# $\pi$-NORMALITY IN TOPOLOGICAL SPACES AND ITS GENERALIZATION

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# $\pi$-NORMALITY IN TOPOLOGICAL SPACES AND ITS GENERALIZATION

by

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Sadeq Ali Saad Thabit

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# DECLARATION

I hereby declare this thesis that submitted to the SCHOOL OF MATHEMATICAL SCIENCES on June 2013 is my own work. I have stated all references used for the completion of my thesis. Moreover, non of this work has been submitted for any other degree or qualification in this or any other higher education institutions. This work has been written using LaTeX template that was created by Lim Lian Tze, from Computer Aided Translation Unit, School of Computer Sciences, according to the Guide of the Preparation, Submission and Examination of Thesis, published by Institute of Postgraduate Studies (IPS), Universiti Sains Malaysia (USM).

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# LIST OF SYMBOLS

| | |
|---|---|
| $\mathbb{R}$ | The set of real numbers |
| $\mathbb{N}$ | The set of natural numbers |
| $\mathbb{Q}$ | The set of rational numbers |
| $\mathbb{P}$ | The set of irrational numbers |
| $X$ | A topological space |
| $\mathscr{T}$ | A topology on a non-empty set |
| $\langle x, y \rangle$ | An ordered pair |
| $A \subseteq X$ | $A$ is subset of a space $X$ |
| $X \setminus A$ | The complement of $A$ in $X$ |
| $\text{int}(A)$ | The interior of $A$ |
| $\overline{A}$ | The closure of $A$ |
| $A^d$ | The derived set of $A$ |
| $\mathscr{T}_M$ | The subspace topology of $M$ |
| $\text{int}_X(A)$ | The interior of $A$ in a space $X$ |
| $\text{int}_M(A)$ | The interior of $A$ in a subspace $M$ |
| $\overline{A}^X$ | The closure of $A$ in $X$ |
| $\overline{A}^M$ | The closure of $A$ in a subspace $M$ |

| | |
|---|---|
| $X \cong Y$ | The space $X$ is homeomorphic to the space $Y$ |
| $\mathscr{U}$ | The usual topology on $\mathbb{R}$ |
| $\mathscr{C}\mathscr{F}$ | The co-finite topology |
| $\mathscr{C}\mathscr{C}$ | The co-countable topology |
| $\mathscr{S}$ | The sorgenfrey topological space |
| $\mathscr{L}$ | The left ray topology |
| $\mathscr{R}$ | The right ray topology |
| $\mathscr{T}_p$ | The particular point topology |
| $\mathscr{R}\mathscr{S}$ | The rational sequence topology |
| $\mathscr{S}^2$ | The sorgenfrey line square topology |
| $\mathscr{B}$ | The base of a space $X$ |
| $\mathscr{P}$ | The subbase of a space $X$ |
| P | A property on a space $X$ |
| $\mathscr{B}(x)$ | The local base at a point $x$ |
| $\mathscr{P}(A)$ | The power set of $A$ |
| $A \sim B$ | $A$ and $B$ are equipotent |
| $\mid A \mid$ | The cardinality of $A$ |
| $\prod_{i=1}^{n} X_i$ | The finite product space |
| $\prod_{i \in M} X_i$ | The infinite product space |

| | |
|---|---|
| $\bigoplus_{s\in S} X_s$ | The free sum space |
| $p\,\mathrm{cl}(A)$ | The pre-closure of a set $A$ in $X$ |
| $p\,\mathrm{int}(A)$ | The pre-interior of $A$ in a space $X$ |
| $PC(X)$ | The family of all pre-closed sets |
| $PO(X)$ | The family of all pre-open sets |
| $p\,\mathrm{cl}_M(A)$ | The pre-closure of $A$ in a subspace $M$ |
| $p\,\mathrm{int}_M(A)$ | The pre-interior of $A$ in a subspace $M$ |
| $\omega_0$ | The first infinite ordinal |
| $\omega_1$ | The first uncountable ordinal |

# LIST OF PUBLICATIONS

## Journals:

**[1]** Thabit, S. A. S. and Kamarulhaili, H., *On $\pi$-Normality, Weak Regularity and the Product of Topological spaces*, European Journal of Scientific Research, Vol. **51**, No. **1** (2011), 29-39. (**Indexed in Scopus**).

**[2]** Thabit, S. A. S. and Kamarulhaili, H., *$\pi$-Closed Sets and Almost Normality of the Niemytzki Plane Topology*, Journal of Mathematical Sciences: Advances and Applications, Vol. **8**, No. **2** (2011), 73-85. (**Indexed in MathSciNet**).

**[3]** Thabit, S. A. S. and Kamarulhaili, H., *On $\pi$-Normality in Topological Spaces*, Journal of Advances and Applications in Mathematical Sciences, Milli Publications, Vol **10**, Issue **5**(2011). Pages: 459-471. (**Indexed in MathSciNet**).

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**[6]** Thabit, S. A. S. and Kamarulhaili, H., *$\pi p$-Normal Topological Spaces*, International Journal of Mathematical Analysis, Vol. **6**, 2012, no. **21**, 1023-1033. (**Indexed in Scopus**, **IF=0.224**).

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Journal of Mathematical Analysis, Vol. **6**, 2012, no. 27, 1301-1311. (**Indexed in Scopus**, **IF=0.224**).

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**[9]** Thabit, S. A. S. and Kamarulhaili, H., *On $\pi$-Closed Sets and $\pi$-Normal Property in Topological Spaces*, The American Institute of Physics (AIP) Conference Proceedings Series, **1557**(2013), pp. 370-375; doi: http://dx.doi.org/10.1063/1.4823938. (**Indexed in Scopus and ISI**).

**[10]** Thabit, S. A. S. and Kamarulhaili, H., *On $\pi p$-Normal Spaces*, will be published in the journal of Matematika, UTM, 2013. (**Indexed in DOAJ**).

## Conference Proceeding:

**[1]** Thabit, S. A. S. and Kamarulhaili, H., "*On almost regularity and $\pi$-normality of topological spaces*", in the 5th international conference on Research and Education in Mathematics, ICREM5, ITB Bandung, Indonesia, October 2011.

**[2]** Thabit, S. A. S. and Kamarulhaili, H., "*Almost regularity, $\pi$-normality and generalized closed sets in topological spaces*", in the 2nd symposium of USM Fellowship 2011, Vistana Hotel, Penang, Malaysia, November 2011.

**[3]** Thabit, S. A. S. and Kamarulhaili, H., "*On $\pi$-closeness and $\pi$-normality in topological spaces*", in the proceeding book of the ICAAA2012, Turkey, pages 228-229.

**[4]** Thabit, S. A. S. and Kamarulhaili, H., "*On π-Closed Sets and π-Normal Property in Topological Spaces*", in the International Conference on Mathematical Sciences and Statistics, ICMSS2013-UPM, Kuala Lumpur, Malaysia, 5-7 February 2013.

