OPTIMAL DESIGNS OF UNIVARIATE AND MULTIVARIATE SYNTHETIC CONTROL CHARTS BASED ON MEDIAN RUN LENGTH

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by

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1. Khoo, M. B. C., Wong, V. H., Wu, Z. and Castagliola, P. (2011). Optimal designs of the multivariate synthetic chart for monitoring the process mean vector based on median run length. *Quality and Reliability Engineering International*, **27**, 981 – 997.

2. Khoo, M. B. C., Wong, V. H., Wu, Z. and Castagliola, P. (2011). Optimal Design of the Synthetic Chart for the Process Mean based on Median Run Length. *IIE Transactions* (in Press – early view available online).

REKABENTUK OPTIMUM CARTA-CARTA KAWALAN SINTETIK UNIVARIAT DAN MULTIVARIAT BERDASARKAN PANJANG LARIAN MEDIAN

ABSTRAK

Carta-carta kawalan univariat dan multivariat biasanya direkabentuk secara optimum dengan menggunakan panjang larian purata (ARL) sebagai ukuran tunggal prestasi carta-carta. Adalah diketahui jelas bahawa bentuk taburan panjang larian carta-carta univariat dan multivariat berubah daripada sangat terpencong apabila proses berada dalam keadaan terkawal kepada hampir simetri untuk anjakan proses yang besar. Oleh itu, panjang larian median (MRL) adalah tafsiran yang lebih bermakna bagi prestasi keadaan terkawal dan terluar kawal carta-carta dan membekalkan maklumat tambahan yang tidak diberikan oleh panjang larian purata (ARL). Tesis ini mencadangkan prosedur rekabentuk optimum untuk carta-carta sintetik univariat dan multivariat berdasarkan MRL dengan menggunakan pendekatan rantai Markov bagi proses-proses keadaan sifar dan mantap. Carta sintetik univariat terdiri daripada gabungan carta-sub $\bar{X}$ dan carta-sub *conforming run length* (CRL) manakala carta sintetik multivariat terdiri daripada gabungan carta-sub T^2 Hotelling dan carta-sub CRL. Program Mathematica telah ditulis untuk mengira parameter optimum carta-carta sintetik univariat dan multivariat berdasarkan MRL terkawal (MRL_0) yang diingini, bagi proses-proses keadaan sifar dan mantap. Prestasi MRL carta sintetik univariat dibandingkan dengan carta-carta kawalan purata bergerak berpemberat eksponen (EWMA) dan $\bar{X}$ Shewhart, manakala prestasi MRL carta sintetik multivariat dibandingkan dengan carta-carta EWMA multivariat (MEWMA) dan T^2 Hotelling. Dua contoh, setiap satu untuk carta-carta sintetik univariat dan multivariat telah diberi untuk menunjukkan bagaimana

prosedur rekabentuk optimum berdasarkan MRL yang dicadangkan digunakan dalam keadaan sebenar.

OPTIMAL DESIGNS OF UNIVARIATE AND MULTIVARIATE SYNTHETIC CONTROL CHARTS BASED ON MEDIAN RUN LENGTH

ABSTRACT

Univariate and multivariate control charts are usually optimally designed using average run length (ARL) as a sole measure of the charts' performances. It is well known that the shape of the run length distribution for the univariate and multivariate charts changes from highly skewed when the process is in-control to approximately symmetric for large process shifts. Therefore, the median run length (MRL) is a more meaningful interpretation of the in-control and out-of-control performances of the charts and provides additional information not given by the average run length (ARL). This thesis proposes optimal design procedures for the univariate and multivariate synthetic charts, based on MRL, using the Markov chain approach for the zero and steady state processes. The univariate synthetic chart consists of an integration of the $\bar{X}$ sub-chart and the conforming run length (CRL) sub-chart while the multivariate synthetic chart consists of a combination of the Hotelling's T^2 sub-chart and the CRL sub-chart. Mathematica programs are written to compute the optimal parameters of the univariate and multivariate synthetic charts, based on desired in-control MRLs (MRL_0s), for the zero and steady state processes. The MRL performance of the univariate synthetic chart is compared with that of the exponentially weighted moving average (EWMA) and Shewhart $\bar{X}$ charts, while the MRL performance of the multivariate synthetic chart is compared with that of the multivariate EWMA (MEWMA) and Hotelling's T^2 charts. Two examples, each for the univariate and multivariate synthetic charts are given to show how the proposed optimal design procedures, based on MRL, are used in a real situation.

CHAPTER 1
INTRODUCTION

1.1 A Brief History of Quality Control Charts

A control chart is the most powerful tool among the seven well known Statistical Process Control (SPC) tools. It was first proposed by Walter A. Shewhart from the Bell Telephone Laboratories in 1924 (Montgomery, 2009). The purpose of a control chart is to determine whether a manufacturing process is in-control and to predict the future performance of the process. In the 1930s, Shewhart published an outline on quality control charting methods in the *Economic Control of Quality of Manufactured Product.* He was also invited by Deming to give lectures in the University of London and to conduct seminars on control charts at the U.S. Department of Agriculture Graduate School (Montgomery, 2009). In 1940, a guide for using control charts to analyze the process data was published by the U.S. Department of War.

The use of control charts greatly expanded after the World War II. The American Society for Quality Control (ASQC) was formed in 1946 to promote the use of quality improvement techniques. Hotelling (1947) extended the control charting applications from univariate to multivariate quality characteristics in the monitoring of manufacturing processes by proposing the Hotelling's T^2 chart. During the 1950s, Page proposed the cumulative sum (CUSUM) chart while Roberts proposed the exponentially weighted moving average (EWMA) chart. The CUSUM and EWMA charts improve the performance of the Shewhart chart for detecting small process shifts (Montgomery, 2009). More recently, the multivariate EWMA (MEWMA) and multivariate CUSUM (MCUSUM) charts were proposed to improve the performance of the Hotelling's T^2 chart towards small shifts in the mean vector.

