because for every $a \in A$, we have

$$\|a\|_A = \|\langle a, a\rangle_A\|^{1/2} = \|a^*a\|^{1/2} = (\|a\|^2)^{1/2} = \|a\|.$$

$A$ is full because if $\{a_\lambda\}$ is an approximate identity for $A$, then $a_\lambda a$ converges to $a$ in the norm of $A$. But for each $\lambda$, $a_\lambda a = a_\lambda^* a = \langle a_\lambda, a\rangle_A$ which shows that the ideal $I = \text{span}\{\langle a, b\rangle_A : a, b \in A\}$ is dense in $A$, and therefore $A$ is full.

**Proposition 2.1** *Let $S$ be any nonempty set and $A$ be a $C^*$-algebra. Then*

$$\ell^2(S, A) := \{f : S \to A : \sum_{s \in S} f(s)^* f(s) \ \ \text{converges in the norm of } A\},$$

*the vector space consisting of all $A$-valued functions of $S$ such that the unordered sum $\sum_{s\in S} f(s)^*f(s)$ converges unconditionally in the norm of $A$, is a full Hilbert $A$-module with*

$$(f \cdot a)(s) := f(s)a \ \ \text{and} \ \ \langle f, g\rangle_A := \sum_{s \in S} f(s)^* g(s)$$

*for every $f, g \in \ell^2(S, A)$ and $a \in A$.*

*Proof.* If $S$ is a finite set, then it follows by the Example 2.14 of [24] that $\ell^2(S, A)$ is a Hilbert $A$-module. Thus we suppose that $S$ is an infinite set which can be either countable or uncountable. However our proof involves the finite subsets of $S$. We therefore need to consider the collection $\mathcal{F}$ of all finite subsets of $S$ which is a directed set with the inclusion '$\subset$'.

First we want to check that $f \cdot a$ belongs to $\ell^2(S,A)$. For any $F \in \mathcal{F}$, we have

$$\begin{aligned}\textstyle\sum_{s\in F}(f(s)a)^*f(s)a &= \textstyle\sum_{s\in F} a^*f(s)^*f(s)a \\ &= a^*(\textstyle\sum_{s\in F} f(s)^*f(s))a \\ &\leq \|\textstyle\sum_{s\in F} f(s)^*f(s)\| a^*a,\end{aligned}$$

because $b^*cb \leq \| c \| b^*b$ is valid for every $b \in A$ and $c$ positive. Since in $A$, $0 \leq b \leq c$ implies $\|b\| \leq \|c\|$, it follows that

$$\|\sum_{s\in F}(f(s)a)^*f(s)a\| \leq \|a\|^2 \|\sum_{s\in F} f(s)^*f(s)\|.$$

We know that since $f \in \ell^2(S,A)$, the partial sums of $\sum_{s\in S} f(s)^*f(s)$ are Cauchy in $A$. Therefore for any $\epsilon > 0$, there is $I \in \mathcal{F}$ such that for every $J \in \mathcal{F}$ with $I \cap J = \emptyset$, we have $\| \sum_{s\in J} f(s)^*f(s)\| < \epsilon/\|a\|^2$. This implies that

$$\|\sum_{s\in J}(f(s)a)^*f(s)a\| \leq \|a\|^2 \|\sum_{s\in J} f(s)^*f(s)\| < \epsilon$$

which means the partial sums of $\sum_{s\in S}(f(s)a)^*f(s)a$ are Cauchy in $A$, and consequently $\sum_{s\in S}(f(s)a)^*f(s)a$ converges in $A$. Thus $f \cdot a$ belongs to $\ell^2(S,A)$.

Next we want to prove that $\langle f,g\rangle_A$ is an inner product on $\ell^2(S,A)$. To do this, first we have to show that $\sum_{s\in S} f(s)^*g(s)$ converges in $A$, thus $\langle f,g\rangle_A$ is well-defined. For every $F \in \mathcal{F}$, the Cauchy-Schwarz inequality implies that

$$\|\sum_{s\in F} f(s)^*g(s)\|^2 \leq \|\sum_{s\in F} f(s)^*f(s)\| \|\sum_{s\in F} g(s)^*g(s)\|. \tag{2.3}$$

Similar to the proof of $f \cdot a \in \ell^2(S,A)$, since $f,g \in \ell^2(S,A)$, for every $\epsilon > 0$, there are $I, I' \in \mathcal{F}$ such that for every $J \in \mathcal{F}$ with $I \cap J = \emptyset$, we have

$$\|\sum_{s\in J} f(s)^* f(s)\| < \epsilon,$$

and for every $K \in \mathcal{F}$ with $I' \cap K = \emptyset$, we have

$$\|\sum_{s\in K} g(s)^* g(s)\| < \epsilon.$$

It follows that for every $J \in \mathcal{F}$ with $J \cap (I \cup I') = \emptyset$,

$$\|\sum_{s\in J} f(s)^* g(s)\|^2 \leq \|\sum_{s\in J} f(s)^* f(s)\| \|\sum_{s\in J} g(s)^* g(s)\| < \epsilon^2$$

which implies that $\|\sum_{s\in J} f(s)^* g(s)\| < \epsilon$. Therefore $\sum_{s\in S} f(s)^* g(s)$ is Cauchy in $A$, and hence it is convergent in $A$.

