STARLIKENESS OF CERTAIN INTEGRAL OPERATORS AND PROPERTIES OF A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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STARLIKENESS OF CERTAIN INTEGRAL OPERATORS AND PROPERTIES OF A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

by

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LIST OF SYMBOLS

$A[f]$	Alexander operator
$\mathcal{A}_n$	Class of all normalized analytic functions f of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad z \in \mathbb{D}$
$\mathcal{A}$	$\mathcal{A}_1$
$\mathbb{C}$	Complex plane
$\mathcal{C}$	Class of convex functions in $\mathcal{A}$
$\mathcal{C}(\alpha)$	Class of convex functions of order α in $\mathcal{A}$
$\mathbb{D}$	Unit disk
$\mathcal{H}$	Class of all analytic functions in $\mathbb{D}$
$\mathcal{H}[a,n]$	Class of all analytic functions f of the form $f(z) = a + \sum_{k=n}^{\infty} a_k z^k, \quad z \in \mathbb{D}$
Im	Imaginary part of a complex number
k	Koebe function, $k(z) = z/(1-z)^2$
$\mathcal{K}$	Class of close-to-convex functions in $\mathcal{A}$
$L[f]$	Libera operator
$L_\gamma[f]$	Bernardi operator
m	Möbius function, $m(z) = (1+z)/(1-z)$
max	Maximum
$\mathbb{N}$	Natural numbers
$\mathcal{P}$	Class of normalized analytic function with positive real part
$\Re$	Real part of a complex number
$\mathcal{S}$	Class of all normalized univalent functions f in $\mathcal{A}$
$\mathcal{S}^*$	Class of starlike functions in $\mathcal{A}$
$\mathcal{S}^*(\alpha)$	Class of starlike functions of order α in $\mathcal{A}$
$\mathcal{S}_s^*$	Class of starlike functions with respect to symmetric points in $\mathcal{A}$
ω	Schwarz function
$\prec$	Subordinate to

KEBAKBINTANGAN BEBERAPA PENGOPERASI KAMIRAN DAN SIFAT SUBKELAS FUNGSI HAMPIR CEMBUNG

ABSTRAK

Disertasi ini mengkaji syarat cukup bagi fungsi analisis bernilai kompleks bak-bintang dalam cakera unit dan ciri-ciri suatu subkelas fungsi hampir cembung. Suatu kajian ringkas mengenai konsep asas dan keputusan dari teori fungsi univalent analitik telah diberikan. Syarat cukup bagi fungsi analitik yang tertakrif dalam cakera unit untuk menjadi bak-bintang peringkat β yang mematuhi ketidaksamaan pembezaan ketiga. Dengan menggunakan ketidaksamaan pembezaan ketiga, kebakbintangan suatu pengoperasi kamiran akan diperoleh. Keputusan yang diperoleh menyatukan hasil kajian terdahulu. Tambahan pula, suatu subklass fungsi hampir cembung yang baru telah diperkenalkan dan beberapa keputusan menarik telah diperoleh seperti sifat rangkuman, anggaran ketidaksamaan Fekete-Szego bagi fungsi tergolong dalam klass, anggaran pekali, dan syarat cukup.

STARLIKENESS OF CERTAIN INTEGRAL OPERATORS AND PROPERTIES OF A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

ABSTRACT

The present dissertation investigates the sufficient conditions for an analytic function to be starlike in the open unit disk $\mathbb{D}$ and some properties of certain subclass of close-to-convex functions. A brief survey of the basic concepts and results from the classical theory of analytic univalent functions are given. Sufficient conditions for analytic functions satisfying certain third-order differential inequalities to be starlike in $\mathbb{D}$ is derived. As a consequence, conditions for starlikeness of functions defined by triple integral operators are obtained. Connections are also made to earlier known results. Furthermore, a new subclass of close-to-convex functions is introduced and studied. Some interesting results are obtained such as inclusion relationships, an estimate for the Fekete-Szegö functional for functions belonging to the class, coefficient estimates, and a sufficient condition.

CHAPTER 1

INTRODUCTION

1.1 A Short History

Geometric function theory is a branch of complex analysis, which studies the geometric properties of analytic functions. The theory of univalent functions is one of the most important subjects in geometric function theory. The study of univalent functions was initiated by Koebe [21] in 1907. One of the major problems in this field had been the Bieberbach [4] conjecture dating from the year 1916, which asserts that the modulus of the nth Taylor coefficient of each normalized analytic univalent function is bounded by n. The conjecture was not completely solved until 1984 by French-American mathematician Louis de Branges [9].

1.2 Basic Definitions And Properties Of The Class Of Univalent Functions

Let $\mathbb{C}$ be the complex plane of complex numbers. A *domain* is an open connected subset of $\mathbb{C}$. A domain is said to be *simply connected* if its complement is connected. Geometrically, a simply connected domain is a domain without any holes in it. A complex-valued function f of a complex variable is said to be *differentiable* at a point $z_0 \in \mathbb{C}$ if it has a derivative

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

at z_0. The function f is *analytic* at z_0 if it is differentiable at every point in some neighborhood of z_0. It is one "miracle" of complex analysis that an analytic function f must have derivatives of all order at z_0 and has a Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!},$$

which converges in some open disk centered at z_0. It is analytic in a domain if it is analytic at every point of the domain.

Definition 1.1. [15] A function f on $\mathbb{C}$ is said to be *univalent* (*one-to-one*) in a domain $\mathcal{D} \subset \mathbb{C}$ if for $z_1, z_2 \in \mathcal{D}$,

$$f(z_1) = f(z_2) \Rightarrow z_1 = z_2,$$

or equivalently

$$z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2).$$

A function f is said to be *locally univalent* at a point $z_0 \in \mathcal{D}$ if it is univalent in some neighborhood of z_0. For analytic functions f, the condition $f'(z_0) \neq 0$ is equivalent to local univalence at z_0. A function f univalent in a domain $\mathcal{D}$ is locally univalent at each of the points in $\mathcal{D}$, but the converse is not true in general. For example, consider the function $f(z) = z^2$ in the domain $\mathbb{C} - \{0\}$. Since $f'(z) = 2z \neq 0$ for $z \neq 0$, it follows that $f(z) = z^2$ is locally univalent in $\mathbb{C} - \{0\}$. But $f(-z) = (-z)^2 = z^2 = f(z)$, so this function is not univalent in the whole domain $\mathbb{C} - \{0\}$. However, $f(z) = z^2$ is univalent on $\{z \in \mathbb{C} : \Re z > 0\}$. (Here, $\Re z$ denote the real part of z.)