To date, numerous works and extensions on the existing charting methods have been carried out by researchers for the sake of suggesting more powerful and robust control charts to meet the requirements of today's manufacturing environment.

1.2 Statistical Process Control (SPC)

SPC is defined as the application of statistical methods to monitor, control and optimize the process. It was introduced by Shewhart in the early 1920s. The main purpose of using SPC is to (Garrity, 1993):

i) eliminate variations in the process and make the process consistent.

ii) identify assignable cause(s) of variation in the process.

iii) reduce errors, scrap and reworks.

iv) simplify and improve work procedures.

v) encourage participation and involvement in quality improvement.

vi) increase the decision making and manage the process by facts but not opinion.

The steps in implementing SPC are as follows (Montgomery, 2009):

i) Identify defined process.

ii) Identify measurable attributes of the process.

iii) Characterize natural variation of attributes.

iv) Track variation in the process.

v) Continue monitoring the process variation if the process is in an in-control state.

vi) When the process is out-of-control, identify and remove the assignable causes. Then continue to monitor the process for future process variation.

The seven basic SPC tools are the cause-and-effect (fishbone) diagram, control chart, scatter diagram, check sheet, flow chart, Pareto chart and histogram. A control chart is the most important tool among the seven SPC tools which allows quality control practitioners to monitor for the presence of trends or unusual process behaviours to ensure that the process is in-control. A control chart consists of the center line (CL), upper control limit (UCL) and lower control limit (LCL). The process parameters, i.e. mean and variance are estimated from an in-control historical dataset to compute the limits of a control chart when the target values of these parameters are unknown. When a control chart declares that a process is out-of-control, investigations must be made to find and remove the assignable cause(s) so that the out-of-control process is brought into an in-control state again.

1.3 Types of Control Charts

Control charts are commonly used to monitor the behaviour of a manufacturing process. There are two basic types of control charts depending on the number of process characteristics to be monitored. The first, referred to as univariate control charts, are used to monitor a single quality characteristic in a manufacturing process while the second, called the multivariate control charts are used to jointly monitor several correlated quality characteristics in a manufacturing process. Section 1.3.1 discusses commonly used univariate control charts with emphasis given to the synthetic $\bar{X}$ chart and its extensions while Section 1.3.2 reviews the common multivariate charts, where special attention is given to the synthetic T^2 chart and its extensions.

1.3.1 Univariate Control Charts

Basically, there are two types of univariate control charts, namely attribute charts and variables charts (Besterfield, 2009). Attribute charts are used to evaluate a process when products are classified as conforming/nonconforming or defective/ nondefective. There are four types of attribute charts, i.e. the *c* chart to monitor the number of nonconformities, *np* chart to deal with the number of nonconforming/ defective units, *p* chart to control the percentage or fraction of nonconforming/ defective units and *u* chart to monitor the average number of nonconformities/defectives per unit of inspection.

Variables charts are used to evaluate variations in a process, where the measurements can be measured on a continuous scale (Montgomery, 2009). Variables charts are more sensitive than attribute charts in the detection of process shifts. Therefore, a variables control chart may alert us to quality problems first before any actual "unacceptables" is detected by an attribute chart (Montgomery, 2009). The most common variables charts used in the monitoring of shifts in the process mean are the Shewhart $\bar{X}$, R and S charts, moving average (MA), exponentially weighted moving average (EWMA) and cumulative sum (CUSUM) charts. The EWMA and CUSUM charts are sensitive to small shifts. On the contrary, the Shewhart $\bar{X}$ chart is slow in detecting small and moderate mean shifts but it is able to detect large shifts quickly.

One of the ways to increase the sensitivity of the $\bar{X}$ chart towards small and moderate shifts in the mean is to use runs rules. To date, numerous works on runs rules for the $\bar{X}$ chart have been made. Champ and Woodall (1987) used the Markov chain approach to derive the average run length (ARL) for the detection of an out-of-control signal on the $\bar{X}$ chart with supplementary runs rules. Hurwitz and Mathur

(1992) proposed a simple 2-of-2 runs rule with the control limits having a width of 1.5 standard deviations from the center line. Klein (2000) developed the 2-of-2 and 2-of-3 runs rules using the Markov chain approach so that the control limits can be adjusted to give a desired in-control ARL. Khoo (2004) presented the 2-of-4, 3-of-3 and 3-of-4 rules using the Markov chain approach. Zhang and Wu (2005) conducted comprehensive simulation studies on 15 combinations (charts) of runs rules by ensuring that the in-control ARL meets a specific value. Khoo and Khotrun (2006) improved the 2-of-2 and 2-of-3 runs rules suggested by Klein (2000) making these rules more sensitive in the detection of moderate and large shifts without sacrificing their sensitivities toward small shifts. Acosta-Mejia (2007) analyzed the statistical characteristics of both the m-of-m rule and (m−1)-of-m rule to supplement the $\bar{X}$ chart. Antzoulakos and Rakitzis (2008) proposed the revised m-of-k rule which improves the performance of the $\bar{X}$ chart in the detection of small to moderate shifts while maintaining the same superiority in detecting large shifts. Kim et al. (2009) investigated the economic-statistical design method for the 2-of-2 and 2-of-3 runs rules. Lim and Cho (2009) studied the economic-statistical design method for the m-of-m rule.