# KENORMALAN-$\pi$ DALAM RUANG TOPOLOGI DAN GENERALISASINYA


## ABSTRAK

Tujuan utama projek ini adalah untuk membuat kajian yang menyeluruh terhadap versi normal yang lebih lemah normal dipanggil normal-$\pi$, yang terletak di antara normal dan hampir normal (kuasi-normal). Pertama, kita berikan beberapa definisi asas, ciri-ciri dan teorem, yang akan digunakan dalam perbincangan tesis ini. Keta berikan satu kajian kaji selidik terhadap set tertutup-$\pi$, terbuka-$\pi$, tertutup-pra dan terbuka-pra. Secara khususnya, kita mengkaji set ini dalam subruang dan juga mengkaji imej dan imej songsang mereka di bawah fungsi yang selanjar. Sebahagian ciri-ciri set ini dibuktikan. Kenormalan-$\pi$ adalah kedua-duanya bersifat topologi dan juga penambahan, tetapi bukan pendaraban dan tidak diwarisi secara umum. Tanggapan terhadap set tertutup-$\pi$ digunakan untuk mendapatkan pelbagai ciri dan teorem pemeliharaan normal-$\pi$. Beberapa ciri tentang ruang hampir kerap, begitu juga hampir kerap sepenuhnya dibentangkan dan beberapa keputusan diperbaiki. Beberapa hubungan di antara normal-$\pi$ dan kedua-dua ruang hampir kekerap dan hampir kerap sepenuhnya diberikan. Keputusan penting adalah mengenai pembentangan beberapa contoh lawan balas, yang pertama adalah mengenai ruang Hausdorff separa normal tetapi tidak normal-$\pi$. Yang kedua adalah mengenai ruang Tychonoff hampir normal tetapi tidak kuasi normal dan yang ketiga adalah mengenai ruang Tychonoff hampir normal tetapi tidak normal-$\pi$. Kami membuktikan bahawa satah Niemytzki dan ruang topologi garisan Sorgenfrey kuasa dua persegi hampir normal tetapi tidak normal-$\pi$ atau separa normal dan juga topologi urutan rasional adalah hampir normal tetapi tidak semi-normal atau separa normal. Kami menunjukkan bahawa setiap ruang hampir normal yang terhingga adalah normal-$\pi$ dan produk terhingga set tertutup-$\pi$ (terbuka-$\pi$, terbuka-pra, tertutup-pra) adalah tertutup-$\pi$ (terbuka-$\pi$, terbuka-pra, tertutup-pra). Salah satu keputusan yang paling penting adalah bahawa terdapat versi normal-$\pi$ seakan Lema Jones untuk ruang normal. Kami memberikan beberapa syarat kepada dua ruang $X$ dan $Y$ supaya ruang produk $X \times Y$ akan menjadi normal-$\pi$. Kami juga telah memberikan beberapa keputusan untuk ruang normal-$\pi$ dan hampir para-padat dan beberapa hubungan antara normal-$\pi$ dan hampir para-padat diberikan. Kami menyiasat tentang ruang $X$ adalah normal-$\pi$ yang terbilangkan hampir para-padat jika dan hanya jika ruang produk $X \times I$ adalah normal-$\pi$, maka subruang $X \times \{0\}$ juga normal-$\pi$.

Sebaliknya, kami memperkenalkan versi normal-pra yang lebih lemah dipanggil normal-pra-$\pi$, yang merupakan normal-$\pi$ yang di itlakkan, dan menunjukkan bahawa normal-pra-$\pi$ adalah kedua-duanya topologikal dan mempunyai ciri aditif, tetapi tidak produktif atau tidak mempunyai ciri warisan secara umum. Sesetengah ciri, contohnya, pengkategorian dan teorem pemeliharaan normal-pra-$\pi$ dibentangkan. Selain itu, kami memperkenalkan klasifikasi baru set terbuka-pra dan tertutup-pra yang dipanggil terbuka-pra-$\pi$ dan tertutup-pra-$\pi$, yang merupakan pengitlakan terbuka-$\pi$ dan tertutup-$\pi$, dan memberikan beberapa ciri asas mereka. Kami membuktikan bahawa kedua-dua sub-maksimum dan extremal terkaitkan-pra adalah ciri warisan berkenaan dengan subset padat dan terbuka. Selain itu, kita menunjukkan bahawa produk dua ruang sub-maksima adalah sub-maksima.

# $\pi$-NORMALITY IN TOPOLOGICAL SPACES AND ITS GENERALIZATION

## ABSTRACT

The main aim of this thesis is to make a comprehensive study of a weaker version of normality called $\pi$-normality, which lies between normality and almost normality (quasi-normality). First, we give some basic definitions, properties and theorems, which we are going to use throughout the thesis. We give a survey study of $\pi$-closed, $\pi$-open, pre-closed and pre-open sets. In particular, we study these sets in subspaces and also study the images and the inverse images of them under continuous functions. Some properties of these sets are given and proved. $\pi$-normality is both a topological and an additive property, but neither a productive nor a hereditary property in general. The notion of $\pi$-generalized closed sets is used to obtain various characterizations and preservation theorems of $\pi$-normality. Some properties of almost regular as well as almost completely regular spaces are presented, and a few results of them are improved. Some relationships between $\pi$-normality and both almost regularity and almost complete regularity are given. The important results are about presenting some counterexamples, the first one is about a semi-normal Hausdorff space but not $\pi$-normal. The second one is about an almost normal Tychonoff space but not quasi-normal and the third one is about an almost normal Tychonoff space but not $\pi$-normal. We prove that the Niemytzki plane and the Sorgenfrey line square topological spaces are almost normal but neither $\pi$-normal nor semi-normal and also

the rational sequence topological space is an almost normal but neither semi-normal nor quasi-normal. We show that every finite almost normal space is $\pi$-normal and a finite product of $\pi$-closed ($\pi$-open, pre-open, pre-closed) sets is $\pi$-closed ($\pi$-open, pre-open, pre-closed), respectively. One of the most important results is that there is a version of $\pi$-normality analogous to the Jones' Lemma for normal spaces. We give some conditions on two spaces $X$ and $Y$ so that the product space $X \times Y$ will be $\pi$-normal. We present some results on $\pi$-normal and nearly paracompact spaces and some relationships between $\pi$-normality and near paracompactness are given. We investigate that a space $X$ is $\pi$-normal countably nearly paracompact if and only if the product space $X \times I$ is $\pi$-normal and if the product space $X \times I$ is $\pi$-normal, then the subspace $X \times \{0\}$ is also $\pi$-normal.