Noshiro [36] and Warschawski [56] independently provides a sufficient condition for an analytic function to be univalent in a convex domain $\mathcal{D}$, which is now known as the Noshiro-Warschawski Theorem. A domain $\mathcal{D}$ is *convex* if the line segment joining any two points in $\mathcal{D}$ lies completely in $\mathcal{D}$, that is, for every $z_1, z_2 \in \mathcal{D}$, we have $z_1 + t(z_2 - z_1) \in \mathcal{D}$ for $0 \leq t \leq 1$. Examples of convex domain are circular disk and half-plane.

Theorem 1.1. (Noshiro-Warschawski Theorem) [36, 56] If f is analytic in a convex domain $\mathcal{D}$, and $\Re\{f'\} > 0$ in $\mathcal{D}$, then f is univalent in $\mathcal{D}$. (Here, $\Re\{f'\}$ denote the real part of f'.)

Proof. We will show that $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in \mathcal{D}$ with $z_1 \neq z_2$. Choose distinct points $z_1, z_2 \in \mathcal{D}$. Since $\mathcal{D}$ is a convex domain, the straight line segment $z = z_1 + t(z_2 - z_1), 0 \leq t \leq 1$, must lie in $\mathcal{D}$. By integrating along this line segment from z_1 to z_2, we have

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(z)\, dz = \int_0^1 f'\Big(z_1 + t(z_2 - z_1)\Big)(z_2 - z_1)\, dt.$$

Dividing by $z_2 - z_1$ and taking the real part, we get

$$\Re\left\{\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right\} = \Re\left\{\int_0^1 f'\Big(z_1 + t(z_2 - z_1)\Big)\, dt\right\}.$$

Since f is analytic in $\mathcal{D}$, f' exists and is analytic in $\mathcal{D}$. It is known that an analytic function is differentiable and continuous in $\mathcal{D}$. It follows that

$$\Re\left\{\int_0^1 f'\Big(z_1 + t(z_2 - z_1)\Big)\, dt\right\} = \int_0^1 \Re\left\{f'\Big(z_1 + t(z_2 - z_1)\Big)\right\} dt.$$

Since $\Re\{f'\} > 0$ for all $z \in \mathbb{D}$, it follows that

$$\Re\left\{\frac{f(z_2)-f(z_1)}{z_2-z_1}\right\} = \int_0^1 \Re\left\{f'\Big(z_1+t(z_2-z_1)\Big)\right\} dt > 0.$$

Hence,

$$\frac{f(z_2)-f(z_1)}{z_2-z_1} \neq 0$$

and so $f(z_1) \neq f(z_2)$. $\square$

Let $\mathcal{H}$ denote the class of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \left\{f \in \mathcal{H} : f(z) = a + \sum_{k=n}^{\infty} a_k z^k,\ z \in \mathbb{D}\right\}$$

and

$$\mathcal{A}_n = \left\{f \in \mathcal{H} : f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k,\ z \in \mathbb{D}\right\},$$

with $\mathcal{A}_1 := \mathcal{A}$. So, $\mathcal{A}$ is the class of analytic functions in $\mathbb{D}$ with normalization $f(0) = 0$ and $f'(0) = 1$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$.

Example 1.1. An important example of functions in the class $\mathcal{S}$ is the Koebe function, given by

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n = z + 2z^2 + 3z^3 + \cdots .$$

It is easy to verify that the Koebe function is analytic, normalized and univalent in $\mathbb{D}$. Since the Koebe function is differentiable at every $z \in \mathbb{D}$, it follows that Koebe function is analytic in $\mathbb{D}$. Also, the Koebe function satisfies the condition $k(0) = 0$ and

$k'(0) = 1$ where $k'(z) = (1+z)/(1-z)^3$. Hence, the Koebe function is normalized in $\mathbb{D}$. To see that the Koebe function is univalent in $\mathbb{D}$, suppose that $k(z_1) = k(z_2)$, that is,

$$\frac{z_1}{(1-z_1)^2} = \frac{z_2}{(1-z_2)^2}, \quad z_1, z_2 \in \mathbb{D}.$$

After a simple computation, we get

$$(z_1 - z_2)(1 - z_1 z_2) = 0.$$

Since $z_1, z_2 \in \mathbb{D}$, we have $|z_1| < 1$ and $|z_2| < 1$ and therefore $|z_1 z_2| = |z_1||z_2| < 1$. This shows that $1 - z_1 z_2 \neq 0$ in $\mathbb{D}$. Thus we must have $z_1 - z_2 = 0$, that is, $z_1 = z_2$. So, the Koebe function, k is univalent in $\mathbb{D}$.

Geometrically, the Koebe function maps $\mathbb{D}$ univalently onto the entire complex plane minus the negative axis from $-1/4$ to infinity. This can be seen by observing that the Koebe function can be written as a composition of three univalent analytic functions, that is,

$$(u_3 \circ u_2 \circ u_1)(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \frac{z}{(1-z)^2},$$

where

$$u_1(z) = \frac{1+z}{1-z}, \quad u_2(z) = z^2, \quad \text{and} \quad u_3(z) = \frac{1}{4}[z-1].$$

It is easy to see that u_1, u_2 and u_3 are analytic and they map univalently on this composition. Since u_1 is the quotient of two analytic functions $1+z$ and $1-z$, therefore it is analytic in $\mathbb{D}$. To see that u_1 is univalent in $\mathbb{D}$, suppose that $u_1(z_1) = u_1(z_2)$, that is,

$$\frac{1+z_1}{1-z_1} = \frac{1+z_2}{1-z_2}, \quad z_1, z_2 \in \mathbb{D}.$$

After simplifying, we obtain $z_1 - z_2 = 0$ or $z_1 = z_2$. Hence, the function $u_1(z) = (1+z)/(1-z)$ is univalent in $\mathbb{D}$. We have

$$\Re\{u_1(z)\} = \Re\left\{\frac{1+z}{1-z}\right\} = \frac{1}{2}\left(\frac{1+z}{1-z} + \overline{\frac{1+z}{1-z}}\right) = \frac{1}{2}\left(\frac{1+z}{1-z} + \frac{1+\bar{z}}{1-\bar{z}}\right) = \frac{1-|z|^2}{|1-z|^2} > 0$$

for $|z| < 1$. Since $u_1(0) = 1$, it follows that $\mathbb{D}$ is mapped univalently onto the right half-plane, $\{z \in \mathbb{C} : \Re\{z\} > 0\}$, under the mapping $u_1(z) = (1+z)/(1-z)$.

