Second Semester Examination
2017/2018 Academic Session

May / June 2018

MSG485 - Finite Element Methods
(Kaedah Unsur Terhingga)

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **ELEVEN (11)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEBELAS (11)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Instructions : Answer **all eight (8)** questions.

*[**Arahan :** Jawab **semua lapan (8)** soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

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Question 1

Consider the problem of finding $u \in H_0^1(0,1)$ that satisfies the boundary value problem

$$L[u] = -\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] = 0, \qquad 0 < x < 1$$
$$u(0) = 0, \quad u(1) = 0.$$

Show that the operator $L[.]$ associated with the problem is self-adjoint.

[20 marks]

Soalan 1

Pertimbangkan masalah mencari $u \in H_0^1(0,1)$ yang memuaskan masalah nilai sempadan

$$L[u] = -\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] = 0, \qquad 0 < x < 1$$
$$u(0) = 0, \quad u(1) = 0.$$

Tunjukkan bahawa pengoperasi $L[.]$ yang bersekutu masalah ini adalah swa-adjoin.

[20 markah]

Question 2

Construct a weak formulation for the following boundary value problem:

$$-\frac{d^2u}{dx^2} + u = x, \quad 0 < x < 1$$
$$u(0) = 0, \quad u(1) = 0.$$

[20 marks]

Soalan 2

Bangunkan suatu formulasi lemah untuk masalah nilai sempadan berikut:

$$-\frac{d^2u}{dx^2} + u = x, \quad 0 < x < 1$$
$$u(0) = 0, \quad u(1) = 0.$$

[20 markah]

Question 3

Given a weak formulation of a boundary value problem of the form

$$\int_0^1 \left\{ -a(x)\frac{dv}{dx}\frac{du}{dx} + b(x)\frac{d^2v}{dx^2}\frac{d^2u}{dx^2} + c(x)uv - vf(x) \right\} dx + M_a v(1) - M_b \left(\frac{dv}{dx}\right)_{x=1} = 0 \ .$$

Write down the associated bilinear and linear form. Show that the bilinear form is symmetric.

[20 marks]

Soalan 3

Diberi formulasi lemah suatu masalah nilai sempadan dalam bentuk

$$\int_0^1 \left\{ -a(x)\frac{dv}{dx}\frac{du}{dx} + b(x)\frac{d^2v}{dx^2}\frac{d^2u}{dx^2} + c(x)uv - vf(x) \right\} dx + M_a v(1) - M_b \left(\frac{dv}{dx}\right)_{x=1} = 0 .$$

Tuliskan bentuk bilinear dan bentuk linear yang bersekutu denganya. Tunjukkan bahawa bentuk bilinear itu adalah simetri.

[20 markah]

Question 4

Consider the 4th order boundary value problem of the form:

$$\frac{d^4w}{dx^4} - p(x) = 0, \quad 0 < x < L,$$

$$w(0) = 0, \ \left(\frac{dw}{dx}\right)_{x=0} = 0, \ \left(\frac{d^2w}{dx^2}\right)_{x=L} = M \neq 0, \ \left(\frac{d^3w}{dx^3}\right)_{x=L} = -V \neq 0.$$

Given an arbitrary weight function $v \in H^2(0,L)$, the weighted integral form of the problem may be written as

$$\int_0^L \left\{ \frac{d^2v}{dx^2}\frac{d^2w}{dx^2} - p(x)v \right\} dx + \left[v\frac{d^3w}{dx^3} - \frac{dv}{dx}\frac{d^2w}{dx^2} \right]_0^L \ .$$

Identify

(a) the essential boundary conditions,

(b) the natural boundary conditions,

of the problem.

[20 marks]

Soalan 4

Pertimbangkan masalah nilai sempadan tahap 4 dalam bentuk:

$$\frac{d^4w}{dx^4} - p(x) = 0, \quad 0 < x < L,$$
$$w(0) = 0, \ \left(\frac{dw}{dx}\right)_{x=0} = 0, \ \left(\frac{d^2w}{dx^2}\right)_{x=L} = M \neq 0, \ \left(\frac{d^3w}{dx^3}\right)_{x=L} = -V \neq 0.$$

Diberi suatu fungsi pemberat sebarangan $v \in H^2(0,L)$ *, bentuk kamiran berpemberat bagi masalah ini boleh ditulis dalam bentuk*

$$\int_0^L \left\{ \frac{d^2v}{dx^2}\frac{d^2w}{dx^2} - p(x)v \right\} dx + \left[v\frac{d^3w}{dx^3} - \frac{dv}{dx}\frac{d^2w}{dx^2} \right]_0^L .$$

Kenalpasti
(a) syarat-syarat sempadan perlu,
(b) syarat-syarat sempadan semula jadi,
bagi masalah ini.