Now we have to check that $\langle \cdot, \cdot \rangle_A$ satisfies all the conditions (i)-(v) in Definition 2.1. Let $f, g, h \in \ell^2(S, A)$, $\lambda, \mu \in \mathbb{C}$, and $a \in A$.

(i) $\langle f, \lambda g + \mu h \rangle_A = \lambda \langle f, g \rangle_A + \mu \langle f, h \rangle_A$:

To see this, we have to show that $\sum_{s\in S} f(s)^*(\lambda g(s) + \mu h(s))$ converges to $\lambda\langle f, g\rangle_A + \mu\langle f, h\rangle_A$. For every $J \in \mathcal{F}$, we have

$$\begin{aligned}
&\|\textstyle\sum_{s\in J} f(s)^*(\lambda g(s) + \mu h(s)) - (\lambda\langle f, g\rangle_A + \mu\langle f, h\rangle_A)\| \\
&= \|\lambda(\textstyle\sum_{s\in J} f(s)^* g(s) - \langle f, g\rangle_A) + \mu(\textstyle\sum_{s\in J} f(s)^* h(s) - \langle f, h\rangle_A)\| \qquad (2.4)\\
&\leq |\lambda| \|\textstyle\sum_{s\in J} f(s)^* g(s) - \langle f, g\rangle_A\| + |\mu| \|\textstyle\sum_{s\in J} f(s)^* h(s) - \langle f, h\rangle_A\|.
\end{aligned}$$

Since $\langle f, g\rangle_A$ and $\langle f, h\rangle_A$ are the limits of all partial sums (finite subsums) of $\sum_{s\in S} f(s)^* g(s)$ and $\sum_{s\in S} f(s)^* h(s)$ respectively, for every $\epsilon > 0$, there exist

$I, I' \in \mathcal{F}$ such that for every $J \in \mathcal{F}$, if $I \subset J$, then

$$\|\sum_{s\in J} f(s)^* g(s) - \langle f, g\rangle_A\| < \epsilon/(2|\lambda|),$$

and if $I' \subset J$, then

$$\|\sum_{s\in J} f(s)^* h(s) - \langle f, h\rangle_A\| < \epsilon/(2|\mu|).$$

Therefore for every $J \in \mathcal{F}$ with $I \cup I' \subset J$, by inequality (2.4), we get

$$\|\sum_{s\in J} f(s)^*(\lambda g(s)+\mu h(s))-(\lambda\langle f, g\rangle_A+\mu\langle f, h\rangle_A)\| < |\lambda|(\epsilon/(2|\lambda|))+|\mu|(\epsilon/(2|\mu|)) = \epsilon$$

Therefore $\sum_{s\in S} f(s)^*(\lambda g(s) + \mu h(s)) = \lambda\langle f, g\rangle_A + \mu\langle f, h\rangle_A$. On the other hand, $\langle f, \lambda g + \mu h\rangle_A = \sum_{s\in S} f(s)^*(\lambda g(s) + \mu h(s))$, so it follows that $\langle f, \lambda g + \mu h\rangle_A = \lambda\langle f, g\rangle_A + \mu\langle f, h\rangle_A$. This shows (i) is valid.

(ii) $\langle f, g \cdot a\rangle_A = \langle f, g\rangle_A a$:

We want to show that $\sum_{s\in S} f(s)^*(g(s)a)$ converges to $\langle f, g\rangle_A a$. For any $J \in \mathcal{F}$, we have

$$\begin{aligned}
\|\textstyle\sum_{s\in J} f(s)^*(g(s)a) - \langle f, g\rangle_A a\| &= \|(\textstyle\sum_{s\in J} f(s)^* g(s))a - \langle f, g\rangle_A a\| \\
&= \|(\textstyle\sum_{s\in J} f(s)^* g(s) - \langle f, g\rangle_A)a\| \\
&\leq \|\textstyle\sum_{s\in J} f(s)^* g(s) - \langle f, g\rangle_A\|\|a\|
\end{aligned}$$

Since $\sum_{s\in S} f(s)^* g(s) = \langle f, g\rangle_A$, for every $\epsilon > 0$ there exists $I \in \mathcal{F}$ such that for every $J \in \mathcal{F}$ with $I \subset J$ we have $\|\sum_{s\in J} f(s)^* g(s) - \langle f, g\rangle_A\| < \epsilon/\|a\|$. This implies that for every $J \in \mathcal{F}$, if $I \subset J$, then

$$\|\sum_{s\in J} f(s)^*(g(s)a) - \langle f, g\rangle_A a\| < (\epsilon/\|a\|)\|a\| = \epsilon.$$

It follows that $\sum_{s\in S} f(s)^*(g(s)a)$ converges to $\langle f,g\rangle_A a$, and therefore $\langle f, g\cdot a\rangle_A = \langle f,g\rangle_A a$.