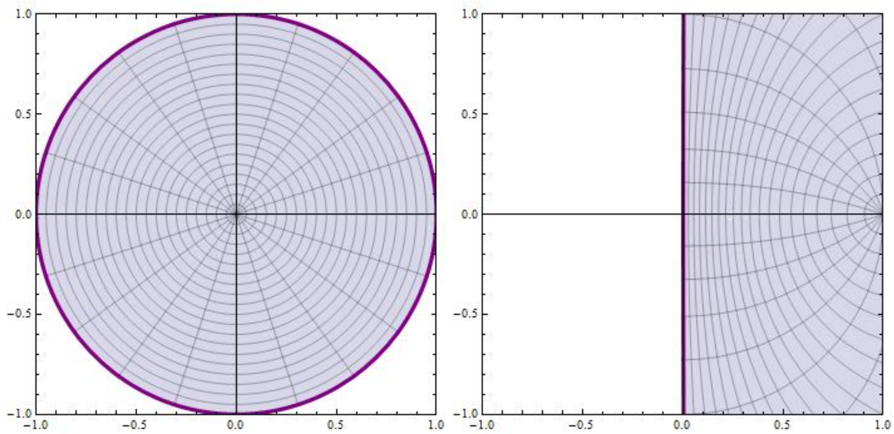

Figure 1.1: The image of unit disk $\mathbb{D}$ under the mapping $u_1(z) = (1+z)/(1-z)$. ■

Since u_2 is the product of two analytic functions z, it follows that u_2 is analytic in the right half plane (a convex domain). For $u_2(z) = z^2$, $\Re\{z\} > 0$, we have

$$\Re\{u_2'(z)\} = 2\Re\{z\} > 0.$$

Hence, by Noshiro - Warschawski Theorem (Theorem 1.1), the function $u_2(z)$ is univalent in the right half plane. Note that the upper right half plane is mapped onto upper half plane, positive real axis is mapped onto positive real axis and the lower right half plane is mapped onto lower half plane. Note that $u_2(0) = 0$ and the imaginary axis is mapped onto the negative real axis. Since the origin and the imaginary axis lies outside of the right half plane, it follows that the function u_2 mapped the right half plane univalently onto the entire complex plane minus the nonnegative real axis.

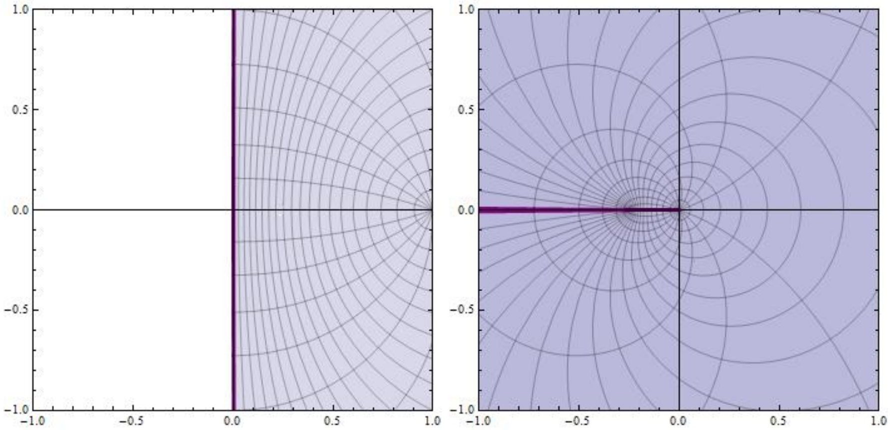

Figure 1.2: The image of right half plane under the mapping $u_2(z) = z^2$.

Clearly, u_3 is analytic in entire complex plane minus the nonnegative real axis. To see that u_3 is univalent, suppose that $u_3(z_1) = u_3(z_2)$, that is,

$$\frac{1}{4}(z_1 - 1) = \frac{1}{4}(z_2 - 1).$$

After simplifying, we obtain $z_1 - z_2 = 0$ or $z_1 = z_2$. Hence, u_3 is univalent in entire complex plane minus the nonnegative real axis. So, u_3 translates the nonnegative real axis one space to the left and multiplies by a factor of $1/4$. Therefore, u_3 maps the entire complex plane except for the nonnegative real axis univalently onto the entire complex plane minus the negative axis from $-1/4$ to infinity.

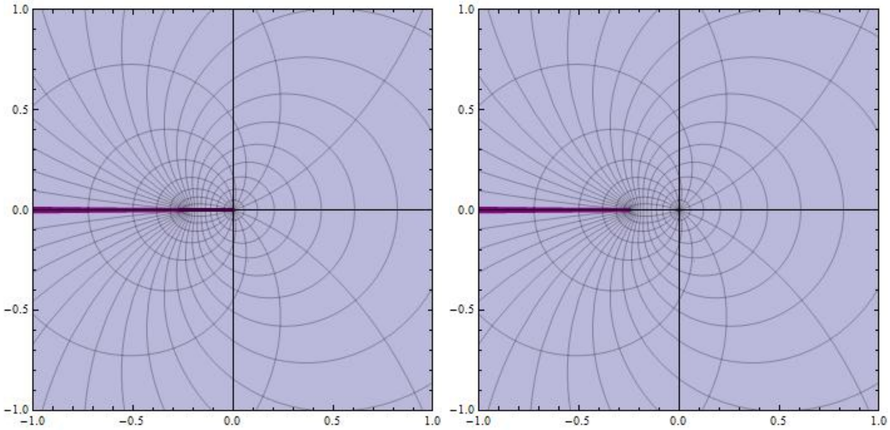

Figure 1.3: The image domain under the mapping $u_3(z) = \frac{1}{4}(z-1)$.

For every function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathcal{S}$, Bieberbach [4] showed that the second coefficient a_2 of the series expansion is bounded by 2, which is now known as Bieberbach's Theorem.

Theorem 1.2. [4] (Bieberbach's Theorem) If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, then $|a_2| \leq 2$, with equality if and only if f is a rotation of the Koebe function.

The extremal property of the Koebe function tempted Bieberbach [4] to conjecture that $|a_n| \leq n$ holds for all f in $\mathcal{S}$. This conjecture was popularly known as Bieberbach's conjecture.

Conjecture 1.1. [4] (Bieberbach's Conjecture) The coefficients of each function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ satisfy $|a_n| \leq n$ for $n = 2, 3, \ldots$. Strict inequality holds for all n unless f is the Koebe function or one of its rotations.

The conjecture had been proven for the case $n = 2, 3, 4, 5, 6$ by some researchers before Louis de Branges [9] proved the general case $|a_n| \leq n$ in 1984. This is summarized in the table below.