In addition, we introduce a weaker version of pre-normality called $\pi$-pre-normality, which is a generalization of $\pi$-normality, and show that $\pi$-pre-normality is both a topological and an additive property, but neither a productive nor a hereditary property in general. Some properties, examples, characterizations and preservation theorems of $\pi$-pre-normality are presented. Also, we introduce new classes of pre-open and pre-closed sets called $\pi$-pre-open and $\pi$-pre-closed, which are the generalizations of $\pi$-open and $\pi$-closed, and present some basic properties of them. We prove that both sub-maximality and extremal pre-disconnectedness are hereditary properties with respect to dense and open subsets. Also, we show that the product of two sub-maximal spaces is sub-maximal.

## CHAPTER 1

# INTRODUCTION

### 1.1 Background of the Study

Topology is a very important branch of pure Mathematics. Its applications are not only in other branches of Mathematics but also in other branches of sciences. The definition of a topological space is very general. It is often desirable for a topologist to be able to assign to a set of objects a topology about which he knows a great deal in advance. This can be done by stipulating that the topology must satisfy axioms in addition to those generally required of topological space.

Two sets $A$ and $B$ of a space $X$ are said to be *separated* if there exist two disjoint open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$, (Dugundji, 1966; Engelking, 1989; Patty, 1993). A subset $A$ of $X$ is said to be a *regularly-open* or an *open domain* if it is the interior of its own closure, or equivalently if it is the interior of some closed set, and $A$ is said to be a *regularly-closed* or a *closed domain* if it is the closure of its own interior, or equivalently if it is the closure of some open set, (Kuratowski, 1958). A subset $A$ of $X$ is called a $\pi$-*closed* if it is a finite intersection of closed domains and $A$ is called a $\pi$-*open* if it is a finite union of open domains, (Zaitsev, 1968). A space $X$ is called a *mildly normal* if any two disjoint closed domains $A$ and $B$ can be separated, (Singal and Singal, 1973). A space $X$ is called an *almost normal* if any two disjoint closed subsets $A$ and $B$, one of which is closed domain, can be separated, (Singal and Arya, 1970). A space $X$ is called a *quasi normal* if any two disjoint $\pi$-closed sets $A$ and

$B$ can be separated, (Zaitsev, 1968). A space $X$ is said to be a *$\pi$-normal*, (Kalantan, 2008), if for every pair of disjoint closed sets $A$ and $B$, one of which is $\pi$-closed, can be separated. A space $X$ is said to be an *almost regular* if any closed domain set $A$ and for each $x \notin A$, there exist two disjoint open sets $U$ and $V$ such that $x \in U$ and $A \subseteq V$. A space $X$ is said to be an *almost completely regular* if for every closed domain set $A$ and for each $x \notin A$, there exists a continuous function $f : X \longrightarrow [0,1]$, where $[0,1]$ is the unit interval with its usual topology such that $f(x) = 0$ and $f(A) = \{1\}$. A *Hausdorff* space is a topological space satisfying the separation axiom $T_2$. A *Tychonoff* space is a topological space satisfying the separation axioms completely regular and $T_1$. A space $X$ is said to be a *semi-normal* if for any closed subset $A$ of $X$ and every open subset $B$ of $X$ with $A \subseteq B$, there exists an open subset $U$ of $X$ such that $A \subseteq U \subseteq \text{int}(\overline{U}) \subseteq B$, (Singal and Arya, 1970). By the definitions of weaker versions of normality, we have:

$$\text{normal} \Longrightarrow \pi\text{-normal} \Longrightarrow \text{almost normal} \Longrightarrow \text{mildly normal}$$

$$\text{normal} \Longrightarrow \pi\text{-normal} \Longrightarrow \text{quasi-normal} \Longrightarrow \text{mildly normal}$$

On the other hand, a subset $A$ of $X$ is said to be a *pre-open*, (Mashhour et al., 1984), if $A \subseteq \text{int}(\overline{A})$. A space $X$ is called a *pre-normal* if any two disjoint closed subsets $A$ and $B$ of $X$ can be separated by two disjoint pre-open subsets of $X$, (Paul and Bhattacharyya, 1995). A space $X$ is called an *almost pre-normal*, (Navalagi, 2000), if any two disjoint closed sets $A$ and $B$, one of which is closed domain, can be separated by two disjoint pre-open subsets. A space $X$ is called a *mildly pre-normal*, (Navalagi, 2000), if any pair of disjoint closed domains $A$ and $B$ can be separated by two disjoint pre-open subsets of $X$.

$\pi$-normality implies to almost normality but the converse is not true in general.

The main problem was,"*Is there an almost normal Tychonoff space which is not $\pi$-normal?*". Kalantan did not give any example about almost normality and not $\pi$-normality. He presented two open problems in his paper, see (Kalantan, 2008), which are:

**Problem 1.** *Is the Niemytzki plane almost normal? $\pi$-normal?.*

**Problem 2.** *Is the Sorgenfrey line square almost normal? $\pi$-normal?.*

Also, Kalantan stated that there is an almost normal space but not $\pi$-normal in finite spaces. Neither a proof nor an example was given. Some results as well as problems on $\pi$-closed sets and $\pi$-normal spaces have been presented in (Thabit, 2008). In this study, we solve all of those problems.

### 1.2 Literature Review

Separation axioms concern the ways of separating points and subsets in topological space. Normality, one of the separation axioms, is an important topological property and hence it is of significance both from intrinsic interest and from applications view point to obtain factorizations of normality in terms of weaker topological properties. Zaitsev (1968) introduced the notion of $\pi$-closed sets and the class of quasi-normal space. Then, Singal and Arya (1970) introduced the class of almost normal space and proved that a space $X$ is normal if and only if it is both a semi-normal and an almost normal. Singal and Singal (1973) introduced a weaker form of normality called mild normality. In the last few years, many authors have studied several forms of normality, as referred in many papers (Dontchev and Noiri, 2000; Ganster et al., 2002; Kohli and Das, 2002; Noiri, 1994; Kalantan, 2008).

$\pi$-normality, which was introduced by Kalantan in 2008, is a weaker version of normality and lies between normality and almost normality (quasi-normality). The importance of this property is that it behaves slightly different from normality and almost normality (quasi-normality). $\pi$-normality is an additive property but neither a productive nor a hereditary property in general. There are many $\pi$-normal topological spaces which are not normal and there are many almost normal (quasi-normal) spaces which are not $\pi$-normal.