A different version of runs rule chart with a headstart feature, called the synthetic $\bar{X}$ chart was proposed by Wu and Spedding (2000a). The synthetic $\bar{X}$ chart integrates the standard $\bar{X}$ chart and the conforming run length (CRL) chart to enhance the speed of the $\bar{X}$ chart in detecting small and moderate shifts in the process mean. Wu and Spedding (2000a) showed that the synthetic $\bar{X}$ chart outperforms the standard $\bar{X}$ chart with or without runs rules, for any level of a mean shift. Besides that, the synthetic $\bar{X}$ chart is also superior to the EWMA chart when the size of a shift in the mean is greater than 0.8σ. Since then, numerous extensions on synthetic charts have been made. Wu and Spedding (2000b) presented a computer program to compute the upper and lower control limits of the synthetic $\bar{X}$ chart, based on a desired size of a shift in the mean that minimizes the out-of-control ARL. Wu et al. (2001) proposed a synthetic *p* chart for attributes data for detecting increases in the fraction nonconforming. Wu and Yeo (2001) provided a C program to determine the control chart's parameters and to calculate the average time to signal (ATS) of the synthetic *p* chart for attribute data. Calzada and Scariano (2001) studied the robustness of the synthetic chart to non-normality for monitoring the process mean. Davis and Woodall (2002) altered the synthetic chart of Wu and Spedding (2000a) to achieve better ARL performance by using the Markov chain model for the zero and steady state cases. Scariano and Calzada (2003) developed a synthetic chart for detecting decreases in the exponential mean which outperforms the Shewhart chart for individuals, EWMA and CUSUM charts. Huang and Chen (2005), and Chen and Huang (2005) proposed synthetic charts for process dispersion, based on the sample standard deviation and sample range, respectively. Costa and Rahim (2006a) suggested a synthetic chart using the non-central chi-square statistic for monitoring the mean and variance. Kotani et al. (2006) presented a synthetic EWMA chart for high yield processes. A variable sampling interval (VSI) synthetic chart for jointly monitoring the mean and standard deviation was developed by Chen and Huang (2006).

Khoo et al. (2008) and Castagliola and Khoo (2009) proposed the weighted variance (WV) and scaled weighted variance (SWV) synthetic mean charts for skewed distributions. Bourke (2008) re-evaluated the synthetic *p* chart by Wu et al. (2001) for detecting increases in the fraction nonconforming. Aparisi and de Luna (2009a) studied the zero and steady state optimization cases of the synthetic $\bar{X}$ chart.

A two stage testing synthetic chart for monitoring the mean and variance was suggested by Costa et al. (2009). A generalized synthetic chart was developed by Scariano and Calzada (2009).

1.3.2 Multivariate Control Charts

In many industrial applications, the quality of a process is often determined by two or more related quality characteristics (Mitra, 1998). For example, in the chemical mechanical planarization process, the quality of a polished wafer depends on several correlated variables, two of which are the remainder thickness of the polished wafer and the uniformity of thickness within the polished wafer (Yeh et. al., 2004). The monitoring or analysis of these data with univariate SPC procedure is often ineffective. The problem of process monitoring involving two or more related variables has led to the work on multivariate quality control which was introduced by Hotelling (1947) who applied it to bombsight data during World War II. The research on multivariate control charts is particularly important today as automatic inspection procedure makes it relatively easy to measure many parameters on each unit of a manufactured product. Several researchers have pointed out that research on multivariate control charts is important in the 21st century (Woodall and Montgomery, 1999; and Stoumbos et al., 2000).

Three most popular multivariate control charts are the Hotelling's T^2, multivariate EWMA (MEWMA) and multivariate CUSUM (MCUSUM) charts. The MEWMA and MCUSUM charts which accumulate information from past observations are sensitive in detecting small and moderate shifts in the mean vector of a multivariate process (Montgomery, 2009). In contrast, the Hotelling's T^2 chart, which is a multivariate Shewhart type control chart that only takes into account of the current information in the process is less sensitive to moderate and small shifts in the process mean vector although it responds quickly to large shifts.

Numerous approaches have been suggested to enhance the sensitivity of a T^2 chart in the detection of small and moderate mean shifts, one of which is via the use of runs rules. Among the works that deal with the use of runs rules on the T^2 chart are as follows: Khoo and Quah (2003) incorporated runs rules into the Hotelling's T^2 chart to improve the T^2 chart's performance. Aparisi et al. (2004) investigated the performance of the T^2 chart supplemented with runs rule and suggested the use of several rules by dividing the T^2 chart into attention zones and zones above and below the mean. Khoo et al. (2005) suggested the combined runs rules for the T^2 chart to increase the sensitivity of the rules suggested by Khoo and Quah (2003). Koutras et al. (2006) introduced a chi-square control chart (CSCC) supplemented with the *m*-of-*m* rule which improves the sensitivity of the standard CSCC in the detection of small and moderate shifts in the mean vector. Rakitzis and Antzoulakos (2011) studied the performance of the T^2 chart supplemented with the *r*-of-*m* rule.

The synthetic T^2 chart, which can be represented as a runs rule chart with a headstart feature was suggested by Ghute and Shirke (2008a) to enhance the performance of the standard T^2 chart towards small and moderate shifts in the mean vector of a multivariate process. The synthetic T^2 chart combines the Hotelling's T^2 and CRL charts and is an extension of the synthetic $\bar{X}$ chart of Wu and Spedding (2000a). Recently, numerous extensions on the synthetic T^2 chart have been made. Machado et al. (2009) presented a synthetic chart, based on the VMAX statistic, using the sample variances of two variables to monitor the covariance matrix of bivariate processes. Ghute and Shirke (2008b) proposed the synthetic $|\boldsymbol{S}|$ chart to monitor the process dispersion by combining the traditional generalized sample variance, $\boldsymbol{S}$ chart and the CRL chart. Aparisi and de Luna (2009b) developed an optimization procedure to compute the optimal parameters of the synthetic T^2 chart, for the zero and steady state cases, based on the desired in-control ARL, sample size and magnitude of shift. Khoo et al. (2009) suggested a multivariate synthetic chart for monitoring the process mean vector of skewed populations using the weighted standard deviation (WSD) method.