Researchers	Result
Bieberbach [4] (1916)	$\lvert a_2 \rvert \leq 2$
Löwner [29] (1923)	$\lvert a_3 \rvert \leq 3$
Garabedian and Schiffer [14] (1955)	$\lvert a_4 \rvert \leq 4$
Pederson [42] (1968), Ozawa [39] (1969)	$\lvert a_6 \rvert \leq 6$
Pederson and Schiffer [41] (1972)	$\lvert a_5 \rvert \leq 5$
de Branges [9] (1984)	$\lvert a_n \rvert \leq n$

Nowadays, the Bieberbach conjecture is also called the de Branges Theorem.

1.2.1 Function With Positive Real Part And Subordination

Definition 1.2. [15] An analytic function of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

in $\mathbb{D}$ with $\Re\{p(z)\} > 0$ is called a *function of positive real part* or *Carathéodory function*. The set of all functions of positive real part in $\mathbb{D}$ is denoted by $\mathcal{P}$.

Example 1.2. The Möbius function

$$m(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots = 1 + 2\sum_{n=1}^{\infty} z^n,$$

is in the class $\mathcal{P}$ since $\Re\{(1+z)/(1-z)\} > 0$, as shown in Example 1.1. ■

Example 1.3. The function

$$w(z) = \frac{1+z^n}{1-z^n}, \quad n = 1, 2, 3, \ldots$$

belongs to $\mathcal{P}$ for $|z| < 1$. To see this, note that $w(0) = 1$. Further, $w(z) = (m \circ \phi)(z)$ where m is the Möbius function and $\phi(z) = z^n$. Since $|\phi(z)| < 1$, it follows from Example 1.2 that $\Re\{w\} > 0$. ■

In 1911, Herglotz [18] obtained an integral formula for functions in the class $\mathcal{P}$.

Theorem 1.3. [18] Let p be an analytic function in $\mathbb{D}$ satisfying $p(0) = 1$. Then $p \in \mathcal{P}$ if and only if

$$p(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu(t),$$

where $d\mu(t) \geq 0$ and $\int_0^{2\pi} d\mu(t) = \mu(2\pi) - \mu(0) = 1$.

The Herglotz formula gives the bounds for the coefficients of functions in $\mathcal{P}$. This result is due to Carathéodory.

Theorem 1.4. [5] If $p \in \mathcal{P}$ with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $z \in \mathbb{D}$, then $|p_n| \leq 2$ for all $n \in \mathbb{N}$. These estimates are sharp.

Proof. Since $p \in \mathcal{P}$, by Theorem 1.3, we have

$$p(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu(t),$$

where $d\mu(t) \geq 0$ and $\int_0^{2\pi} d\mu(t) = \mu(2\pi) - \mu(0) = 1$. Therefore,

$$\begin{aligned} p(z) &= \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu(t) \\ &= \int_0^{2\pi} (1 + 2ze^{-it} + 2z^2 e^{-2it} + 2z^3 e^{-3it} + \cdots) d\mu(t) \\ &= 1 + \sum_{n=1}^{\infty} \left(2 \int_0^{2\pi} e^{-int} d\mu(t) \right) z^n. \end{aligned}$$

Now comparing this with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ yields

$$p_n = 2\int_0^{2\pi} e^{-int} d\mu(t).$$

Hence,

$$\begin{aligned}
|p_n| &= 2\left|\int_0^{2\pi} e^{-int} d\mu(t)\right| \\
&\leq 2\int_0^{2\pi} |e^{-int}||d\mu(t)| \\
&= 2\int_0^{2\pi} d\mu(t) \\
&= 2.
\end{aligned}$$

The Möbius function in Example 1.2 showed that the bound $|p_n| \leq 2$ is sharp. $\square$

Closely related to the class $\mathcal{P}$ is the class of functions with positive real part of order $\alpha, 0 \leq \alpha < 1$.

Definition 1.3. [15] An analytic function p with the normalization $p(0) = 1$ in $\mathbb{D}$ is said to be a *function of positive real part of order* α, $0 \leq \alpha < 1$ if $\Re\{p(z)\} > \alpha$. The set of all functions of positive real part of order α is denoted by $\mathcal{P}(\alpha)$. Observe that for $\alpha = 0$, we have $\mathcal{P}(0) = \mathcal{P}$.

Example 1.4. Consider the function $f(z) = 1/(1-z)$, $z \in \mathbb{D}$. Since f is differentiable for all $z \in \mathbb{D}$, it is analytic in $\mathbb{D}$. Clearly, $f(0) = 1$. Furthermore,

$$\Re\left\{\frac{1}{1-z}\right\} = \Re\left\{\frac{1}{2}\left(\frac{1+z}{1-z}+1\right)\right\} = \frac{1}{2}\Re\left\{\frac{1+z}{1-z}\right\} + \frac{1}{2} > 0 + \frac{1}{2} = \frac{1}{2}.$$

Therefore, the function $f(z) = 1/(1-z)$ belongs to $\mathcal{P}(1/2)$.

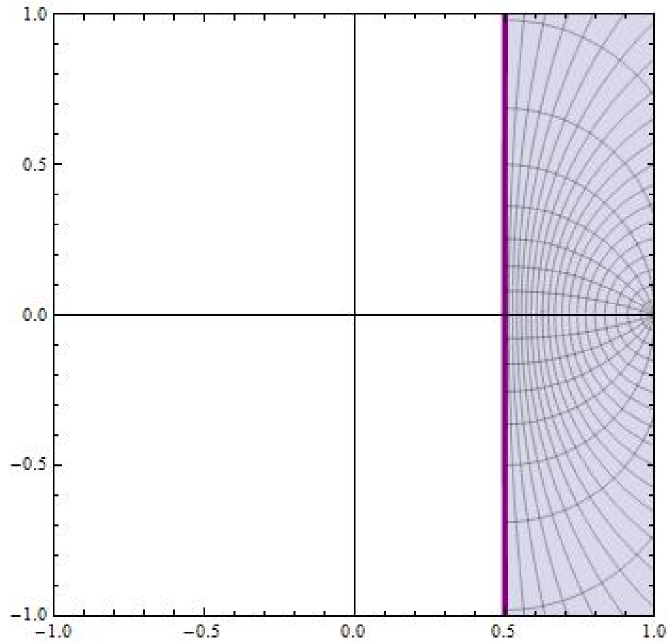

Figure 1.4: The real part of $f(z) = 1/(1-z)$. ■

Example 1.5. The function

$$f(z) = \frac{1+(1-2\alpha)z}{1-z} = (1-\alpha)\left(\frac{1+z}{1-z}\right) + \alpha = 1 + 2(1-\alpha)\sum_{n=1}^{\infty} z^n$$

is in the class $\mathcal{P}(\alpha)$ for $0 \le \alpha < 1$. Clearly, $f(0) = 1$. Also,

$$\Re\left\{(1-\alpha)\left(\frac{1+z}{1-z}\right) + \alpha\right\} = (1-\alpha)\Re\left\{\frac{1+z}{1-z}\right\} + \alpha > \alpha$$

using the fact that $\Re\{(1+z)/(1-z)\} > 0$ as in Example 1.1. For $\alpha = 0$, we have the inequality

$$\Re\{f(z)\} = \Re\left\{\frac{1+z}{1-z}\right\} > 0$$

which has been discussed in Example 1.1.