On the other hand, the notion of pre-open sets, (Mashhour et al., 1982), plays a significant role in general topology. The most important generalizations of regularity (normality) are the notions of pre-regularity, (Benchalli et al., 2009), and strong regularity (pre-normality, strong normality (Mashhour et al., 1984)), respectively. Levine (1963) started the study of generalized open sets with the introduction of semi-open sets. Then, Njastad (1965) studied $\alpha$-open sets. Mashhour et al. (1982) introduced pre-open and pre-continuity in topology. Since then many topologists have utilized these concepts to the various notions of subsets, weak separation axioms, weak regularity, weak normality and weaker and stronger forms of covering axioms in the literature. The concepts of $s$-normal and $s$-regular spaces were introduced and studied by Maheshwari and Prasad (1975, 1978). Arya and Nour (1990) obtained some characterizations of $s$-normal spaces. Munshi (1986) introduced and studied the notions of $g$-regular and $g$-normal spaces using $g$-closed sets. Further, Noiri and Popa (1999) investigated the concepts that introduced by Munshi (1986). Veerakumar (2002) defined the notions of $g^*$-pre-closed sets, $g^*$-pre-continuity and $g^*$-pre-irresolute mappings. Nour (1989) used pre-open sets to define pre-normal spaces. Navalagi (2000) has continued the study of further properties of pre-normal spaces, and also defined and investigated mildly pre-normal as well as almost pre-normal, which are generalizations of both mildly normal and almost normal spaces.

In this thesis, we make a comprehensive study of $\pi$-normality. We introduce and study a weaker version of pre-normality called $\pi$-pre-normality, which is a generalization of $\pi$-normality. We also introduce and study new classes of pre-open and pre-closed sets called $\pi$-pre-open and $\pi$-pre-closed.

## 1.3 Research Questions

Kalantan (2008) and Thabit (2008) presented many problems on $\pi$-normality. Now, we list out those problems as follows:

**Problems:**

(1) Is there a semi-normal Hausdorff space which is not $\pi$-normal?.

(2) Is there an almost normal Tychonoff space which is not quasi-normal?.

(3) Is there an almost normal Tychonoff space which is not $\pi$-normal?.

(4) Is the rational sequence topological space almost normal? quasi-normal? $\pi$-normal?.

(5) Is every finite almost normal space, $\pi$-normal?.

(6) Is there a version of $\pi$-normality analogous to Jones' Lemma for normal spaces?.

(7) What are the conditions that should be given on two spaces $X$ and $Y$ so that the product space $X \times Y$ will be $\pi$-normal?.

(8) Is a finite product of $\pi$-open ($\pi$-closed) sets, $\pi$-open ($\pi$-closed)?.

(9) If $M$ is a $\pi$-open subspace of $X$ and $A \subseteq M$. Is the statement "*$A$ is a $\pi$-open in $M$ if and only if $A$ is a $\pi$-open in $X$*", true?.

(10) Is any almost regular Lindelöf space, $\pi$-normal?.

(11) Is any almost regular space with $\sigma$-locally finite base, $\pi$-normal?.

(12) Is any $\pi$-closed ($\pi$-open) set in an almost regular space with $\sigma$-locally finite base an $F_\sigma$-set (a $G_\delta$-set), respectively?.

(13) Is a $\pi$-closed ($\pi$-open, open domain) subspace of a $\pi$-normal space, $\pi$-normal?.

## 1.4 Research Objectives

The objectives of this study are:

(i) To make a comprehensive study of $\pi$-normality with other topological aspects such as addition, product, quotient, subspace, images and pre-images of functions.

(ii) To give various characterizations of $\pi$-normality by using $\pi$-generalized closed sets and establish preservation properties under continuous or some generalized sense of continuous mappings as well as some relationships between $\pi$-normality and other weaker versions of both regularity and complete regularity.

(iii) To distinguish between $\pi$-normality and other weaker versions of normality by giving counterexamples and improve some previous results on almost regularity, almost complete regularity and almost normality.

(iv) To give some conditions on two spaces $X$ and $Y$ so that the product space $X \times Y$ will be $\pi$-normal.

(v) To introduce and study a new concept of topological properties called $\pi$-pre-normality and present some properties, examples, characterizations and preservation theorems of it.

(vi) To introduce and study new classes of pre-open and pre-closed sets called $\pi$-pre-open and $\pi$-pre-closed.

## 1.5 Research Methodology

We use the basic definitions and the theorems in the Chapter 2, and some definitions and results in the references (Kalantan, 2008; Thabit, 2008; Singal and Singal, 1973, 1968; Singal and Arya, 1970, 1969a; Shchepin, 1972)...ect., to solve the listed main problems by proving or giving counterexamples.

## 1.6 Thesis Contribution

Most results in this thesis are included in the chapters 3,4,5,6,7 and 8. The most important results can be listed as follows:

(1) There exists a semi-normal Hausdorff space but not $\pi$-normal.

(2) There is an almost normal Tychonoff space but not quasi-normal.

(3) There is an almost normal Tychonoff space but not $\pi$-normal.

(4) The Niemytzki plane and the Sorgenfrey line square topological spaces are almost normal but neither $\pi$-normal nor semi normal.

(5) The rational sequence topological space is almost normal but neither quasi-normal nor semi-normal.

(6) Every finite almost normal space is $\pi$-normal.

(7) There is a version of $\pi$-normality (quasi normality) analogous to Jones' Lemma for normal spaces.

(8) A finite product of $\pi$-open ($\pi$-closed, pre-open, pre-closed) sets is $\pi$-open ($\pi$-closed, pre-open, pre-closed), respectively.

(9) An almost regular, Lindelöf space (or with $\sigma$-locally finite base) is not necessarily $\pi$-normal.

(10) Any $\pi$-closed ($\pi$-open) set in an almost regular space with $\sigma$-locally finite base is an $F_\sigma$-set (a $G_\delta$-set), respectively.

(11) We study both $\pi$-closed and $\pi$-open sets in subspaces and prove the following results:

- Let $M$ be a $\pi$-open subspace of $X$ and $A \subseteq M$. $A$ is a $\pi$-open in $M$ if and only if $A$ is a $\pi$-open in $X$.
- Let $M$ be an open (dense) subspace of $X$ and $A \subseteq M$. If $A$ is an open domain (closed domain, $\pi$-open, $\pi$-closed) in $X$, then $A$ is an open domain (closed domain, $\pi$-open, $\pi$-closed) in $M$, respectively.
- Let $M$ be an open (dense) subspace of $X$ and $A \subseteq X$. If $A$ is an open domain (closed domain, $\pi$-open, $\pi$-closed) in $X$, then $M \bigcap A$ is an open domain (closed domain, $\pi$-open, $\pi$-closed) in $M$, respectively.

(12) A $\pi$-open subspace of a $\pi$-normal space is not necessarily $\pi$-normal.

(13) We give some conditions on two spaces $X$ and $Y$ so that the product space $X \times Y$ will be $\pi$-normal, where we prove the following results:

- If $X$ is a $\pi$-normal, countably compact and $M$ is a paracompact first countable, then the product space $X \times M$ is a $\pi$-normal.
- The product $X \times Y$ of a countably nearly paracompact, $\pi$-normal space $X$ and a nearly compact second countable space $Y$, is a $\pi$-normal.
- Let $X \times I$ be the product of a space $X$ and the closed unit interval $I$ with its usual topology. If $X \times I$ is a $\pi$-normal, then $X \times \{0\}$ is a $\pi$-normal subspace of $X \times I$.
- A space $X$ is a $\pi$-normal, countably nearly paracompact if and only if the product space $X \times I$ is a $\pi$-normal.