1.4 Measures of Performance of a Control Chart

1.4.1 Average Run Length (ARL)

The performance of a control chart can be evaluated by its run length (RL) value, which is defined as the number of sample points that must be plotted on a chart until the first out-of-control signal is detected, i.e. when the first sample point plots beyond the control limits of the control chart. Then ARL is defined as the average number of sample points that must be plotted before the first out-of-control is signaled by the chart.

1.4.2 Median Run Length (MRL)

The MRL is defined as the 50th percentage point of the probability distribution of the run length. Palm (1990) pointed out that the in-control run length distribution is highly skewed, and the skewness of the run length distribution decreases with the magnitude of a shift in the process. Thus, the main setback of the ARL is its difficulty of interpretation. As pointed out by Gan (1992), a CUSUM chart with an in-control ARL $\left(\mathrm{ARL}_0\right)$ of 500, will have 50% of all the run lengths less than 348, i.e. the in-control MRL $\left(\mathrm{MRL}_0\right)$ is 348, and about 63% of all the run lengths are less than 500. The difficulty of interpretation becomes even more complicated as the shape of the run length distribution changes with the magnitude of the shift, making interpretation based on ARL alone confusing and misleading. Barnard (1959), Bissell (1969), Woodall (1983), Waldmann (1986a and 1986b) and Gan (1992), to name a few, have all recommended the use of MRL in conjunction with the ARL, in measuring a chart's performance.

1.4.3 Percentage Points of the Run Length Distribution

The performance of a control chart is usually measured based on its ARL values. Palm (1990) stated that this single parameter does not contain enough information to make ARL particulary useful in practical applications because the run lengths are geometrically distributed. Practitioners are more interested in the percentage points (percentile) of the run length distribution which provides more information regarding the expected behaviour of the run lengths. The percentage points of the run length distribution is defined as the cumulative percent of signals given by the number of plotted statistics following the shift. For example, if the 30^{th} percentage point of the run length distribution of a chart, for a shift with a certain magnitude is, say 25, then this magnitude of shift will be detected by the 25^{th} sample point, by the chart, in 30 percent of the time.

Although the MRL (50^{th} percentage point of the run length distribution) is suggested by researchers to overcome the weakness of the ARL but it could not address another practical problem, i.e. the occurrence of an early false alarm when the process is in-control, which is usually represented by the low percentage points, for example, the 5^{th} and 10^{th} percentiles, of the run length distribution (Klein, 1997). As pointed out by Crowder (1987), practitioners might be interested with the probability of early false alarm for a given control scheme. Setting the in-control ARL at a desired level may not ensure that the probability of an early false signal is acceptable. Therefore, an analysis of the percentage points of the run length distribution must be considered.

1.5 Objectives of the Study

The objectives of this thesis are as follows:

i) To develop the optimal designs of the univariate and multivariate synthetic charts, based on MRL, that minimizes the out-of-control MRL for a specified shift of interest once the in-control MRL is fixed at a desired level. The optimal designs of the univariate and multivariate synthetic charts currently available in the literature are only based on the ARL. Thus, this thesis extends the optimal designs of the charts by considering the MRL as a criterion for evaluating the charts' performances. Step-by-step approaches to obtain the optimal parameters of the univariate and multivariate synthetic charts, based on MRL are provided.

ii) To provide computer programs to help practitioners to compute the optimal parameters for the univariate and multivariate synthetic charts, designed based on MRL, for the zero and steady state cases. These programs incorporate the Markov chain approach, used in the optimal designs of the charts.

iii) To compare the MRL performances of (a) univariate synthetic chart with the EWMA and Shewhart $\bar{X}$ charts, (b) multivariate synthetic chart with the MEWMA and Hotelling's T^2 charts.

1.6 Organization of the Thesis

Chapter 1 provides a brief history of control charts and an overview of SPC. It explains the various types of univariate and multivariate charts, where emphasis is given to the synthetic type charts. Several measures of the performance of a control chart are also described. This chapter is concluded by highlighting the objectives and organization of the thesis.

In Chapter 2, the normal distribution and some related univariate control charts are discussed. A brief review on the $\bar{X}$, conforming run length (CRL), synthetic $\bar{X}$ and EWMA charts is also given.

Chapter 3 gives a discussion on the multivariate normal distribution, as well as, the Hotelling's T^2, synthetic T^2 and MEWMA charts.

A review of the basic theory of the Markov chain approach and the Markov chain representation in the computation of the MRL of the EWMA and MEWMA charts is given in Chapter 4, while the Markov Chain theory for computing the ARL of the synthetic $\bar{X}$ and synthetic T^2 charts is reviewed in Chapter 5.

Chapters 6 and 7 explain the procedures to obtain the optimal parameters of the univariate synthetic $\bar{X}$ and multivariate synthetic T^2 charts, respectively, when the charts are designed based on MRL. The descriptions of the optimization programs which are used to compute the optimal parameters, for the zero and steady state cases are also provided. Two examples, each for the synthetic $\bar{X}$ and synthetic T^2 charts are given in Chapters 6 and 7, respectively, to illustrate the use of the optimal design procedure, based on MRL, in a real life application. Finally, the conclusions of this thesis and some suggestions for further research are discussed in Chapter 8.

CHAPTER 2
SOME PRELIMINARIES AND REVIEW ON UNIVARIATE CONTROL CHARTS

2.1 Normal Distribution

The normal distribution, also called the Gaussian distribution, is the most important and widely used distribution to describe the behavior of a continuous quality characteristic in univariate statistical quality control. A normally distributed random variable X with mean, μ and variance, σ^2, denoted as $X \sim N(\mu, \sigma^2)$ has probability density function (pdf) given by (Montgomery, 2009)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \tag{2.1}$$

where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. The pdf of a normal distribution is a symmetric, unimodal and bell shaped curve as shown in Figure 2.1.