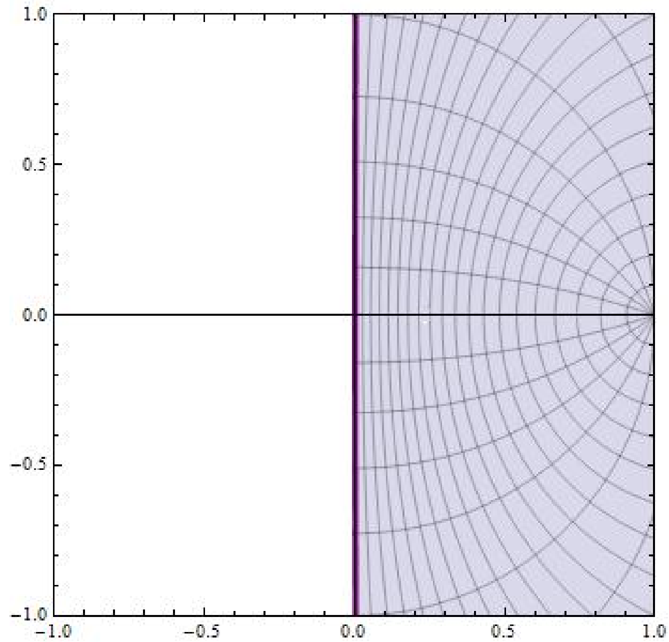

Figure 1.5: The real part of $f(z) = (1+z)/(1-z)$. ■

Definition 1.4. A function ω which is analytic in $\mathbb{D}$ and satisfies the properties $\omega(0) = 0$ and $|\omega(z)| < 1$ is called a *Schwarz function*. The class of all Schwarz functions is denoted by Ω.

Definition 1.5. For analytic functions f and g on $\mathbb{D}$, we say that f is subordinate to g, denoted $f \prec g$, if there exists a Schwarz function ω in $\mathbb{D}$ such that

$$f(z) = g(\omega(z)), \quad z \in \mathbb{D}.$$

Example 1.6. The function z^2 is subordinate to z in $\mathbb{D}$. Referring to Definition 1.5, we can choose $\omega(z) = z^2$. Clearly, ω is analytic in $\mathbb{D}$ and $\omega(0) = 0$. Also, $|\omega(z)| = |z^2| = |z|^2 < 1$ since $z \in \mathbb{D}$. ■

Example 1.7. The function z^4 is subordinate to z^2 in $\mathbb{D}$. Referring to Definition 1.5, we can choose $\omega(z) = z^2$. Clearly, ω is analytic in $\mathbb{D}$ and $\omega(0) = 0$. Also, $|\omega(z)| = |z^2| = |z|^2 < 1$ since $z \in \mathbb{D}$. In general, we have $z^{2n} \prec z^2$ in $\mathbb{D}$ for n a positive integer. ■

Theorem 1.5. Let f and g be analytic in $\mathbb{D}$. If g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(\mathbb{D}) \subset g(\mathbb{D})$ and $f(0) = g(0)$.

Proof. Suppose $f \prec g$. By Definition 1.5, there exists a Schwarz function ω such that $f(z) = g(\omega(z))$. Since $\omega(\mathbb{D}) \subset \mathbb{D}$, it follows that $f(\mathbb{D}) = g(\omega(\mathbb{D})) \subset g(\mathbb{D})$. Also $f(0) = g(\omega(0)) = g(0)$.

Conversely, suppose $f(\mathbb{D}) \subset g(\mathbb{D})$ and $f(0) = g(0)$. Since g is univalent in $\mathbb{D}$, it follows that g maps $\mathbb{D}$ one-to-one onto its image $g(\mathbb{D})$. Therefore, the inverse g^{-1} exists in $g(\mathbb{D})$ and maps $g(\mathbb{D})$ onto $\mathbb{D}$. Since g is analytic in $\mathbb{D}$, the inverse g^{-1} is also analytic in $g(\mathbb{D})$. Since $f(\mathbb{D}) \subset g(\mathbb{D})$, it follows that the function

$$\omega(z) := g^{-1}(f(z))$$

is analytic in $\mathbb{D}$ and $|\omega(z)| < 1$. Thus, we obtain $f(z) = g(\omega(z))$. From this, we have $g(\omega(0)) = f(0) = g(0)$. Since g is univalent, this forces $\omega(0) = 0$ by Definition 1.1. So, ω is a Schwarz function such that $f(z) = g(\omega(z))$ for $z \in \mathbb{D}$. Therefore, $f \prec g$. $\square$

1.2.2 Subclasses Of Univalent Functions

In the course of tackling the Bieberbach conjecture, new classes of analytic and univalent functions were defined and some nice properties of these classes were widely investigated. Examples of such classes are the classes of starlike, convex and close-to-convex functions.

A domain $\mathcal{D} \subset \mathbb{C}$ is said to be *starlike with respect to a point* w_0 in $\mathcal{D}$ if every line joining the point w_0 to every other point w in $\mathcal{D}$ lies entirely inside $\mathcal{D}$. A domain which is starlike with respect to the origin is simply called a *starlike* domain. Geometrically, a starlike domain is a domain whose all points can be seen from the origin. A function $f \in \mathcal{A}$ is called a *starlike* function if $f(\mathbb{D})$ is a starlike domain. The subclass of $\mathcal{S}$ consisting of all starlike functions is denoted by $\mathcal{S}^*$.