(14) Every weakly regular (almost regular) paracompact space is a $\pi$-normal.

(15) Any regular, nearly paracompact space is a $\pi$-normal but an almost regular, nearly paracompact space is not necessarily $\pi$-normal.

(16) We study both pre-closed and pre-open sets in subspaces and prove the following results:

- Let $M$ be a closed domain subspace of $X$ and $A \subseteq M$. $A$ is a pre-closed in $M$ if and only if $A$ is a pre-closed in $X$.
- Let $M$ be a closed subspace of $X$ and $A \subseteq M$. If $A$ is a pre-closed in $M$, then $A$ is a pre-closed in $X$.
- Let $M$ be an open (or dense) subspace of $X$ and $A \subseteq M$. $A$ is a pre-open in $M$ if and only if $A$ is a pre-open in $X$.

- Let $M$ be a closed domain subspace of $X$. If $A$ is a pre-closed (pre-open) in $X$, then $A \bigcap M$ is a pre-closed (pre-open) in $M$, respectively.
- Let $M$ be an open (or dense) subspace of $X$. If $A$ is a pre-open (pre-closed) in $X$, then $A \bigcap M$ is a pre-open (pre-closed) in $M$, respectively.
- If $M$ is an open (or dense) subspace of $X$ and $A \subseteq M$, then $p\,\mathrm{cl}_M(A) = p\,\mathrm{cl}_X(A) \bigcap M$.

(17) A closed domain subspace of a $\pi$-pre-normal space is a $\pi$-pre-normal.

(18) The image of a pre-closed (pre-open) subset under a closed-and-open bijective continuous function is a pre-closed (pre-open), respectively.

(19) The inverse image of a pre-closed (pre-open) subset under an open continuous function is a pre-closed (pre-open), respectively.

(20) An open subspace of a sub-maximal space is a sub-maximal, and the product of two sub-maximal spaces is a sub-maximal.

(21) $\pi$-pre-normality is both a topological and an additive property but neither a productive nor a hereditary property in general.

(22) There is a version of $\pi$-pre-normality analogous to the Urysohn's Lemma for normal spaces by adding some conditions.

## 1.7 Thesis Organization

This thesis is organized as follows:

**Chapter 1**; Introduction: This chapter is an introduction of the thesis.

**Chapter 2**; Preliminaries: This chapter contains some basic definitions, theorems and some classical results in the General Topology as well as in the Set Theory, which we are going to use throughout the thesis.

**Chapter 3**; $\pi$-Open and $\pi$-Closed Sets: In this chapter, we give a survey study of the notions of $\pi$-closed and $\pi$-open sets. These kinds of sets are used to define the notions of both $\pi$-normality and $\pi$-pre-normality. We study these notions in subspaces, in free sum and in product spaces. Also, we study the images and the inverse images of these under continuous functions. We prove some various properties of them and present some examples.

**Chapter 4**; $\pi$-Normal Topological Spaces: In this chapter, we study the notion of $\pi$-normality. We obtain various characterizations, properties and examples concerning it and present its relationships with other types of separation properties weaker than normality as well as regularity. We present some properties of almost regular spaces and improve a few of them. We show that an almost regular Lindelöf space (or with $\sigma$-locally finite base) is not necessarily $\pi$-normal by giving two counterexamples. We give some conditions to assure that the product of two spaces will be $\pi$-normal and that the quotient space of a $\pi$-normal space will be $\pi$-normal.

**Chapter 5**; New Results on $\pi$-Normality: In this chapter, we present the most important results on $\pi$-normality. We show that there is a version of $\pi$-normality analogous to Jones' Lemma for normal spaces and prove that both the Niemytzki plane and the Sorgenfrey line square topological spaces are almost normal but neither $\pi$-normal nor semi-normal. Also, we prove that the rational sequence topological space is almost normal but neither semi-normal nor quasi normal and that every finite almost normal space is $\pi$-normal. We present some characterizations of almost completely regular spaces by using the notions of $\pi$-closed as well as zero-sets.

**Chapter 6**; $\pi$-Normal and Nearly Paracompact Spaces: In this chapter, we present some results on $\pi$-normal and nearly paracompact spaces. We give other conditions on two spaces $X$ and $Y$ so that the product space $X \times Y$ will be $\pi$-normal. We prove that if the product space $X \times I$ is $\pi$-normal, then the subspace $X \times \{0\}$ is $\pi$-normal. We also show that $X$ is $\pi$-normal countably nearly paracompact if and only if the product space $X \times I$ is $\pi$-normal.

**Chapter 7**; $\pi$-Pre-normal Topological Spaces: In this chapter, we introduce and study a weaker version of pre-normality called $\pi$-pre-normality. We show that this notion is both a topological and an additive property, and prove that it is hereditary only with respect to closed domain subspaces. Some properties, examples, characterizations and preservation theorems of this property are presented. We study the notions of pre-open (pre-closed) sets in subspaces as well as their images and inverse images under continuous functions. We give some characterizations of almost pre-regularity by using the notion of $\pi$-closed sets and present its relationships with $\pi$-pre-normality.

**Chapter 8**; $\pi$-Pre-open and $\pi$-Pre-closed Sets: In this chapter, we introduce new classes of pre-open and pre-closed sets called $\pi$-pre-open and $\pi$-pre-closed, respectively, which are the generalizations of $\pi$-open and $\pi$-closed. We present and prove some basic properties of them. We prove the facts that an open subspace of a sub-maximal (resp. an extremally pre-disconnected) space is a sub-maximal (resp.

an extremally pre-disconnected) and that the product of two sub-maximal spaces is a sub-maximal. We also prove that a finite product of pre-closed (pre-open) sets is a pre-closed (pre-open). We investigate that $\pi$-pre-normality is neither a productive nor a hereditary property in general by giving two counterexamples. We also show that there is a version of $\pi$-pre-normality analogous to the Urysohn's Lemma for normal spaces by adding some conditions.

**Chapter 9**; Conclusion: This chapter summarizes the most important results that have been found throughout the study. We also present some problems that found during this research, and we still do not have the answers for them until now.

**References**; We list the references that have been used in the research.

# CHAPTER 2

# PRELIMINARIES

In this chapter, some basic definitions, theorems and some classical results in general topology as well as in set theory, which we are going to use throughout the thesis, are presented (without proof). The closed unit interval $[0,1]$ is denoted by $I$ and it will be considered with its usual topology. Throughout this thesis, a space $X$ always means a topological space on which no separation axioms are assumed, unless explicitly stated. We will denote an ordered pair by $\langle x,y\rangle$. $A$ is an open in $X$ means $A$ is an open set in $X$ or $A$ is an open subset of $X$. Similarly, if $A$ is closed in $X$. The main references of this chapter are (Arhangel'skii, 1963; Dugundji, 1966; Engelking, 1989; Patty, 1993; Steen and Seebach, 1995). At first, we give the definitions of a topology and an open set.