(iii) $\langle f,g\rangle_A^* = \langle g,f\rangle_A$:

To see this, since $\sum_{s\in S} f(s)^*g(s) = \langle f,g\rangle_A$, so for every $\epsilon > 0$ there is a $I \in \mathcal{F}$ such that for every $J \in \mathcal{F}$ with $I \subset J$ we have $\| \sum_{s\in J} f(s)^*g(s) - \langle f,g\rangle_A\| < \epsilon$. But

$$\begin{aligned}\| \textstyle\sum_{s\in J} g(s)^*f(s) - \langle f,g\rangle_A^*\| &= \| \textstyle\sum_{s\in J}(f(s)^*g(s))^* - \langle f,g\rangle_A^*\| \\ &= \|(\textstyle\sum_{s\in J} f(s)^*g(s))^* - \langle f,g\rangle_A^*\| \\ &= \|(\textstyle\sum_{s\in J} f(s)^*g(s) - \langle f,g\rangle_A)^*\| \\ &= \| \textstyle\sum_{s\in J} f(s)^*g(s) - \langle f,g\rangle_A\|,\end{aligned}$$

therefore

$$\| \sum_{s\in J} g(s)^*f(s) - \langle f,g\rangle_A^*\| = \| \sum_{s\in J} f(s)^*g(s) - \langle f,g\rangle_A\| < \epsilon.$$

This implies that the finite subsums of $\sum_{s\in S} g(s)^*f(s)$ converge to $\langle f,g\rangle_A^*$, and hence $\langle f,g\rangle_A^* = \langle g,f\rangle_A$.

(iv) $\langle f,f\rangle_A \geq 0$ in $A$:

If $f = 0$, then it is clear, because $\langle f,f\rangle_A = 0$. Now assume $f \neq 0$ and $\langle f,f\rangle_A \leq 0$. So it follows that $-\langle f,f\rangle_A \geq 0$. Since $f \neq 0$, there is a $t \in S$ such that $f(t) \neq 0$ in $A$. Thus for $\epsilon = \|f(t)\|^2/2 > 0$, there exists $I \in \mathcal{F}$ such that for every $J \in \mathcal{F}$, if $I \subset J$, then $\| \sum_{s\in J} f(s)^*f(s) - \langle f,f\rangle_A\| < \|f(t)\|^2/2$. Let $J' = I \cup \{t\}$, since $I \subset J'$, we have

$$\| \sum_{s\in J'} f(s)^*f(s) - \langle f,f\rangle_A\| < \|f(t)\|^2/2$$

which ia a contradiction. Because

$$\sum_{s\in J'} f(s)^*f(s) - \langle f,f\rangle_A \geq f(t)^*f(t) - \langle f,f\rangle_A \geq f(t)^*f(t) > 0$$

which implies that

$$\|\sum_{s\in J'} f(s)^*f(s) - \langle f,f\rangle_A\| \geq \|f(t)^*f(t)\| = \|f(t)\|^2 > \|f(t)\|^2/2.$$

Therefore $\langle f,f\rangle_A \geq 0$ for every $f \in \ell^2(S,A)$.

(v) If $\langle f,f\rangle_A = 0$, then $f = 0$:

Suppose $\langle f,f\rangle_A = 0$ but $f \neq 0$. Thus there is a $t \in S$ such that $f(t) \neq 0$. Since $\langle f,f\rangle_A = 0$, so for $\epsilon = \|f(t)\|^2/2 > 0$ there exists $I \in \mathcal{F}$ such that for every $J \in \mathcal{F}$ with $I \subset J$ we have $\|\sum_{s\in J} f(s)^*f(s)\| < \|f(t)\|^2/2$. Now if $J' = I \cup \{t\} \supset I$, then

$$\|\sum_{s\in J'} f(s)^*f(s)\| < \|f(t)\|^2/2.$$

But this is a contradiction because

$$\sum_{s\in J'} f(s)^*f(s) \geq f(t)^*f(t) > 0$$

which implies that

$$\|\sum_{s\in J'} f(s)^*f(s)\| \geq \|f(t)^*f(t)\| = \|f(t)\|^2 > \|f(t)\|^2/2.$$

Thus (v) is also satisfied, and therefore $\ell^2(S,A)$ is an inner product $A$-module.