Theorem 1.6. [10, Theorem 2.10] Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}^*$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

Example 1.8. Recall from Example 1.1, the Koebe function $k(z) = z/(1-z)^2$ is analytic and normalized in $\mathbb{D}$. Moreover, k is in $\mathcal{S}^*$ since

$$\Re\left\{\frac{zk'(z)}{k(z)}\right\} = \Re\left\{\frac{z(1+z)}{(1-z)^3}\frac{(1-z)^2}{z}\right\} = \Re\left\{\frac{1+z}{1-z}\right\} > 0. \qquad \blacksquare$$

Example 1.9. The function

$$f(z) = \frac{z}{1-z^2} = \sum_{n=0}^{\infty} z^{2n+1}$$

is analytic in $\mathbb{D}$ since f is differentiable at all $z \in \mathbb{D}$. Clearly, $f(0) = 0$. Since $f'(z) = (1+z^2)/(1-z^2)^2$, it follows that $f'(0) = 1$. Also,

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} = \Re\left\{\frac{z(1+z^2)}{(1-z^2)^2}\frac{(1-z^2)}{z}\right\} = \Re\left\{\frac{1+z^2}{1-z^2}\right\} > 0.$$

The last inequality follows from Example 1.3. Hence, the function $f(z) = z/(1-z^2)$ is starlike on $\mathbb{D}$.

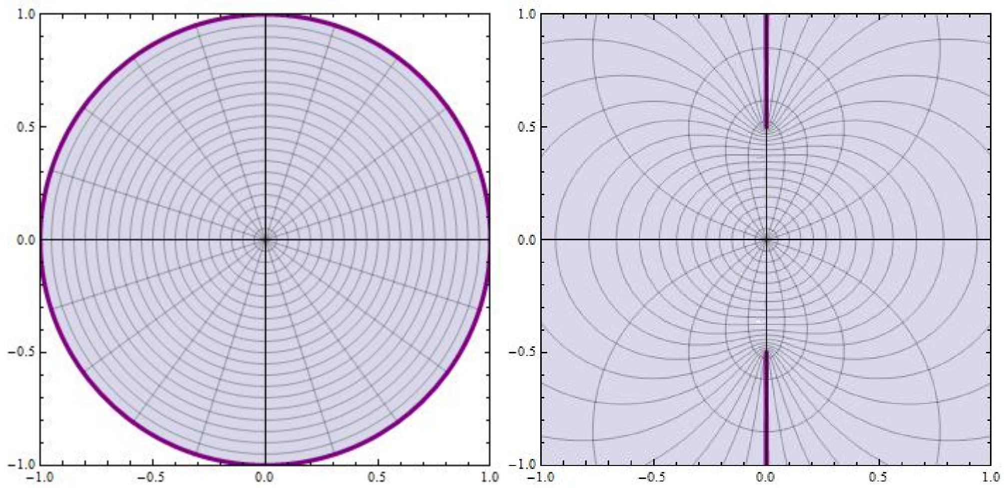

Figure 1.6: The image of $\mathbb{D}$ under the mapping $f(z) = z/(1-z^2)$. ■

A domain $\mathcal{D} \subset \mathbb{C}$ is said to be *convex* if every linear segment joining any two points in $\mathcal{D}$ lies completely inside $\mathcal{D}$. In other words, the domain $\mathcal{D}$ is convex if and only if it is starlike with respect to every point in $\mathcal{D}$. A function $f \in \mathcal{A}$ is said to be *convex* if $f(\mathbb{D})$ is a convex domain. The subclass of $\mathcal{S}$ consisting of all convex functions is denoted by $\mathcal{C}$.

Theorem 1.7. [10, Theorem 2.11] Let $f \in \mathcal{A}$. Then $f \in \mathcal{C}$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

Example 1.10. The identity function $f(z) = z$ is a convex function. Note that $f''(z) = 1$ and $f''(z) = 0$. Hence,

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = 1 > 0. \qquad ■$$

Example 1.11. The function

$$f(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n$$

is analytic in $\mathbb{D}$ since f is differentiable in $\mathbb{D}$. Clearly, $f(0)=0$. Since $f'(z)=1/(1-z)^2$, it follows that $f'(0)=1$. Also, $f''(z)=2/(1-z)^3$. Hence,

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}=\Re\left\{1+\frac{2z(1-z)^2}{(1-z)^3}\right\}=\Re\left\{\frac{1+z}{1-z}\right\}>0.$$

Therefore, the function $z/(1-z)$ is convex in $\mathbb{D}$.

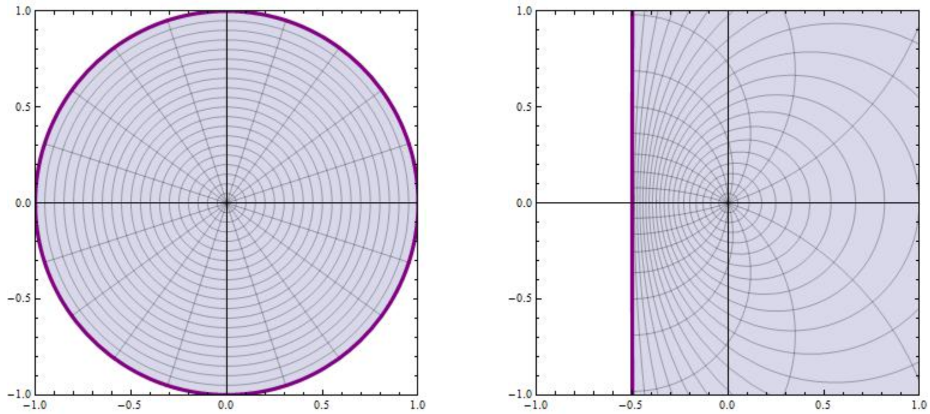

Figure 1.7: The image of unit disk $\mathbb{D}$ under the mapping $f(z)=z/(1-z)$. ■

Example 1.12. The function

$$f(z)=-\log(1-z)=\sum_{n=1}^{\infty}\frac{z^n}{n}$$

is analytic in $\mathbb{D}$ since f is differentiable at every $z\in\mathbb{D}$. Clearly, $f(0)=0$. Since $f'(z)=1/(1-z)$, it follows that $f'(0)=1$. Also, $f''(z)=1/(1-z)^2$. So,

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}=\Re\left\{1+\frac{z(1-z)}{(1-z)^2}\right\}=\Re\left\{\frac{1}{1-z}\right\}.$$

From Example 1.4, it has been shown that $\Re\{1/(1-z)\}>1/2$. It follows that $\Re\{1/(1-z)\}>0$. Hence, the function $f(z)=-\log(1-z)$ is convex on $\mathbb{D}$.