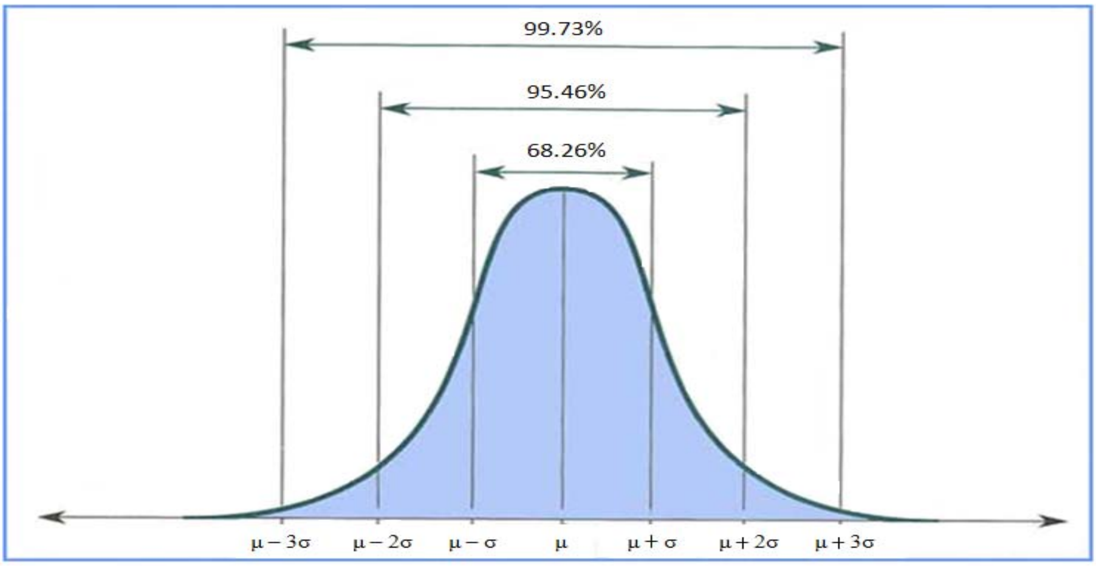

Figure 2.1. The density function of a normal distribution

(Source: Montgomery, 2009)

The normal distribution has several important properties (Montgomery, 2009):

i) The mean, mode and median are equal.

ii) The total area under the pdf of the normal distribution is one.

iii) Approximately 68.26% of the population values fall between the $\mu \pm 1\sigma$ limits.

iv) Approximately 95.46% of the population values fall between the $\mu \pm 2\sigma$ limits.

v) Approximately 99.73% of the population values fall between the $\mu \pm 3\sigma$ limits.

2.2 Shewhart $\bar{X}$ Control Chart

The Shewhart $\bar{X}$ chart is the most widely used chart to monitor the process mean in industries. It consists of three important lines, i.e. the upper control limit (UCL), center line (CL) and lower control limit (LCL). Assume that a process follows a normal distribution with mean μ, and standard deviation σ, where both μ and σ are known. Also, let $X_1, X_2, ..., X_n$ be a sample of size *n* taken from the process with a $N\left(\mu, \sigma^2\right)$ distribution. Then the sample mean is

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \,, \tag{2.2}$$

where $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ (Montgomery, 2009). The limits of the Shewhart $\bar{X}$ chart are computed as

$$\text{UCL} = \mu_0 + k\frac{\sigma}{\sqrt{n}} \tag{2.3a}$$

and

$$\text{LCL} = \mu_0 - k\frac{\sigma}{\sqrt{n}} \,, \tag{2.3b}$$

where *k* is a constant controlling the width of the control limits of the $\bar{X}$ chart, while μ_0 and σ are the in-control mean and standard deviation, respectively.

In practice, the values of μ_0 and σ are usually unknown and are estimated from an in-control historical data set consisting of *m* samples, each of size, *n*. Let

$\bar{X}_1$, $\bar{X}_2$, ..., $\bar{X}_m$ denote the means of the *m* in-control samples in a Phase–I process, then μ is estimated as follows (Montgomery, 2009):

$$\bar{\bar{X}} = \frac{\bar{X}_1 + \bar{X}_2 + \ldots + \bar{X}_m}{m}, \tag{2.4}$$

where $\bar{\bar{X}}$ is the average of the sample means and is the center line of the $\bar{X}$ chart. The standard deviation σ can be estimated from either the sample ranges or sample standard deviations of the *m* in-control Phase-I samples. Let R_i, i = 1, 2, …, m represent the ranges of the *m* samples. Then, the average sample range is

$$\bar{R} = \frac{1}{m}\sum_{i=1}^{m} R_i \tag{2.5}$$

Thus, the limits of the $\bar{X}$ chart when parameters are estimated are computed as follows (Montgomery, 2009):

$$\text{UCL} = \bar{\bar{X}} + A_2\bar{R} \tag{2.6a}$$

and

$$\text{LCL} = \bar{\bar{X}} - A_2\bar{R}. \tag{2.6b}$$

The value of the constant, A_2 which depends on the sample size, *n*, can be obtained from most statistical quality control textbooks.

If the sample standard deviation, *S* is used to estimate σ, where

$$S = \sqrt{\frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}}, \tag{2.7}$$

then the average sample standard deviation estimated from *m* preliminary samples in Phase–I is

$$\bar{S} = \frac{1}{m}\sum_{j=1}^{m} S_j. \tag{2.8}$$

It follows that the limits of the $\bar{X}$ chart when parameters are estimated are (Montgomery, 2009)

$$\text{UCL} = \bar{\bar{X}} + A_3\bar{S} \tag{2.9a}$$

and

$$\text{LCL} = \bar{\bar{X}} - A_3\bar{S}\,, \tag{2.9b}$$

where A_3 is the control limit constant whose value can be obtained from most quality control reference books.