As we know, this inner product induces a norm on $\ell^2(S,A)$ given by $\|f\|_A = \|\langle f,f\rangle_A\|^{1/2} = \|\sum_{s\in S} f(s)^*f(s)\|^{1/2}$. Now we want to show that $\ell^2(S,A)$ is complete in this norm, and hence it is a Hilbert $A$-module. To see this, we show that every Cauchy sequence in $\ell^2(S,A)$ is convergent.

Let $\{f_n\}_{n=0}^{\infty}$ be a Cauchy sequence in $\ell^2(S,A)$. For every $t \in S$, $\{f_n(t)\}_{n=0}^{\infty}$ is also a Cauchy sequence in $A$. To see this, first note that if $f \in \ell^2(S,A)$, then for every $\epsilon > 0$ there exists $I \in \mathcal{F}$ such that for every $J \in \mathcal{F}$ with $I \subset J$ we have

$$\|\sum_{s \in J} f(s)^* f(s) - \langle f, f\rangle_A\| < \epsilon.$$

Thus if $t \in S$, then for $J' = I \cup \{t\} \supset I$ we have

$$\|\sum_{s \in J'} f(s)^* f(s) - \langle f, f\rangle_A\| < \epsilon.$$

Now since $0 \leq f(t)^* f(t) \leq \sum_{s \in J'} f(s)^* f(s)$, $\|f(t)^* f(t)\| \leq \|\sum_{s \in J'} f(s)^* f(s)\|$, and consequently

$$\begin{aligned} \|f(t)\|^2 - \|\langle f, f\rangle_A\| &\leq \|\textstyle\sum_{s \in J'} f(s)^* f(s)\| - \|\langle f, f\rangle_A\| \\ &\leq \|\textstyle\sum_{s \in J'} f(s)^* f(s) - \langle f, f\rangle_A\| < \epsilon. \end{aligned}$$

It follows that $\|f(t)\|^2 < \|\langle f, f\rangle_A\| + \epsilon$ for every $\epsilon > 0$. This implies that $\|f(t)\|^2 \leq \|\langle f, f\rangle_A\| = \|f\|_A^2$, and hence $\|f(t)\| \leq \|f\|_A$ for every $t \in S$. From this we conclude that for any $\epsilon > 0$, since $\{f_n\}_{n=0}^{\infty}$ is Cauchy, there exists $N > 0$ such that for every $m$ and $n$ with $n \geq m \geq N$ we have

$$\|f_n(t) - f_m(t)\| = \|(f_n - f_m)(t)\| \leq \|f_n - f_m\|_A < \epsilon,$$

for every $t \in S$. Thus $\{f_n(t)\}_{n=0}^{\infty}$ is a Cauchy sequence in $A$ and therefore it is convergent in $A$. Let $f : S \to A$ be a map such that $f(t) = \lim_{n \to \infty} f_n(t)$. We claim that $f \in \ell^2(S, A)$ and $f_n \to f$ in $\ell^2(S, A)$, thus it follows that $\ell^2(S, A)$ is complete.

To see $f \in \ell^2(S, A)$, we will prove that the partial sums of $\sum_{s \in S} f(s)^* f(s)$ are Cauchy in $A$ and therefore it is convergent in $A$. First we need these materials that for every map $g : S \to A$ and $F \in \mathcal{F}$, define

$$\|g\|_F := \|\sum_{s\in F} g(s)^* g(s)\|^{1/2}.$$

Note that for every $F \in \mathcal{F}$, the vector space

$$\ell^2(F, A) := \{h : F \to A : \sum_{s\in F} h(s)^* h(s) \text{ converges in the norm of } A\}$$

is a Hilbert $A$-module by [24, Example 2.14] as we have mentioned in the beginning of the proof for a finite set $S$. Therefore the module norm on $\ell^2(F, A)$ is $\|\cdot\|_F$, because we can in fact view $\ell^2(F, A)$ as the subspace of $\ell^2(S, A)$. It is the subspace of all $h \in \ell^2(S, A)$ such that $h(s) = 0$ whenever $s \notin F$. Moreover, if $g \in \ell^2(S, A)$, then the map $h : F \to A$ given by

$$h(s) = \begin{cases} g(s) & \text{if } s \in F \\ 0 & \text{if } s \notin F \end{cases}$$

belongs to $\ell^2(F, A)$ such that $\|h\|_A = \|g\|_F$. Similar to the proof of $\|g(t)\| \leq \|g\|_A$ for every $g \in \ell^2(S, A)$ and $t \in S$ in above, we can show that$\|g\|_F \leq \|g\|_A$ for every $g \in \ell^2(S, A)$.