## 2.1 Topological spaces, open and closed sets

**Definition 2.1** A topological space is a pair $(X,\mathscr{T})$ consisting of a non-empty set $X$ and a family $\mathscr{T}$ of subsets of $X$ satisfying the following conditions:

$(\mathbf{T_1})$ $X, \emptyset \in \mathscr{T}$.

$(\mathbf{T_2})$ If $U \in \mathscr{T}$ and $V \in \mathscr{T}$, then $U \bigcap V \in \mathscr{T}$.

$(\mathbf{T_3})$ If $U_\alpha \in \mathscr{T}$ for each $\alpha$ in an index set $\Lambda$, then $\bigcup_{\alpha\in\Lambda} U_\alpha \in \mathscr{T}$.

The set $X$ is called a space, the elements of $X$ are called points of the space and the subsets of $X$ belonging to $\mathscr{T}$ are called *open* sets. The family $\mathscr{T}$ of open subsets of $X$ is also called a *topology* on $X$. An *open neighborhood* of an $x \in X$ is just any open set $U$ containing $x$.

**Definition 2.2** A subset $A$ of $X$ is called a *closed* subset if its complement $X \setminus A$ is an open. A subset $A$ is called a *clopen* subset if it is an open and a closed subset at the same time.

**Definition 2.3** Let $A$ be a subset of $X$.

(i) The *interior* of $A$ is denoted by $\text{int}(A)$ and defined as:

$\text{int}(A) = \{x \in X : \text{ there is an open set } U \in \mathscr{T} \text{ such that } x \in U \subseteq A\}$.

(ii) The *closure* of $A$ is denoted by $\overline{A}$ and defined as:

$\overline{A} = \{x \in X : U \bigcap A \neq \emptyset, \text{ for each } U \in \mathscr{T} \text{ with } x \in U\}$.

(iii) A point $x \in X$ is called a *limit point* (or an *accumulation* point) of $A$ if for any open neighborhood $U$ of $x$, we have $U \bigcap (A \setminus \{x\}) \neq \emptyset$. The set of all limit points of $A$ is called *derived* set and denoted by $A^d$.

## 2.2 Properties of the closure and the interior

Now, we give some properties of $\text{int}(A)$ and $\overline{A}$, which are in (Engelking, 1989; Patty, 1993).

**Proposition 2.4** *Let $X$ be a space and $A, B \subseteq X$. Then,*

(1) $\overline{A} = A \bigcup A^d$.

(2) $\text{int}(A \bigcap B) = \text{int}(A) \bigcap \text{int}(B)$.

(3) $\text{int}(A \bigcup B) \supseteq \text{int}(A) \bigcup \text{int}(B)$.

(4) *If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.*

(5) $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$.

(6) $\overline{A \bigcap B} \subseteq \overline{A} \bigcap \overline{B}$.

(7) $\overline{\overline{A}} = \overline{A}$.

(8) $\overline{X \setminus A} = X \setminus \text{int}(A)$

(9) $\text{int}(X \setminus A) = X \setminus \overline{A}$.

(10) *$A$ is an open if and only if $A = \text{int}(A)$.*

(11) *$A$ is a closed if and only if $A = \overline{A}$.*

### 2.3 Subspaces, bases and subbases for a topology

**Definition 2.5** Let $M$ be a non-empty subset of a topological space $(X, \mathscr{T})$. The *subspace* topology or *relative* topology on $M$ determined by $\mathscr{T}$ is the collection $\mathscr{T}_M = \{U \bigcap M : U \in \mathscr{T}\}$.

If $X$ and $Y$ are two spaces and $A \subseteq X \bigcap Y$, then we denote the interior of $A$ with respect to the space $X$ by $\text{int}_X(A)$, the interior of $A$ in the space $Y$ by $\text{int}_Y(A)$, the closure of $A$ in $X$ by $\overline{A}^X$ and the closure of $A$ in $Y$ by $\overline{A}^Y$. Also, if $U \subseteq X \bigcap Y$, then we say $U$ is an $X$-open if it is an open in $X$ and similarly, for $Y$-open.

**Remark 2.6** (Engelking, 1989) Let $M$ be a subspace of $X$ and $A \subseteq M$. Then:

(i) $\mathrm{int}_X(A) \subseteq \mathrm{int}_M(A)$.

(ii) $\overline{A}^M \subseteq \overline{A}^X$.

(iii) $\mathrm{int}_M(A) = M \setminus (\overline{M \setminus A}^X)$.

**Definition 2.7** Let $(X, \mathscr{T})$ be a topological space. A subfamily $\mathscr{B}$ of $\mathscr{T}$ is called a *base* of $X$ if every non-empty open subset of $X$ can be represented as a union of a subfamily of $\mathscr{B}$. Any open set $B \in \mathscr{B}$ is called a *basic open* subset of $X$. Any base $\mathscr{B}$ of $X$ satisfies the following conditions:

$[\mathbf{B_1}]$ $X = \bigcup\{B : B \in \mathscr{B}\}$.

$[\mathbf{B_2}]$ If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \bigcap B_2$, then there exists $B \in \mathscr{B}$ such that $x \in B \subseteq B_1 \bigcap B_2$.

**Proposition 2.8** *Let $\mathscr{B}$ be a family of subsets of $X$, which has properties $[\mathbf{B_1}]$ and $[\mathbf{B_2}]$. Define $\mathscr{T}$ on $X$ by $U \in \mathscr{T}$ if and only if $U = \emptyset$ or $U = \bigcup \mathscr{B}_0$, for a subfamily $\mathscr{B}_0 \subseteq \mathscr{B}$. Then, $\mathscr{T}$ is a unique topology on $X$, which has $\mathscr{B}$ as a base. The topology $\mathscr{T}$ is called the topology generated by $\mathscr{B}$.*

**Definition 2.9** A family $\mathscr{P} \subset \mathscr{T}$ is called a *subbase* for a topological space $(X, \mathscr{T})$ if the family of all finite intersection $U_1 \bigcap U_2 \bigcap U_3 \bigcap \ldots \bigcap U_n$, $U_i \in \mathscr{P}$ for $i = 1, 2, 3, \ldots, n$, is a base for $(X, \mathscr{T})$.

## 2.4 Some special topological spaces

Now, we give the definitions of some famous topological spaces.