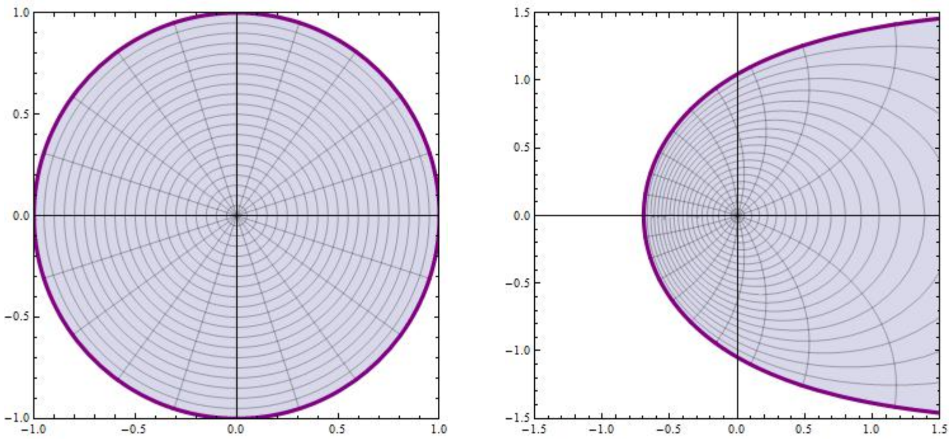

Figure 1.8: The image of $\mathbb{D}$ under the mapping $f(z) = -\log(1-z)$. ■

Example 1.13. The function

$$f(z) = \frac{1}{2}\log\frac{1+z}{1-z} = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1}$$

is analytic in $\mathbb{D}$ since f is differentiable at every $z \in \mathbb{D}$. Clearly, $f(0) = 0$. Note that $f'(z) = 1/(1-z^2)$ and therefore $f'(0) = 1$. Also, $f''(z) = 2z/(1-z^2)^2$. Hence,

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = \Re\left\{1 + \frac{2z^2(1-z^2)}{(1-z^2)^2}\right\} = \Re\left\{\frac{1+z^2}{1-z^2}\right\} > 0,$$

by Example 1.3. Therefore, $f(z) = (1/2)[\log(1+z)/(1-z)]$ is convex in $\mathbb{D}$.

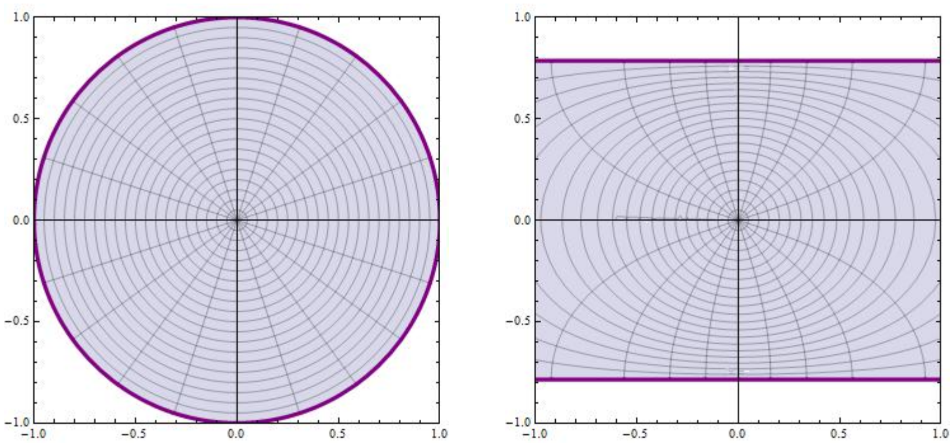

Figure 1.9: The image of $\mathbb{D}$ under the mapping $f(z) = (1/2)[\log(1+z)/(1-z)]$. ■

Remark 1.1. Every convex function f in $\mathbb{D}$ is evidently starlike because the convex domain $f(\mathbb{D})$ is also a starlike domain (starlike with respect to the origin) since f always maps origin to origin. The converse is not true in general as shown by the Koebe function, $k(z) = z/(1-z)^2$. We have seen in Example 1.8 that k is a starlike function. Now, note that since $k'(z) = (1+z)/(1-z)^3$ and $k''(z) = 2(z+2)/(1-z)^4$, it follows that

$$\Re\left\{1+\frac{zk''(z)}{k'(z)}\right\} = \Re\left\{1+\frac{2z(z+2)}{(1-z)^4}\frac{(1-z)^3}{(1+z)}\right\} = \Re\left\{\frac{z^2+4z+1}{1-z^2}\right\}.$$

For $z = -1/2 \in \mathbb{D}$, we have

$$\Re\left(\frac{z^2+4z+1}{1-z^2}\right) = -1 < 0.$$

Hence, Koebe function is not convex in $\mathbb{D}$. Alternatively, we can also show the Koebe function is not convex in geometric view. Recall that the Koebe function maps $\mathbb{D}$ maps $\mathbb{D}$ one-to-one and onto the entire complex plane minus the part of the negative axis from $-1/4$ to infinity. Consider the two points $-1/4+i$ and $-1/4-i$ in the image domain. Clearly, the line segment joining $-1/4+i$ and $-1/4-i$ does not lie inside the image domain. Therefore, the Koebe function is not convex in $\mathbb{D}$.

The two preceding theorems, that is, Theorem 1.6 and Theorem 1.7, provide a connection between starlikeness and convexity. This was first observed by Alexander [2] in 1915.

Theorem 1.8. (Alexander's Theorem) [2] A function $f \in \mathcal{A}$ is convex in $\mathbb{D}$ if and only if the function g defined by $g(z) = zf'(z)$ is starlike in $\mathbb{D}$.

Proof. If $g(z) = zf'(z)$, then

$$\frac{zg'(z)}{g(z)} = \frac{z(zf''(z)+f'(z))}{zf'(z)} = 1+\frac{zf''(z)}{f'(z)}.$$

If the function f is convex, by Theorem 1.7, we have $\Re\{1+zf''(z)/f'(z)\} > 0$. Since $\Re\{zg'(z)/g(z)\} = \Re\{1+zf''(z)/f'(z)\} > 0$, the function g is starlike. The converse follows similarly from above. □

The Alexander's Theorem (Theorem 1.8) can be rephrased in the form $f \in \mathcal{S}^*$ if and only if the function

$$g(z) = \int_0^z \frac{f(t)}{t}\, dt$$

is convex in $\mathbb{D}$.

Example 1.14. Consider the function $f(z) = z/(1-z)$. Since f is convex by Example 1.11, the function

$$g(z) = zf'(z) = \frac{z[(1-z)-z(-1)]}{(1-z)^2} = \frac{z}{(1-z)^2}$$

is starlike in $\mathbb{D}$. Notice that g is the Koebe function. ■

The Bieberbach conjecture for the class $\mathcal{S}^*$ of starlike functions holds true and it was proved by Nevalinna [35] in 1921.

Theorem 1.9. [35, see also 15] If $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*$, then $|a_n| \le n$ for all n. The inequality is sharp, as shown by the Koebe function, $k(z) = z/(1-z)^2$.

Using Alexander's Theorem (Theorem 1.8), the coefficient bound for class $\mathcal{C}$ of convex functions is easily deduced. This result was proved by Löewner [27] in 1917.