Now since $\{f_n\}_{n=0}^{\infty}$ is Cauchy, for given $\epsilon > 0$ there exists $N > 0$ such that for every $m$ and $n$ with $m \geq n \geq N$ we have $\|f_m - f_n\|_A < \epsilon/3$. On the other hand, for $F \in \mathcal{F}$, we can find $M \geq N$ such that $\|f - f_M\|_F < \epsilon/3$. Because the sum in $\|f - f_M\|_F = \|\sum_{s\in F}(f(s) - f_M(s))^*(f(s) - f_M(s))\|^{1/2}$ is finite and as we have seen it before, $f_i(s) \to f(s)$ for every $s \in S$. Moreover, since $\sum_{s\in S} f_N(s)^* f_N(s)$ converges in $A$, for $\epsilon > 0$ there exits $I \in \mathcal{F}$ such that for every $F \in \mathcal{F}$ with $I \cap F = \emptyset$, we have $\|f_N\|_F = \|\sum_{s\in F} f_N(s)^* f_N(s)\| < \epsilon/3$. Thus for every $F \in \mathcal{F}$

with $I \cap F = \emptyset$, if we choose $M \geq N$ such that $\|f - f_M\|_F < \epsilon/3$, then we have

$$\begin{aligned}
\|f\|_F &= \|f - f_M + f_M - f_N + f_N\|_F \\
&\leq \|f - f_M\|_F + \|f_M - f_N\|_F + \|f_N\|_F \\
&\leq \|f - f_M\|_F + \|f_M - f_N\|_A + \|f_N\|_F < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}$$

Therefore $\| \sum_{s \in F} f(s)^* f(s) \| = \|f\|_F^2 < \epsilon^2$, and since the existence of $I \in \mathcal{F}$ depends only on $\epsilon$, we conclude that the partial sums of $\sum_{s \in S} f(s)^* f(s)$ are Cauchy in $A$. This implies that $f \in \ell^2(S, A)$.

Now we want to show that $f_n \to f$ in $\ell^2(S, A)$. For any $\epsilon > 0$, there exists $N > 0$ such that for every $m \geq n \geq N$, we have $\|f_m - f_n\|_A < \epsilon$. Thus for every $F \in \mathcal{F}$,

$$\| \sum_{s \in F} (f_m(s) - f_n(s))^* (f_m(s) - f_n(s)) \| < \epsilon^2,$$

and if $m \to \infty$, then we get

$$\| \sum_{s \in F} (f(s) - f_n(s))^* (f(s) - f_n(s)) \| < \epsilon^2 \quad \text{for every } n \geq N. \tag{2.5}$$

We know that $f - f_n \in \ell^2(S, A)$, because $f \in \ell^2(S, A)$. Thus $\sum_{s \in S} (f(s) - f_n(s))^* (f(s) - f_n(s))$ converges in $A$, and since the inequality (2.5) is valid for every $F \in \mathcal{F}$, $\|f - f_n\|_A < \epsilon$ for every $n \geq N$. This implies that $f_n \to f$ in $\ell^2(S, A)$.

Finally we show that $\ell^2(S, A)$ is full. Let $a \in A$ and $\{a_\lambda\}$ is an approximate identity in $A$. Since $S$ is nonempty, there is a $t \in S$. Now define the maps $f, g_\lambda : S \to A$ such that

$$f(s) = \begin{cases} a & \text{if } s = t \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g_\lambda(s) = \begin{cases} a_\lambda & \text{if } s = t \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $f, g_\lambda \in \ell^2(S, A)$, and since $\langle g_\lambda, f\rangle_A = a_\lambda a$ converges to $a$ in $A$, it follows that the ideal $I = \text{span}\{\langle f, g\rangle_A : f, g \in \ell^2(S, A)\}$ is dense in $A$. Thus $\ell^2(S, A)$ is a full Hilbert $A$-module. This completes the proof. ∎

**Remark 2.2** We will see in Proposition 2.2 that $\ell^2(S, A)$ is naturally isomorphic to the (tensor product) Hilbert $A$-module $\ell^2(S) \otimes A$.

### 2.1.2 Internal Tensor Product of Hilbert $C^*$-Modules

Let $A$ and $B$ be two $C^*$-algebras. Suppose $E$ is a (right) Hilbert $A$-module, $F$ is a (right) Hilbert $B$-module and $\phi : A \to \mathcal{L}(F)$ is a $*$-homomorphism of $A$ into the $C^*$-algebra of adjointable operators on $F$. We can consider $F$ as a $A - B$ bimodule, the left action of $A$ is given by

$$a \cdot y = \phi(a)y \ \text{ for } a \in A, y \in F$$

The algebraic tensor product of $E$ and $F$ over $A$, $E \odot_A F$, is the quotient of vector space tensor product $E \odot F$ by the subspace

$$N = \text{span}\{(x \cdot a) \otimes y - x \otimes \phi(a)y : x \in E, y \in F \text{ and } a \in A\},$$

meaning that $E \odot_A F = (E \odot F)/N$. Let $x \otimes_A y$ denote the coset $(x \otimes y) + N$, then $(x \cdot a) \otimes_A y = x \otimes_A \phi(a)y$. $E \odot_A F$ is a right $B$-module with the action of $B$ given by

$$(x \otimes_A y) \cdot b = x \otimes_A (y \cdot b) \ \text{ for } x \in E, y \in F \text{ and } b \in B$$