2.3 Conforming Run Length (CRL) Chart

The count of conforming control chart (CCC chart) which is a type of the conforming run length (CRL) chart was first studied by Calvin (1983) to monitor processes with zero-defects. This chart was further studied by Goh (1987). The CRL chart was later introduced by Bourke (1991) for attribute quality control to detect shifts in the fraction nonconforming, *p*. In 100% inspection, the CRL value is defined as the number of inspected samples between two consecutive nonconforming samples including the ending nonconforming sample. Figure 2.2 illustrates an example with three CRL samples, where the white and black dots represent the conforming and nonconforming samples, respectively.

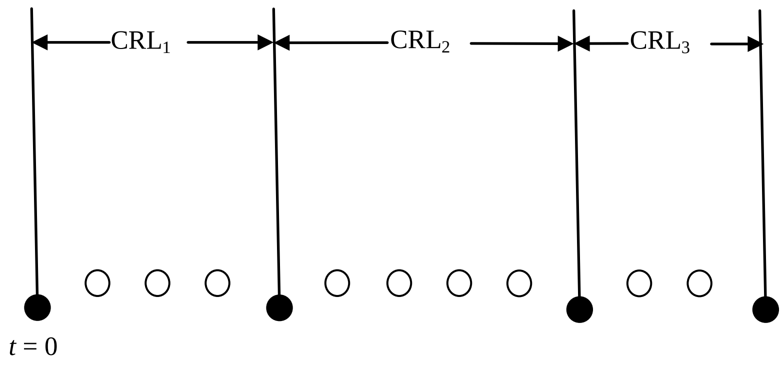

Figure 2.2. Conforming run length

Suppose that a process starts at $t = 0$, then the CRL values in Figure 2.2 are $\text{CRL}_1 = 4$, $\text{CRL}_2 = 5$ and $\text{CRL}_3 = 3$. The basic idea of the CRL chart is that the value of CRL will change when the fraction nonconforming, p, changes, i.e. the CRL value decreases as p increases and vice versa.

The random variable CRL follows a geometric distribution. Consequently, the expected value, μ_{CRL} and the cumulative distribution function of CRL, i.e. $\Pr(\text{CRL} \le x) = F_p(x)$ are defined as follows (Bourke, 1991):

$$\mu_{\text{CRL}} = \frac{1}{p} \tag{2.10}$$

and

$$F_p(x) = 1 - (1-p)^x, \; x = 1, 2, \ldots. \tag{2.11}$$

CRL is shortened as p increases and lengthened as p decreases. The average number of CRL samples required to detect an out-of-control fraction nonconforming, p is denoted as (Bourke, 1991)

$$\text{ARL}_{\text{CRL}} = \frac{1}{F_p(L)} = \frac{1}{1-(1-p)^L}, \tag{2.12}$$

where L is the lower limit of the CRL chart.

2.4 Synthetic $\bar{X}$ Control Chart

The synthetic $\bar{X}$ chart for the process mean which integrates the Shewhart $\bar{X}$ and conforming run length (CRL) charts was proposed by Wu and Spedding (2000a). It comprises a $\bar{X}/S$ sub-chart and a CRL/S sub-chart and it improves the performance of the Shewhart $\bar{X}$ chart for the detection of small and moderate shifts in the process mean. Besides that, it also surpasses the exponentially weighted moving average (EWMA) and the joint $\bar{X}-\text{EWMA}$ charts, in detecting a mean shift of greater than 0.8σ. The CRL value is denoted as the total number of inspected $\bar{X}$ samples between the current and the last nonconforming $\bar{X}$ samples, including the current nonconforming one.

The synthetic $\bar{X}$ chart is constructed based on the following six steps procedure (Wu and Spedding, 2000a):

Step 1: Determine the lower control limit, L, of the CRL/S sub-chart and calculate the upper and lower control limits, $\text{UCL}_{\bar{X}/S}$ and $\text{LCL}_{\bar{X}/S}$, respectively, of the $\bar{X}/S$ sub-chart using the following formulae:

$$\text{UCL}_{\bar{X}/S} = \mu_0 + k\sigma_{\bar{X}} \tag{2.13a}$$

and

$$\text{LCL}_{\bar{X}/S} = \mu_0 - k\sigma_{\bar{X}} \tag{2.13b}$$

where μ_0 is the in-control process mean, k is the constant controlling the width of the control limits and $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$ is the in-control standard deviation of the sample mean, $\bar{X}$ with a sample size of n.

Step 2: Take a random sample of size, n, at each inspection point and calculate the sample mean, $\bar{X}$.

Step 3: If $\text{LCL}_{\bar{X}/S} < \bar{X} < \text{UCL}_{\bar{X}/S}$, the sample is classified as conforming and the control flow returns to Step 2. Otherwise, the sample is considered as nonconforming and the control flow proceeds to Step 4.

Step 4: Count the number of samples between the current (included in the count) and last (excluded in the count) nonconforming samples as the CRL value of the CRL/S sub-chart.

Step 5: If the CRL value is greater than L, i.e. $\mathrm{CRL} > L$, the process is considered as in-control and the control flow returns to Step 2. Otherwise, the process is considered as out-of-control and the control flow advances to Step 6.

Step 6: Signal an out-of-control. Take corrective actions to return the out-of-control process into an in-control state again. Then return to Step 2.

Note that the synthetic $\bar{X}$ chart does not give an out-of-control signal when a sample point, $\bar{X}$ plots beyond the limits of the $\bar{X}/S$ sub-chart, given in Equations (2.13a) and (2.13b), instead the synthetic $\bar{X}$ chart just indicates a nonconforming sample. An out-of-control is signaled by the synthetic $\bar{X}$ chart only when $\mathrm{CRL} \le L$.

