Second Semester Examination
2017/2018 Academic Session

May / June 2018

MGM563 - Statistical Inference
(Pentaabiran Statistik)

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **NINE (9)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEMBILAN (9)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Instructions : Answer **all five (5)** questions.

*[**Arahan :** Jawab **semua lima (5)** soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

Question 1

(a) In a particular game, three six-sided fair dice are rolled. The score of the game is given based on the following criteria:

- 1 score is given if a '5' shows up
- 1 score is deducted if no '5' shows up
- Assume that X is the score of the game and $P(X = x)$ is the probability.

(i) Fill up the following table:

X	-1	1	2	3
$P(X = x)$				

(ii) Find the probability mass function and cumulative distribution function for $P(X = x)$, x=-1, 1, 2, 3.

(iii) Calculate the mean and variance of X.

(b) Show that 63/512 is the probability that the fifth head is observed on the tenth independent flip of a fair coin.

(c) Let X have a Poisson distribution such that $3P(X = 1) = P(X = 2)$. Find $P(X \leq 1)$.

[20 marks]

Soalan 1

(a) *Dalam permainan tertentu, tiga dadu adil yang mempunyai enam sisi dilambung. Skor permainan diberi berdasarkan kriteria berikut:*

- *Skor 1 diberi jika '5' muncul*
- *Skor 1 ditolak jika tiada '5' muncul*
- *Andaikan X merupakan skor permainan dan* $P(X=x)$ *ialah kebarangkalian.*

X	-1	1	2	3
$P(X=x)$				

(i) *Isi jadual berikut:*

(ii) *Cari fungsi jisim kebarangkalian dan fungsi taburan longgokan untuk* $P(X=x)$, *x=-1, 1, 2, 3.*

(iii) *Kira min dan varians untuk X.*

(b) *Tunjukkan bahawa 63/512 adalah kebarangkalian kepala ke-5 dicerap dalam jentikan tak bersandar ke-sepuluh suatu syiling adil.*

(c) *Biarkan X mempunyai taburan Poisson sedemikian sehingga* $3P(X=1)=P(X=2)$. *Cari* $P(X\leq 1)$.

[20 markah]

Question 2

(a) Let the independent random variables X and Y have binomial distributions with parameters $n,\ p_1 = \frac{1}{2}$ and $m,\ p_2 = \frac{1}{2}$, respectively. Show that $W = X - Y + m$ has a binomial distribution with parameters $N = n + m$, p=1/2

(b) Let X and Y have the joint pdf $f(x,y) = \begin{cases} 2\exp\{-(x+y)\} & 0 < x < y < \infty \\ 0 & elsewhere \end{cases}$.

Find the following:

(i) $f(x)$ and $g(y)$

(ii) $f(y|x)$

(iii) $E(Y|x)$

(iv) Are X and Y independent?

[20 marks]

Soalan 2

(a) Biarkan pembolehubah rawak tak bersandar X and Y mempunyai taburan binomial dengan parameter n, $p_1 = \frac{1}{2}$ dan m, $p_2 = \frac{1}{2}$ masing-masing. Tunjukkan bahawa $W = X - Y + m$ mempunyai taburan binomial dengan parameter $N = n + m$, p=1/2.

(b) Biarkan X dan Y mempunyai fungsi ketumpatan kebarangkalian tercantum $f(x,y) = \begin{cases} 2\exp\{-(x+y)\} & 0 < x < y < \infty \\ 0 & selainnya \end{cases}$

Cari yang berikut:

(i) $f(x)$ *dan* $g(y)$

(ii) $f(y|x)$

(iii) $E(Y|x)$

(iv) Adakah X dan Y tak bersandar?

[20 markah]

Question 3

(a) If *X*, *Y* and *Z* are independent random variables having identical density functions, $f\left(x\right)=e^{-x}$, $0<x<\infty$. Derive the joint distribution of $U=X+Y$, $V=X+Z$ and $W=Y+Z$.

(b) A company is interested in studying its client's behavior. For this purpose, the mean time between consecutive demands of service is modeled by a random variable whose density function defined as:

$$f\left(x;\theta\right)=\frac{1}{\theta}e^{-\frac{x-2}{\theta}}, \quad x\geq 2 \text{ and } \theta>0$$

The estimator provided by the method of moments is $\hat{\theta}=\overline{X}-2$. Given also $E\left(X\right)=\theta+2$ and $\text{Var}\left(X\right)=\theta^2$.

(i) Is $\hat{\theta}$ an unbiased estimator of θ?

(ii) Calculate the mean square error of $\hat{\theta}$. Is $\hat{\theta}$ a consistent estimator of θ?

[20 marks]

Soalan 3

(a) Jika X, Y dan Z adalah pembolehubah rawak tak bersandar yang mempunyai fungsi ketumpatan secaman, $f\left(x\right)=e^{-x}$, $0<x<\infty$. *Terbitkan taburan tercantum untuk* $U=X+Y$, $V=X+Z$ *dan* $W=Y+Z$.