Proposition 4.5 of [15] says that $E \odot_A F$ is an inner product (right) $B$-module under the $B$-valued inner product given on simple tensor by

$$\langle x_1 \otimes_A y_1, x_2 \otimes_A y_2 \rangle_B = \langle y_1, \phi(\langle x_1, x_2 \rangle_A) y_2 \rangle_B \tag{2.6}$$

for every $x_1, x_2 \in E$ and $y_1, y_2 \in F$. This $B$-valued inner product induces a norm on $E \odot_A F$, and the completion of $E \odot_A F$ with respect to this norm is a (right) Hilbert $B$-module which is called the *Internal tensor product* of $E$ and $F$ and is denoted by $E \otimes_A F$.

Particularly, when $E$ is a Hilbert space $H$ and $F$ is a $C^*$-algebra $A$, we have an important case of the internal tensor product of Hilbert $C^*$-modules. Since every Hilbert space $H$ is a (right) Hilbert $\mathbb{C}$-module and every $C^*$-algebra $A$ can be regarded as a (right) Hilbert $A$-module over itself, it follows that the vector space tensor product $H \odot A$ is a (right) inner product $A$-module with

$$\langle h \otimes a, k \otimes b \rangle_A = \langle k \mid h \rangle a^* b \quad \text{and} \quad (h \otimes a) \cdot b = h \otimes ab \tag{2.7}$$

for every $h, k \in H$ and $a, b \in A$. So we have a (right) Hilbert $A$-module $H \otimes A$ which is the completion of $H \odot A$ with respect to the associated module norm induced by the $A$-valued inner product [15, pages 6 and 41].

In the Proposition below, we elaborate one of the most important and special realizations in the internal tensor product of Hilbert $C^*$-modules. We will see that for any set $S$, the Hilbert $A$-module $\ell^2(S) \otimes A$ is isomorphic to the Hilbert $A$-module $\ell^2(S, A)$ explained in Proposition 2.1. We remind the readers that two Hilbert $A$-modules $E$ and $F$ are isomorphic if there is a linear map $u : E \to F$ which is isometric, surjective, and $A$-linear.

**Proposition 2.2** *Let $S$ be any nonempty set and $A$ be a $C^*$-algebra. The Hilbert $A$-module $\ell^2(S) \otimes A$ is obtained by completing the vector space tensor product*

*$\ell^2(S) \odot A$ over the $A$-valued inner product $\langle \xi \otimes a, \eta \otimes b\rangle_A = \langle \eta \mid \xi\rangle a^*b$ and the module action $(\xi \otimes a) \cdot b = \xi \otimes ab$ for every $\xi, \eta \in \ell^2(S)$ and $a, b \in A$. This module is naturally isomorphic to the Hilbert $A$-module*

$$\ell^2(S, A) = \{f : S \to A : \sum_{s\in S} f(s)^*f(s) \text{ converges in the norm of } A\}$$

*equipped with the module structures $(f \cdot a)(s) = f(s)a$ and $\langle f, g\rangle_A = \sum_{s\in S} f(s)^*g(s)$ for every $f, g \in \ell^2(S, A)$, $a \in A$ and $s \in S$.*

*Proof.* To see this, first let $\mathcal{F}$ be the collection of all finite subsets of $S$. Then define the map $\varphi : \ell^2(S) \times A \to \ell^2(S, A)$ such that $(\varphi(\xi, a))(s) = \xi(s)a$ for all $\xi \in \ell^2(S)$, $a \in A$ and $s \in S$. Note that $\varphi(\xi, a) \in \ell^2(S, A)$, because for every $F \in \mathcal{F}$,

$$\begin{aligned} \|\textstyle\sum_{s\in F}(\varphi(\xi, a))(s)^*(\varphi(\xi, a))(s)\| &= \|\textstyle\sum_{s\in F} \overline{\xi(s)}\xi(s)a^*a\| \\ &= \|(\textstyle\sum_{s\in F} \overline{\xi(s)}\xi(s))a^*a\| \\ &= |\textstyle\sum_{s\in F} \overline{\xi(s)}\xi(s)|\|a^*a\|. \end{aligned}$$

Since the partial sums of $\sum_{s\in S} \overline{\xi(s)}\xi(s)$ are Cauchy in $\mathbb{C}$, similar to the proof in Proposition 2.1, we can show that the partial sums of $\sum_{s\in S}(\varphi(\xi, a))(s)^*(\varphi(\xi, a))(s)$ is Cauchy in $A$, and hence it is convergent in $A$. The map $\varphi$ is bilinear, therefore there is a well-defined linear map $\widetilde{\varphi} : \ell^2(S) \odot A \to \ell^2(S, A)$ such that $\widetilde{\varphi}(\xi \otimes a)(s) = \xi(s)a$. Now we show that $\widetilde{\varphi}$ is an isometric $A$-linear map whose extension on $\ell^2(S) \otimes A$ is surjective.