Theorem 1.10. [27, see also 15] If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}$, then $|a_n| \leq 1$ for all n. The inequality is sharp for all n.

Proof. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $\mathcal{C}$, by Theorem 1.8,

$$zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$$

is in $\mathcal{S}^*$. By Theorem 1.9, we have $n|a_n| \leq n$. Hence, $|a_n| \leq 1$. Since $z/(1-z) = z + z^2 + z^3 + \cdots$, and it is convex by Example 1.11, the bound $|a_n| \leq 1$ is sharp. □

In 1936, Robertson [45] introduced the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ of starlike and convex functions of order $\alpha, 0 \leq \alpha < 1$, respectively, which are defined as

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \right\}$$

and

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \right\}.$$

For $\alpha = 0$, we have $\mathcal{S}^*(0) := \mathcal{S}^*$ and $\mathcal{C}(0) := \mathcal{C}$. As α increases, both classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ become smaller. For $0 \leq \alpha < 1$, the geometrical interpretation of the notion of convexity of order α is that the ratio of the angle between two adjacent tangents to the unit circle to the angle between the two corresponding tangents of the image of the unit circle is less than $1/\alpha$ and comes arbitrarily close to $1/\alpha$ for some point of the unit circle [45]. Unfortunately, the class $\mathcal{S}^*(\alpha)$ do not admit any clear geometric interpretation for $0 \leq \alpha < 1$.

Example 1.15. Consider the function $k_\alpha(z) = z/(1-z)^{2(1-\alpha)}$, where $0 \leq \alpha < 1$. The function k_α is analytic in $\mathbb{D}$ since it is differentiable at all $z \in \mathbb{D}$. Clearly, $k_\alpha(0) = 0$. Since $k'_\alpha(z) = [1+(1-2\alpha)z]/(1-z)^{3-2\alpha}$, it follows that $k'_\alpha(0) = 1$. Note that

$$\Re\left\{\frac{zk'_\alpha(z)}{k_\alpha(z)}\right\} = \Re\left\{\frac{z(1+(1-2\alpha)z)}{(1-z)^{3-2\alpha}}\frac{(1-z)^{2(1-\alpha)}}{z}\right\} = \Re\left\{\frac{1+(1-2\alpha)z}{1-z}\right\} > \alpha.$$

Hence, k_α is in $\mathcal{S}^*(\alpha)$. This function k_α is called the Koebe function of order α, as $k_0(z) = z/(1-z)^2 = k(z)$, the Koebe function. ■

For $\alpha = 1/2$, we have the class of starlike functions of order $1/2$, that is,

$$\mathcal{S}^*(1/2) = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{2}\right\}.$$

Marx [30] and Strohhäcker [50] independently established the connection between the classes $\mathcal{C}$ and $\mathcal{S}^*(1/2)$.

Theorem 1.11. [30, 50] If $f \in \mathcal{C}$, then $f \in \mathcal{S}^*(1/2)$. This result is sharp, that is, the constant $1/2$ cannot be replaced by a larger constant.

Example 1.16. From Example 1.11, we know that the function $f(z) = z/(1-z)$ is convex. Hence, by Theorem 1.11, we can conclude that $f(z) = z/(1-z)$ is also starlike of order $1/2$. Alternatively, we can show directly that $\Re\{zf'(z)/f(z)\} > 1/2$. Note that $f'(z) = 1/(1-z)^2$. Hence

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} = \Re\left\{\frac{z}{(1-z)^2}\frac{(1-z)}{z}\right\} = \Re\left\{\frac{1}{1-z}\right\} > \frac{1}{2},$$

where the inequality follows from Example 1.4. ■

For $f \in \mathcal{S}^*(1/2)$, Schild [48] obtained the coefficient estimates as follows.

Theorem 1.12. [48] If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(1/2)$, then $|a_n| \leq 1$. The inequality is sharp, as shown by the function $z/(1-z)$.

Another important subclass of univalent analytic functions is the class of close-to-convex functions, which was introduced by Kaplan [19].

Definition 1.6. [19] A function $f \in \mathcal{A}$ is said to be *close-to-convex* in $\mathbb{D}$ if there exists a convex function g in $\mathbb{D}$ such that

$$\Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0,\ z \in \mathbb{D}. \tag{1.1}$$

We denote by $\mathcal{K}$ the class of close-to-convex functions in $\mathbb{D}$.

Every convex function is obviously close-to-convex in $\mathbb{D}$. Indeed, if f is convex in $\mathbb{D}$, then by choosing $g = f$ in (1.1), we have

$$\Re\left\{\frac{f'(z)}{g'(z)}\right\} = \Re\left\{\frac{f'(z)}{f'(z)}\right\} = 1 > 0.$$

Equivalently, the condition (1.1) can be written in the form

$$\Re\left\{\frac{zf'(z)}{h(z)}\right\} > 0,\ z \in \mathbb{D} \tag{1.2}$$

where $h(z) = zg'(z)$ is a starlike function on $\mathbb{D}$ by Alexander's Theorem (Theorem 1.8). In other words, a function $f \in \mathcal{A}$ is said to be *close-to-convex* in $\mathbb{D}$ if there exists a starlike function h such that the inequality (1.2) holds.

Suppose f is a starlike function in $\mathbb{D}$. If choose $h = f$ in (1.2), we have

$$\Re\left\{\frac{zf'(z)}{h(z)}\right\} = \Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

Hence, we can conclude that every starlike function is close-to-convex in $\mathbb{D}$.

Therefore, we have the following inclusion

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K}.$$

From this, instant examples of close-to-convex functions are $z/(1-z)$ and the Koebe functions, $k(z) = z/(1-z)^2$. Now, it is also natural to ask if close-to-convex functions are univalent. Kaplan [19] showed that they are indeed so.

Theorem 1.13. [19] Every close-to-convex function is univalent.

Proof. Suppose f is close-to-convex in $\mathbb{D}$. By Definition 1.6, there exists a convex function g in $\mathbb{D}$ in such that $\Re\{f'(z)/g'(z)\} > 0$. Since g is convex, it follows that g maps $\mathbb{D}$ one-to-one and onto convex domain $g(\mathbb{D})$. Therefore, g^{-1} exists in $g(\mathbb{D})$. Consider the function

$$h(w) = f(g^{-1}(w)),\ w \in g(\mathbb{D}). \tag{1.3}$$

Since g is analytic $\mathbb{D}$, it follows that g^{-1} is also analytic in $g(\mathbb{D})$. Using the fact that the composition of two analytic functions is analytic, the function h is analytic in $\mathbb{D}$.