(b) Sebuah syarikat berminat dalam mengkaji tingkah-laku pelanggannya. Untuk tujuan ini, min masa antara permintaan terhadap perkhidmatan berturutan dimodelkan oleh pembolehubah rawak dengan fungsi ketumpatan yang ditakrifkan sebagai:

$$f\left(x;\theta\right)=\frac{1}{\theta}e^{-\frac{x-2}{\theta}}, \quad x\geq 2 \textit{ and } \theta>0$$

Penganggar yang diberikan oleh kaedah momen adalah $\hat{\theta}=\overline{X}-2$. *Diberi juga bahawa* $E\left(X\right)=\theta+2$ *dan* $Var\left(X\right)=\theta^2$.

(i) Adakah $\hat{\theta}$ *merupakan suatu penganggar saksama untuk* θ*?*

(ii) Kirakan min ralat kuasadua untuk $\hat{\theta}$. *Adakah* $\hat{\theta}$ *merupakan penganggar konsisten bagi* θ*?*

[20 markah]

Question 4

Let $X_1,...,X_n$ be a simple random sample that follows an exponential distribution given by $f(x) = \lambda e^{-\lambda x}$, $x \in (0,\infty) = \mathbb{R}^+$ and $\lambda \in \mathbb{R}^+$.

(a) Apply the method of moments to find an estimator of the parameter λ .

(b) Apply the maximum likelihood method to find an estimator of the parameter λ .

(c) Assume that $\sum_{j=1}^{55} x_j = 598d$. Give a final estimate of λ you obtained from (a) and (b).

(d) Based on the result obtained in (c), calculate $E(X)$.

(e) Assume that $\theta = \lambda^{-1}$ and the exponential density function be defined as $f(x;\theta) = \frac{1}{\theta} e^{-\frac{1}{\theta}x}$, $E(X) = \theta$ and $\text{Var}(X) = \theta^2$. Show that the variance of the maximum likelihood estimator of $\theta = \lambda^{-1}$ attains the Cramer-Rao's Lower Bound.

[20 marks]

Soalan 4

Biarkan $X_1,...,X_n$ sebagai suatu sampel rawak yang mengikuti taburan eksponen yang diberi oleh $f(x) = \lambda e^{-\lambda x}$, $x \in (0,\infty) = \mathbb{R}^+$ dan $\lambda \in \mathbb{R}^+$.

(a) Gunakan kaedah momen untuk mencari satu penganggar bagi parameter λ .

(b) Gunakan kaedah kebolehjadian maksimum untuk mencari penganggar bagi parameter λ .

(c) Andaikan bahawa $\sum_{j=1}^{55} x_j = 598d$. Berikan anggaran akhir untuk λ yang anda dapati dari (a) dan (b).

(d) Berdasarkan keputusan yang didapati dalam (c), cari $E(X)$.

(e) Andaikan bahawa $\theta = \lambda^{-1}$ dan fungsi ketumpatan eksponen didefinisikan sebagai $f(x;\theta) = \frac{1}{\theta} e^{-\frac{1}{\theta}x}$, $E(X) = \theta$ dan $Var(X) = \theta^2$. Tunjukkan bahawa varians penganggar kebolehjadian maksimum bagi $\theta = \lambda^{-1}$ mencapai batas bawah Cramer-Rao.

[20 markah]

Question 5

(a) Let X be a single observation from a density function $f(x;\theta)=\theta x^{\theta-1}, \quad 0<x<1$ and $\theta>0$. If $-\theta\log X$ is a pivotal quantity, use it to build a confidence interval for θ with the level of confidence, $\gamma=1-\alpha$.

(b) Let $X_1,...,X_n$ be a simple random sample from a normal distribution with the $N(0,\theta)$ distribution, where θ is unknown. The simple hypothesis is performed by setting $H_0:\theta=\theta_1$ versus $H_1:\theta=\theta_2$. Based on the likelihood ratio test, determine the best critical region.

(c) A 64-element simple random sample of petrol consumption (litres per 100 kilometers) in private cars has been taken, yielding a mean consumption of 9.36 and a standard deviation of 1.4.

(i) Provide a 96% confidence interval for the mean consumption and the maximum error E_i *given* $z_{0.02}=2.054$.

(ii) Assume that the variance is $\sigma^2=2$. What is the sample size if, with the same confidence, we now set the new maximum error E_{ii} to be $\frac{1}{4}$ of that in part (i) denoted by E_i , i.e. $E_{ii}=\frac{1}{4}E_i$?

[20 marks]

Soalan 5

(a) *Biarkan X sebagai suatu cerapan tunggal dari fungsi ketumpatan* $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$ *dan* $\theta > 0$. *Jika* $-\theta \log X$ *adalah suatu kuantiti pangsian, gunakannya untuk membina satu selang keyakinan bagi* θ *dengan selang keyakinan* $\gamma = 1 - \alpha$.