To see $\widetilde{\varphi}$ is $A$-linear, let $\gamma = \sum_{i=1}^{n} \xi_i \otimes a_i \in \ell^2(S) \odot A$ and $b \in A$. We want to show that $\widetilde{\varphi}(\gamma \cdot b) = \widetilde{\varphi}(\gamma) \cdot b$. But first see that for a simple element $\xi \otimes a \in \ell^2(S) \odot A$, we have

$$\widetilde{\varphi}((\xi \otimes a) \cdot b)(s) = \widetilde{\varphi}(\xi \otimes ab)(s) = \xi(s)ab = \widetilde{\varphi}(\xi \otimes a)(s)b = (\widetilde{\varphi}(\xi \otimes a) \cdot b)(s)$$

for every $s \in S$. This implies that $\widetilde{\varphi}((\xi \otimes a) \cdot b) = \widetilde{\varphi}(\xi \otimes a) \cdot b$. Thus for $\gamma$, by linearity of $\widetilde{\varphi}$, it follows that

$$\begin{aligned}
\widetilde{\varphi}(\gamma \cdot b) = \widetilde{\varphi}((\sum_{i=1}^{n} \xi_i \otimes a_i) \cdot b) &= \widetilde{\varphi}(\sum_{i=1}^{n} (\xi_i \otimes a_i) \cdot b) \\
&= \sum_{i=1}^{n} \widetilde{\varphi}((\xi_i \otimes a_i) \cdot b) \\
&= \sum_{i=1}^{n} \widetilde{\varphi}(\xi_i \otimes a_i) \cdot b \\
&= (\sum_{i=1}^{n} \widetilde{\varphi}(\xi_i \otimes a_i)) \cdot b = \widetilde{\varphi}(\sum_{i=1}^{n} \xi_i \otimes a_i) \cdot b = \widetilde{\varphi}(\gamma) \cdot b.
\end{aligned}$$

Therefore $\widetilde{\varphi}$ is $A$-linear.

$\widetilde{\varphi}$ is an isometry, because if $\alpha = \sum_{i=1}^{n} \xi_i \otimes a_i$ and $\beta = \sum_{j=1}^{m} \eta_j \otimes b_j$ in $\ell^2(S) \odot A$, then we have

$$\begin{aligned}
\langle \widetilde{\varphi}(\alpha), \widetilde{\varphi}(\beta) \rangle_A &= \langle \sum_{i=1}^{n} \widetilde{\varphi}(\xi_i \otimes a_i), \sum_{j=1}^{m} \widetilde{\varphi}(\eta_j \otimes b_j) \rangle_A \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \widetilde{\varphi}(\xi_i \otimes a_i), \widetilde{\varphi}(\eta_j \otimes b_j) \rangle_A \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} [\sum_{s \in S} \widetilde{\varphi}(\xi_i \otimes a_i)(s)^* \widetilde{\varphi}(\eta_j \otimes b_j)(s)] \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} [\sum_{s \in S} \overline{\xi_i(s)} \eta_j(s) a_i^* b_j] \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \eta_j \mid \xi_i \rangle a_i^* b_j \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \xi_i \otimes a_i, \eta_j \otimes b_j \rangle_A \\
&= \langle \sum_{i=1}^{n} \xi_i \otimes a_i, \sum_{j=1}^{m} \eta_j \otimes b_j \rangle_A = \langle \alpha, \beta \rangle_A.
\end{aligned}$$

So for every $\alpha \in \ell^2(S) \odot A$, $\|\widetilde{\varphi}(\alpha)\|_A = \|\langle \widetilde{\varphi}(\alpha), \widetilde{\varphi}(\alpha) \rangle_A\|^{1/2} = \|\langle \alpha, \alpha \rangle_A\|^{1/2} = \|\alpha\|_A$ which means $\widetilde{\varphi}$ is an isometry. Thus it extends to an isometric $A$-linear map

$\overline{\varphi}$ of $\ell^2(S) \otimes A$ into $\ell^2(S, A)$.