2.5 Exponentially Weighted Moving Average (EWMA) Control Chart

The EWMA chart was introduced by Roberts (1959) as a superior alternative to the Shewhart $\bar{X}$ chart in detecting small process shifts. The chart's statistics is given as (Montgomery, 2009)

$$Z_i = \lambda \bar{X}_i + (1-\lambda) Z_{i-1}, \text{ for } i = 1, 2, \ldots, \tag{2.14}$$

where λ $(0 < \lambda \le 1)$ is the smoothing constant and $\bar{X}_i$ is the mean of sample *i*. The EWMA statistics can be expressed as a weighted linear combination of current and past sample means. The smaller the value of λ, the greater the influence of the past sample means (Montgomery, 2009). When choosing the value of λ used for weighting, it is recommended to use small values to detect small shifts, and large values for large shifts. The EWMA chart with $\lambda = 1$ is actually the Shewhart $\bar{X}$ chart. Crowder (1989) provided plots to help the user in selecting the optimal value of λ for a quick detection of a desired magnitude of a mean shift. Continuing to substitute recursively for Z_{i-j} , $j = 2,3,...,t$, in Equation (2.14), it can be shown that (Montgomery, 2009)

$$Z_i = \lambda \sum_{j=0}^{i-1} (1-\lambda)^j \bar{X}_{i-j} + (1-\lambda)^i Z_0 . \qquad (2.15)$$

The weights $\lambda(1-\lambda)^j$ in Equation (2.15), decrease geometrically with the age of the sample mean and these weights sum to unity because (Montgomery, 2009)

$$\lambda \sum_{j=0}^{i-1} (1-\lambda)^j = \lambda \left[\frac{1-(1-\lambda)^i}{1-(1-\lambda)} \right] = 1-(1-\lambda)^i . \qquad (2.16)$$

The exact control limits for the EWMA chart are

$$\text{UCL}_{\text{EWMA}} = \mu_0 + w'\sigma \sqrt{\frac{\lambda}{n(2-\lambda)} \left[1-(1-\lambda)^{2i} \right]} \qquad (2.17a)$$

and

$$\text{LCL}_{\text{EWMA}} = \mu_0 - w'\sigma \sqrt{\frac{\lambda}{n(2-\lambda)} \left[1-(1-\lambda)^{2i} \right]} , \qquad (2.17b)$$

where μ_0 and σ are the in-control mean and in-control standard deviation respectively, while w' is the constant controlling the width of the limits. As i becomes larger, the term $\left[1-(1-\lambda)^{2i}\right]$ approaches unity. Therefore, based on Equations (2.17a) and (2.17b), the asymptotic limits of the EWMA chart are

$$\text{UCL}_{\text{EWMA}} = \mu_0 + w'\sigma \sqrt{\frac{\lambda}{n(2-\lambda)}} \qquad (2.18a)$$

and

$$\text{LCL}_{\text{EWMA}} = \mu_0 - w'\sigma \sqrt{\frac{\lambda}{n(2-\lambda)}} . \qquad (2.18b)$$

In the discussion hereafter, we let $w = w' \sqrt{\frac{\lambda}{n(2-\lambda)}}$.

CHAPTER 3
SOME PRELIMINARIES AND REVIEW ON MULTIVARIATE CONTROL CHARTS

3.1 Multivariate Normal Distribution

The multivariate normal distribution is an extension of the one dimensional (univariate) normal distribution to higher dimensions. Assume that there are *p* variables, $X_1, X_2, ..., X_p$ in a *p*-component vector, $\boldsymbol{X} = \left(X_1, X_2, \ldots, X_p\right)'$. The values assumed by variable X_j, $j = 1, 2, \ldots, p$, is $-\infty < x_j < \infty$. Let the mean vector of $\boldsymbol{X}$ be $\boldsymbol{\mu} = \left(\mu_1, \mu_2, \ldots, \mu_p\right)'$ and $\boldsymbol{\Sigma}$ be the covariance matrix of $\boldsymbol{X}$, where the main diagonal elements of $\boldsymbol{\Sigma}$ are the variances of X_j, for $j = 1, 2, \ldots, p$, and the off-diagonal elements are the covariances (Montgomery, 2009). The squared standardized (generalized) distance from $\boldsymbol{X}$ to $\boldsymbol{\mu}$ is

$$\left(\boldsymbol{X} - \boldsymbol{\mu}\right)' \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{X} - \boldsymbol{\mu}\right). \tag{3.1}$$

The multivariate normal probability density function (pdf) is obtained by replacing the standardized distance, $\left(X - \mu\right)\left(\sigma^2\right)^{-1}\left(X - \mu\right)$ in the univariate normal distribution with the multivariate generalized distance in Equation (3.1) and changing the constant term, $\frac{1}{\sqrt{2\pi\sigma^2}}$ to a more general form that makes the area under the pdf unity, regardless of the value of *p*. Thus, the multivariate normal pdf is given as (Montgomery, 2009)

$$f\left(\boldsymbol{x}\right) = \frac{1}{\left(2\pi\right)^{p/2} \left|\boldsymbol{\Sigma}\right|^{1/2}} e^{-\frac{1}{2}\left(\boldsymbol{x} - \boldsymbol{\mu}\right)' \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{x} - \boldsymbol{\mu}\right)}, \tag{3.2}$$

where $\boldsymbol{X} = \left(X_1, X_2, \ldots, X_p\right)'$, $-\infty < x_j < \infty$ and $j = 1, 2, ..., p$.