(b) *Biarkan* $X_1, ..., X_n$ *sebagai suatu sampel rawak ringkas dari taburan normal* $N(0, \theta)$, *di mana* θ *tidak diketahui. Hipotesis ringkas dijalankan dengan menetapkan* $H_0 : \theta = \theta_1$ *lawan* $H_1 : \theta = \theta_2$. *Berdasarkan ujian nisbah kebolehjadian, tentukan rantau genting terbaik.*

(c) *Suatu sampel rawak ringkas 64-elemen bagi penggunaan petrol (liter per 100 kilometer) dalam kereta persendirian telah diambil, menghasilkan min penggunaan 9.36 dan sisihan piawai 1.4.*

(i) *Berikan selang keyakinan 96% bagi min penggunaan dan ralat maksimum* E_i *diberi* $z_{0.02} = 2.054$.

(ii) *Andaikan bahawa varians ialah* $\sigma^2 = 2$. *Apakah saiz sampel jika, dengan keyakinan yang sama, kita menetapkan ralat maksimum baru,* E_{ii} *sebagai* $\frac{1}{4}$ *daripada yang didapati di bahagian (i) diwakili oleh* E_i *iaitu* $E_{ii} = \frac{1}{4} E_i$?

[20 markah]

Continuous Distributions

Beta $0 < \alpha$ $0 < \beta$	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1$ $\mu = \frac{\alpha}{\alpha+\beta}, \quad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$
Chi-square $\chi^2(r)$ $r = 1, 2, \ldots$	$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \quad 0 \le x < \infty$ $M(t) = \frac{1}{(1-2t)^{r/2}}, \quad t < \frac{1}{2}$ $\mu = r, \quad \sigma^2 = 2r$
Exponential $0 < \theta$	$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 \le x < \infty$ $M(t) = \frac{1}{1-\theta t}, \quad t < \frac{1}{\theta}$ $\mu = \theta, \quad \sigma^2 = \theta^2$
Gamma $0 < \alpha$ $0 < \theta$	$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \quad 0 \le x < \infty$ $M(t) = \frac{1}{(1-\theta t)^\alpha}, \quad t < \frac{1}{\theta}$ $\mu = \alpha\theta, \quad \sigma^2 = \alpha\theta^2$
Normal $N(\mu, \sigma^2)$ $-\infty < \mu < \infty$ $0 < \sigma$	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$ $M(t) = e^{\mu t + \sigma^2 t^2/2}$ $E(X) = \mu, \quad \mathrm{Var}(X) = \sigma^2$
Uniform $U(a, b)$ $-\infty < a < b < \infty$	$f(x) = \frac{1}{b-a}, \quad a \le x \le b$ $M(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}, \quad t \ne 0; \quad M(0) = 1$ $\mu = \frac{a+b}{2}, \quad \sigma^2 = \frac{(b-a)^2}{12}$

Discrete Distributions

Bernoulli $0 < p < 1$	$f(x) = p^x(1-p)^{1-x}, \quad x = 0, 1$ $M(t) = 1 - p + pe^t$ $\mu = p, \quad \sigma^2 = p(1-p)$
Binomial $b(n, p)$ $0 < p < 1$	$f(x) = \frac{n!}{x!(n-x)!} p^x(1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n$ $M(t) = (1 - p + pe^t)^n$ $\mu = np, \quad \sigma^2 = np(1-p)$
Geometric $0 < p < 1$	$f(x) = (1-p)^{x-1}p, \quad x = 1, 2, 3, \ldots$ $M(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\ln(1-p)$ $\mu = \frac{1}{p}, \quad \sigma^2 = \frac{1-p}{p^2}$
Hypergeometric $N_1 > 0, \ N_2 > 0$ $N = N_1 + N_2$	$f(x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}, \quad x \le n, x \le N_1, n - x \le N_2$ $\mu = n\left(\frac{N_1}{N}\right), \quad \sigma^2 = n\left(\frac{N_1}{N}\right)\left(\frac{N_2}{N}\right)\left(\frac{N-n}{N-1}\right)$
Negative Binomial $0 < p < 1$ $r = 1, 2, 3, \ldots$	$f(x) = \binom{x-1}{r-1} p^r(1-p)^{x-r}, \quad x = r, r+1, r+2, \ldots$ $M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}, \quad t < -\ln(1-p)$ $\mu = r\left(\frac{1}{p}\right), \quad \sigma^2 = \frac{r(1-p)}{p^2}$
Poisson $0 < \lambda$	$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$ $M(t) = e^{\lambda(e^t-1)}$ $\mu = \lambda, \quad \sigma^2 = \lambda$
Uniform $m > 0$	$f(x) = \frac{1}{m}, \quad x = 1, 2, \ldots, m$ $\mu = \frac{m+1}{2}, \quad \sigma^2 = \frac{m^2-1}{12}$

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