**Example 2.10** (The usual topology on $\mathbb{R}$)

Consider the real numbers $\mathbb{R}$ and let $\mathscr{B} = \{(a,b) : a,b \in \mathbb{R}, a < b\}$ be the set of all bounded open intervals. Then, $\mathscr{B}$ is a base for a unique topology on $\mathbb{R}$. This topology is called the *usual* topology on $\mathbb{R}$ and denoted by $\mathscr{U}$.

**Example 2.11** (The Sorgenfrey line topology on $\mathbb{R}$)

Consider the set of real numbers $\mathbb{R}$ and let $\mathscr{B} = \{[a,b) : a,b \in \mathbb{R}, a < b\}$. Then, $\mathscr{B}$ is a base for a unique topology on $\mathbb{R}$. This topology is called the *Sorgenfrey* line and denoted by $\mathscr{S}$. The Sorgenfrey line square is the square of the Sorgenfrey line topological space.

**Example 2.12** (The left ray topology on $\mathbb{R}$)

Consider the set of real numbers $\mathbb{R}$ and let $\mathscr{B} = \{(-\infty, a) : a \in \mathbb{R}\}$, then $\mathscr{B}$ is a base for a unique topology on $\mathbb{R}$. This topology is called the *left ray* topology and denoted by $\mathscr{L}$.

**Example 2.13** (The right ray topology on $\mathbb{R}$)

Consider the set of real numbers $\mathbb{R}$ and let $\mathscr{B} = \{(a, +\infty) : a \in \mathbb{R}\}$, then $\mathscr{B}$ is a base for a unique topology on $\mathbb{R}$. This topology is called the *right ray* topology and denoted by $\mathscr{R}$.

**Example 2.14** (The particular point topology)

Let $X$ be any set having more than two points. Fix a point $p \in X$. Define $\mathscr{T}_p \subseteq \mathscr{P}(X)$ as follows:

$$\mathscr{T}_p = \{\emptyset\} \bigcup \{U \subseteq X : \ p \in U\}$$

If $X = \mathbb{R}$, then $(\mathbb{R}, \mathscr{T}_p)$ is called the *particular point* topology on $\mathbb{R}$.

**Example 2.15** (The co-countable topology on $\mathbb{R}$)

The *co-countable* topology on $\mathbb{R}$ is denoted by $\mathscr{C}\mathscr{C}$ and defined as $U \in \mathscr{C}\mathscr{C}$ if and only if $U = \emptyset$ or $\mathbb{R} \setminus U$ is countable.

**Example 2.16** (The co-finite topology on $\mathbb{R}$)

The *co-finite* topology on $\mathbb{R}$ is denoted by $\mathscr{C}\mathscr{F}$ and defined as $U \in \mathscr{C}\mathscr{F}$ if and only if $U = \emptyset$ or $\mathbb{R} \setminus U$ is finite.

### 2.5 Local base and famous examples

The following definitions are in (Engelking, 1989; Patty, 1993)

**Definition 2.17** A family $\mathscr{B}(x)$ of open neighborhoods of $x$ is called a *local base* of $X$ at the point $x$ if for any open neighborhood $V$ of $x$, there exists an open set $U \in \mathscr{B}(x)$ such that $x \in U \subseteq V$.

**Definition 2.18** Let $X$ be a space and suppose that for every $x \in X$ a local base $\mathscr{B}(x)$ of $X$ at $x$ is given, then the collection $\{\mathscr{B}(x) : \ x \in X\}$ is called a *neighborhood system*. Any neighborhood system $\{\mathscr{B}(x) : \ x \in X\}$ of $X$ satisfies the following conditions:

[$\mathbf{BP_1}$] For each $x \in X$, $\mathscr{B}(x) \neq \emptyset$ and for each $U \in \mathscr{B}(x)$, we have $x \in U$.

[$\mathbf{BP_2}$] If $x \in U \in \mathscr{B}(y)$, then there exists an open set $V \in \mathscr{B}(x)$ such that $V \subseteq U$.

[$\mathbf{BP_3}$] For each $U_1, U_2 \in \mathscr{B}(x)$, there exists an open set $U \in \mathscr{B}(x)$ such that $U \subseteq U_1 \bigcap U_2$.

**Proposition 2.19** *Let $X$ be a non-empty set and $\{\mathscr{B}(x) : x \in X\}$ be a collection of families of subsets of $X$, which satisfies properties* [$\mathbf{BP_1}$], [$\mathbf{BP_2}$] *and* [$\mathbf{BP_3}$]. *Let $\mathscr{T}$ be the family of all subsets of $X$ that are unions of subfamilies of $\bigcup_{x \in X} \mathscr{B}(x)$. Then, $\mathscr{T}$ is a unique topology on $X$ while the collection $\{\mathscr{B}(x) : x \in X\}$ is a neighborhood system of $X$. The topology $\mathscr{T}$ is called the topology generated by the neighborhood system $\{\mathscr{B}(x) : x \in X\}$.*

**Definition 2.20** A *sequence* of a space $X$ is a function $a : X \longrightarrow \mathbb{N}$ such that $a(k) = a_k$. For each $m \geq 1$, a set $A_m = \{a_k : k \geq m\}$ is called a *tail* of $(a_n)_{n \in \mathbb{N}}$.

**Definition 2.21** Let $X$ be a space. We say that a sequence $(a_n)_{n \in \mathbb{N}}$ *converges* to $x \in X$ and denoted by $a_n \longrightarrow x$ if for any open neighborhood $U_x$ of $x$, there exists a tail $A_m$ of $(a_n)_{n \in \mathbb{N}}$ such that $A_m \subseteq U_x$. That means for any open neighborhood $U_x$ of $x$, there exists an $m \in \mathbb{N}$ such that $a_k \in U_x$ for each $k \geq m$.

Now, we recall the definitions of two famous topological spaces, which are the Niemytzki plane and the rational sequence, (Steen and Seebach, 1995).

**Example 2.22** Let $\mathbb{P} = \{\langle x, y \rangle : x, y \in \mathbb{R}, y > 0\}$ be the open upper half-plane with the usual Euclidean topology and $\mathbb{L}$ be the $x$-axis. We generate a topology $\mathscr{T}$ on $X = \mathbb{P} \bigcup \mathbb{L}$

by the following neighborhood system: the basic open neighborhood of $\langle x,y\rangle \in \mathbb{P}$ is an open disc $D$ in $\mathbb{P}$. The basic open neighborhood of $\langle x,0\rangle \in \mathbb{L}$ is of the form $\{\langle x,0\rangle\}\bigcup D$, where $D$ is an open disc in $\mathbb{P}$, which is tangent to $\mathbb{L}$ at the point $\langle x,0\rangle$. This topology is called the *Niemytzki plane* or the *Moore plane*.