Next we prove that $\overline{\varphi}$ is surjective, so it is an isomorphism from $\ell^2(S) \otimes A$ onto $\ell^2(S, A)$. Let $f \in \ell^2(S, A)$, and for every $r \in S$, the map $\delta_r : S \to \mathbb{C}$ be the point mass defined by

$$\delta_r(s) = \begin{cases} 1 & \text{for } s = r \\ 0 & \text{otherwise.} \end{cases}$$

Each $\delta_r$ clearly belongs to $\ell^2(S)$ and the family $\{\delta_r : r \in S\}$ is orthogonal, meaning that

$$\langle \delta_s \mid \delta_r \rangle = \begin{cases} 1 & \text{for } s = r \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\sum_{r \in S} \delta_r \otimes f(r)$ converges to some $\alpha$ in $\ell^2(S) \otimes A$ and $\overline{\varphi}(\alpha) = f$. For every $F \in \mathcal{F}$, we have

$$\begin{aligned} \| \sum_{r \in F} \delta_r \otimes f(r) \|_A^2 &= \| \langle \sum_{r \in F} \delta_r \otimes f(r), \sum_{s \in F} \delta_s \otimes f(s) \rangle_A \| \\ &= \| \sum_{r \in F} \sum_{s \in F} \langle \delta_r \otimes f(r), \delta_s \otimes f(s) \rangle_A \| \\ &= \| \sum_{r \in F} \sum_{s \in F} \langle \delta_s \mid \delta_r \rangle f(r)^* f(s) \| \\ &= \| \sum_{r \in F} f(r)^* f(r) \|. \end{aligned}$$

The partial sums of $\sum_{r \in S} f(r)^* f(r)$ are Cauchy in $A$ because $f \in \ell^2(S, A)$. Therefore the partial sums of $\sum_{r \in S} \delta_r \otimes f(r)$ are Cauchy in $\ell^2(S) \otimes A$. Thus it follows that $\sum_{r \in S} \delta_r \otimes f(r)$ converges to some $\alpha$ in $\ell^2(S) \otimes A$. To see $\overline{\varphi}(\alpha) = f$, it is enough to prove that the finite subsums $\widetilde{\varphi}(\sum \delta_r \otimes f(r)) = \sum \widetilde{\varphi}(\delta_r \otimes f(r))$ converge to $f$ in $\ell^2(S, A)$. First note that we have

$$\| \sum_{r \in F} \widetilde{\varphi}(\delta_r \otimes f(r)) - f \|_A^2 = \| \langle f, f \rangle_A - \sum_{r \in F} f(r)^* f(r) \|$$

for every $F \in \mathcal{F}$ which is the key point of the proof. To verify this, we let $\beta = \sum_{r\in F} \widetilde{\varphi}(\delta_r \otimes f(r))$ for convenience in our following computations. Then we have

$$\|\beta - f\|_A^2 = \|\langle \beta - f, \beta - f\rangle_A\| = \|\langle \beta, \beta\rangle_A - \langle \beta, f\rangle_A - \langle f, \beta\rangle_A + \langle f, f\rangle_A\|. \quad (2.8)$$

Now we evaluate $\langle \beta, \beta\rangle_A$, $\langle \beta, f\rangle_A$, and $\langle f, \beta\rangle_A$ separately. We have

$$\begin{aligned}\langle \beta, \beta\rangle_A &= \sum_{r\in F}\sum_{s\in F}\langle \widetilde{\varphi}(\delta_r \otimes f(r)), \widetilde{\varphi}(\delta_s \otimes f(s))\rangle_A \\ &= \sum_{r\in F}\sum_{s\in F}\langle \delta_r \otimes f(r), \delta_s \otimes f(s)\rangle_A \\ &= \sum_{r\in F}\sum_{s\in F}\langle \delta_s \mid \delta_r\rangle f(r)^* f(s) = \sum_{r\in F} f(r)^* f(r),\end{aligned} \quad (2.9)$$

and

$$\begin{aligned}\langle \beta, f\rangle_A = \sum_{r\in F}\langle \widetilde{\varphi}(\delta_r \otimes f(r)), f\rangle_A &= \sum_{r\in F}\sum_{s\in S}\widetilde{\varphi}(\delta_r \otimes f(r))(s)^* f(s) \\ &= \sum_{r\in F}\sum_{s\in S}\overline{\delta_r(s)} f(r)^* f(s) \\ &= \sum_{r\in F} f(r)^* f(r).\end{aligned} \quad (2.10)$$

For $\langle f, \beta\rangle_A$, since $\langle f, \beta\rangle_A = \langle \beta, f\rangle_A^*$, it follows from our calculation in (2.10) that

$$\langle f, \beta\rangle_A = (\sum_{r\in F} f(r)^* f(r))^* = \sum_{r\in F}(f(r)^* f(r))^* = \sum_{r\in F} f(r)^* f(r). \quad (2.11)$$

Thus we have

$$\langle \beta, \beta\rangle_A = \langle \beta, f\rangle_A = \langle f, \beta\rangle_A = \sum_{r\in F} f(r)^* f(r).$$