We will now give a brief description on the sample mean vector and sample variance-covariance matrix of a random sample from a multivariate normal distribution. Suppose that we have a random sample of size *n*, i.e., $\boldsymbol{X}_1, \boldsymbol{X}_2, ..., \boldsymbol{X}_n$ from a multivariate normal distribution, where the i^{th} vector, $\boldsymbol{X}_i$, for $1 \le i \le n$ contains observations on each of the *p* variables, $X_{i1}, X_{i2}, ..., X_{ip}$. Then the sample mean vector is (Montgomery, 2009)

$$\bar{\boldsymbol{X}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i \tag{3.3}$$

and the sample covariance matrix is

$$\boldsymbol{S} = \frac{1}{n-1}\sum_{i=1}^{n}\left(\boldsymbol{X}_i - \bar{\boldsymbol{X}}\right)\left(\boldsymbol{X}_i - \bar{\boldsymbol{X}}\right)'. \tag{3.4}$$

The sample variances on the main diagonal of matrix $\boldsymbol{S}$ are computed as

$$S_j^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_{ij} - \bar{X}_j\right)^2, \text{ for } j = 1, 2, \ldots, p, \tag{3.5}$$

and the sample covariances are

$$S_{jk} = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_{ij} - \bar{X}_j\right)\left(X_{ik} - \bar{X}_k\right), \tag{3.6}$$

for $j = 1, 2, \ldots, p$, $k = 1, 2, \ldots, p$ and $j \ne k$. Here, the sample mean of variable *j* is computed as follows:

$$\bar{X}_j = \frac{1}{n}\sum_{i=1}^{n} X_{ij}. \tag{3.7}$$

It can be shown that the sample mean vector, $\bar{\boldsymbol{X}}$ and the sample covariance matrix, $\boldsymbol{S}$ are unbiased estimators of the corresponding population quantities, i.e. $E\left(\bar{\boldsymbol{X}}\right) = \boldsymbol{\mu}$ and $E\left(\boldsymbol{S}\right) = \boldsymbol{\Sigma}$ (Montgomery, 2009). The sample covariance matrix in correlation form is made up of elements r_{jk}, for $j, k = 1, 2, \ldots, p$ representing the pairwise correlation coefficient between quality characteristics X_j and X_k in vector

$\boldsymbol{X}$, i.e. the element in the j^{th} row and k^{th} column of the sample covariance matrix in correlation form is given by (Montgomery, 2009)

$$r_{jk} = \frac{S_{jk}}{S_j . S_k} . \quad (3.8)$$

For the case where only two variables, X_1 and X_2 are involved, i.e. the bivariate case, the bivariate normal pdf is defined as (Montgomery, 2009)

$$f(x_1, x_2) = \frac{1}{2\pi \, \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \left\{ \left(\frac{x_1 - \mu_1}{\sigma_1} \right)^2 - 2\rho \left(\frac{x_1 - \mu_1}{\sigma_1} \right) \left(\frac{x_2 - \mu_2}{\sigma_2} \right) + \left(\frac{x_2 - \mu_2}{\sigma_2} \right)^2 \right\} \right], \quad (3.9)$$

where ρ is the correlation coefficient between X_1 and X_2.

3.2 Multivariate Hotelling's T^2 Control Chart

The Hotelling's T^2 chart was introduced by Hotelling (1947) to monitor two or more quality characteristics at the same time in a manufacturing process. It is a multivariate extension of the Shewhart $\bar{X}$ chart which takes the correlation among two or more variables into account. Assume that $\boldsymbol{X}_1, \boldsymbol{X}_2, ..., \boldsymbol{X}_n$ represent a sequence of multivariate observations from a random sample having a *p*-variate normal distribution with in-control mean vector, $\boldsymbol{\mu}_0$ and covariance matrix, $\boldsymbol{\Sigma}_0$. The T^2 statistics plotted on the chart is defined as follows (Montgomery, 2009):

$$T_i^2 = n\left(\bar{\boldsymbol{X}}_i - \boldsymbol{\mu}_0\right)' \boldsymbol{\Sigma}_0^{-1} \left(\bar{\boldsymbol{X}}_i - \boldsymbol{\mu}_0\right) = \boldsymbol{Y}_i' \, \boldsymbol{\rho}^{-1} \, \boldsymbol{Y}_i, \text{ for } i = 1, 2, \ldots, \quad (3.10)$$

where $\boldsymbol{\rho}$ denotes the correlation matrix for $\boldsymbol{X}$ and $\overline{\boldsymbol{X}}_i = \left(\overline{X}_{i1}, \overline{X}_{i2}, ..., \overline{X}_{ip}\right)'$ is the i^{th} sample mean vector. Here, $\overline{X}_{ij} = \frac{\sum_{k=1}^{n} X_{ijk}}{n}$ represents the sample mean for quality characteristic *j* in sample *i*, while X_{ijk} denotes the k^{th} observation on the j^{th} quality characteristic in sample *i*. Note that $\boldsymbol{Y}_i = \left(Y_{i1}, Y_{i2}, ..., Y_{ip}\right)'$ is the i^{th} standardized mean vector, where the j^{th} element of $\boldsymbol{Y}_i$ is $Y_{ij} = \frac{\overline{X}_{ij} - \mu_j}{\sigma_j / \sqrt{n}}$. An out-of-control is signaled by the chart when the T^2 statistic exceeds the control limit, $h = \chi^2_{\alpha}(p)$, where α is a specified size of the Type-I error and $\chi^2_{\alpha}(p)$ is the $100(1-\alpha)^{\text{th}}$ percentile of the χ^2 distribution with *p* degrees of freedom (Montgomery, 2009).

3.3 Multivariate Synthetic T^2 Control Chart

The synthetic T^2 chart which integrates the Hotelling's T^2 and CRL charts was introduced by Ghute and Shirke (2008a) as a superior alternative to the T^2 chart, for the detection of small and moderate shifts in the multivariate process mean. The CRL value is denoted as the total number of inspected T^2 samples between the current and the last nonconforming T^2 samples, including the current nonconforming sample.

The synthetic T^2 chart is implemented as follows (Ghute and Shirke, 2008a):

1) Determine the upper control limit, UCL_{T^2} of the T^2 sub-chart and the lower control limit, *L* of the CRL sub-chart. Ghute and Shirke (2008a), and Aparisi and de Luna (2009b) provided formulae for computing optimal limits of UCL_{T^2} and *L*, by fixing the in-control ARL (ARL_0) and minimizing the out-of-control ARL,