**Example 2.23** Let $X=\mathbb{R}$. For each $x\in\mathbb{P}$, where $\mathbb{P}$ is the irrational numbers, fix a sequence $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{Q}$ such that $x_n \longrightarrow x$, where the convergency is taken in $(\mathbb{R},\mathscr{U})$. Let $A_n(x)$ denote the $n^{\text{th}}$-tail of the sequence, where $A_n(x)=\{x_j : j\geq n\}$. For each $x\in\mathbb{P}$, let $\mathscr{B}(x)=\{U_n(x) : n\in\mathbb{N}\}$, where $U_n(x)=A_n(x)\bigcup\{x\}$. For each $x\in\mathbb{Q}$, let $\mathscr{B}(x)=\{\{x\}\}$. Then, $\{\mathscr{B}(x)\}_{x\in\mathbb{R}}$ is a neighborhood system. The unique topology on $\mathbb{R}$ generated by $\{\mathscr{B}(x)\}_{x\in\mathbb{R}}$ is called the *rational sequence* on $\mathbb{R}$ and denoted by $\mathscr{RS}$.

## 2.6 Continuous functions and homeomorphism

The following definitions and results are in (Engelking, 1989; Patty, 1993)

**Proposition 2.24** *Let $X$ and $Y$ be two sets, $f : X\longrightarrow Y$ be a function, $A,B\subseteq X$ and $C,D\subseteq Y$, then:*

(i) $f(A\bigcup B)=f(A)\bigcup f(B)$.

(ii) $f^{-1}(C\bigcup D)=f^{-1}(C)\bigcup f^{-1}(D)$.

(iii) $f(A\bigcap B)\subseteq f(A)\bigcap f(B)$ *and* $f(A\bigcap B)=f(A)\bigcap f(B)$ *if and only if $f$ is one-to-one.*

(iv) $f^{-1}(C\bigcap D)=f^{-1}(C)\bigcap f^{-1}(D)$.

(v) $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

(vi) $A \subseteq f^{-1}(f(A))$ *and* $A = f^{-1}(f(A))$ *if* $f$ *is one-to-one.*

(vii) $f(f^{-1}(D)) \subseteq D$ *and* $f(f^{-1}(D)) = D$ *if* $f$ *is onto.*

**Definition 2.25** Let $X$ and $Y$ be two spaces. Then,

(1) A function $f : X \longrightarrow Y$ is called a *continuous* at a point $x \in X$ if for each open set $V$ in $Y$ with $f(x) \in V$, there exists an open set $U$ in $X$ such that $x \in U$ and $f(U) \subseteq V$. The function $f$ is called continuous on $X$ if it is continuous at each point $x \in X$.

(2) A function $f : X \longrightarrow Y$ is called an *open* (resp. a *closed*) if the image of any open (resp. closed) set in $X$ is an open (resp. closed) set in $Y$.

(3) A function $f : X \longrightarrow Y$ is called a *clopen* if it is a closed and open.

Observe that we do not require continuity in the definitions of open and closed functions.

**Theorem 2.26** *Let* $X$ *and* $Y$ *be two spaces and* $f : X \longrightarrow Y$ *be a function, then the following statements are equivalent:*

(a) $f$ *is continuous.*

(b) *For each open set* $V \subseteq Y$, $f^{-1}(V)$ *is an open in* $X$.

(c) *The inverse image of all members of a subbase* $\mathscr{P}$ *for* $Y$ *are open sets in* $X$.

(d) *For each basic open set* $W \subseteq Y$, $f^{-1}(W)$ *is an open in* $X$.

(e) *For each closed set $M \subseteq Y$, $f^{-1}(M)$ is a closed in $X$.*

(f) *For each $A \subseteq X$, we have $f(\overline{A}^X) \subseteq \overline{f(A)}^Y$.*

(g) *For every $B \subseteq Y$, we have $\overline{f^{-1}(B)}^X \subseteq f^{-1}(\overline{B}^Y)$.*

(h) *For every $B \subseteq Y$, we have $f^{-1}(\mathrm{int}_Y(B)) \subseteq \mathrm{int}_X(f^{-1}(B))$.*

**Theorem 2.27** *Let $f : X \longrightarrow Y$ be a continuous function, $A \subseteq X$ and $B \subseteq Y$, then:*

(1) *$f$ is a closed if and only if $\overline{f(A)}^Y = f(\overline{A}^X)$.*

(2) *$f$ is an open if and only if $\mathrm{int}_X(f^{-1}(B)) = f^{-1}(\mathrm{int}_Y(B))$.*

(3) *$f$ is an open if and only if $\overline{f^{-1}(B)}^X = f^{-1}(\overline{B}^Y)$.*

(4) *If $f$ is an open, then $f(\mathrm{int}_X(A)) \subseteq \mathrm{int}_Y(f(A))$.*

(5) *If $f$ is an open and onto, then $f(\mathrm{int}_X(A)) = \mathrm{int}_Y(f(A))$.*

**Definition 2.28** A function $f : X \longrightarrow Y$ is called a *homeomorphism* if it is continuous, one-to-one, onto and $f^{-1}$ is continuous. Two spaces $X$ and $Y$ are *homeomorphic* if there exists a homeomorphism $f$ from $X$ onto $Y$ and denoted by $X \cong Y$.

**Definition 2.29** A property P is said to be a *topological property* if whenever one space possesses the property P, any space homeomorphic to it also possesses the same property. If every subspace has the property P whenever a space does, then the property P is said to be a *hereditary property*.

**Proposition 2.30** *Let $f$ be a bijective continuous function from a space $X$ onto a space $Y$, then the following conditions are equivalent:*

(1) *$f$ is a homeomorphism.*

(2) *$f$ is a closed.*

(3) *$f$ is an open.*

(4) *The set $f(A)$ is a closed in $Y$ if and only if $A$ is a closed in $X$.*

(5) *The set $f^{-1}(B)$ is an open in $X$ if and only if $B$ is an open in $Y$.*

(6) *The set $f^{-1}(B)$ is a closed in $X$ if and only if $B$ is a closed in $Y$.*

(7) *The set $f(A)$ is an open in $Y$ if and only if $A$ is an open in $X$.*

## 2.7 Separability, second and first countability

The following definitions and propositions are in (Engelking, 1989; Patty, 1993).

**Definition 2.31** A point $x$ in a space $X$ is called an *isolated* point if and only if $\{x\}$ is open. Indeed, the singleton $\{x\}$ is open if and only if $\{x\} = X \setminus \overline{X \setminus \{x\}}$, i.e., $x \notin \overline{X \setminus \{x\}}$.

**Definition 2.32** A set $A$ is called a *finite* if it is an empty or there exists a bijective function $f : A \to I_n$, where $I_n = \{a_1, a_2, ..., a_n\}$, $n \in \mathbb{N}$. A set that is not finite is called an *infinite*. A set $A$ is called a *countably infinite* if there exists a bijective function $f : A \to \mathbb{N}$. A set $A$ is called a *countable* if it is finite or countably infinite. A set that is not countable is said to be *uncountable*.