[20 markah]

Question 5

Consider the problem

$$-u'' - u + x^2 = 0, \quad 0 < x < 1,$$
$$u(0) = 0, \quad u(1) = 0.$$

The associated variational problem is given by

$$B(v,u) - l(v) = 0, \qquad u, v \in H^1(0,1) ,$$

where

$$B(v,u) = \int_0^1 \left\{ \frac{du}{dx}\frac{dv}{dx} - v(x)u \right\} dx, \quad l(v) = -\int_0^1 v(x)x^2 dx.$$

Consider finding the N -parameter Galerkin approximation of the form

$$U_N = \sum_{j=1}^{N} c_j\phi_j(x) + \phi_0(x) .$$

(a) Identify suitable basis functions $\phi_0, \phi_1, \phi_2, \ldots, \phi_N$ to be used in the approximation.

(b) Show that the unknown parameters $c_1, c_2, \ldots, c_N$ satisfy the $N \times N$ linear system

$$\mathbf{Kc} = \mathbf{F} \ ,$$

where the (i, j) entry of $\mathbf{K}$ is

$$\mathbf{K}_{ij} = B(\phi_i, \phi_j), \quad i, j = 1, 2, \ldots, N \ ,$$

the i th entry of $\mathbf{F}$ is

$$\mathbf{F}_i = l(\phi_i) - B(\phi_i, \phi_0), \quad i = 1, 2, \ldots, N \ ,$$

and $\mathbf{c}$ is the unknown vector

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} .$$

[20 marks]

Soalan 5

Pertimbangkan masalah

$$-u'' - u + x^2 = 0, \quad 0 < x < 1,$$
$$u(0) = 0, \quad u(1) = 0.$$

Masalah bervariasi bersekutunya diberikan oleh

$$B(v, u) - l(v) = 0, \qquad u, v \in H^1(0,1) \ ,$$

dengan

$$B(v, u) = \int_0^1 \left\{ \frac{du}{dx}\frac{dv}{dx} - v(x)u \right\} dx, \quad l(v) = -\int_0^1 v(x)x^2 dx.$$

Pertimbangkan mencari penghampiran Galerkin N -parameter dalam bentuk

$$U_N = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x) \ .$$

(a) *Kenalpasti fungsi-fungsi asas sesuai $\phi_0, \phi_1, \phi_2, \ldots, \phi_N$ untuk digunakan dalam penghampiran tersebut.*

(b) *Tunjukkan bahawa parameter-parameter anu* $c_1, c_2, \ldots, c_N$ *memuaskan sistem linear* $N \times N$

$$\mathbf{Kc} = \mathbf{F} \ ,$$

yang mana pemasukkan (i, j) *bagi* $\mathbf{K}$ *ialah*

$$\mathbf{K}_{ij} = B(\phi_i, \phi_j), \quad i, j = 1, 2, \ldots, N \ ,$$

pemasukkan ke i *bagi* $\mathbf{F}$ *ialah*

$$\mathbf{F}_i = l(\phi_i) - B(\phi_i, \phi_0), \quad i = 1, 2, \ldots, N \ ,$$

dan $\mathbf{c}$ *ialah vektor anu*

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} .$$

[20 markah]

Question 6

Consider the problem of finding $u \in C^2(0,1)$ such that

$$-u''(x) = f(x), \quad 0 < x < 1,$$
$$u(0) = 0; \quad u(1) = 0.$$

The weak formulation associated with this problem is: Given $f \in L_2(0,1)$, find $u \in H_0^1(0,1)$ such that

$$\int_0^1 u'v' = \int_0^1 fv, \quad \forall v \in H_0^1(0,1). \qquad \text{(Problem A)}$$

If u solves Problem A, show that

$$F(u) \le F(v), \quad \forall v \in H_0^1(0,1),$$

where F is the quadratic functional

$$F(v) = \frac{1}{2}\int_0^1 (v')^2 - \int_0^1 fv \ .$$

[HINT: Let $v = w + u$ for some $w \in H_0^1(0,1)$]

[20 marks]

Soalan 6

Pertimbangkan masalah mencari $u \in C^2(0,1)$ *sedemikian rupa sehinggakan*

$$-u''(x) = f(x), \quad 0 < x < 1,$$
$$u(0) = 0; \quad u(1) = 0.$$

Formulasi lemah yang bersekutu dengan masalah ini ialah: Diberi $f \in L_2(0,1)$, *cari* $u \in H_0^1(0,1)$ *sedemikian rupa sehinggakan*

$$\int_0^1 u'v' = \int_0^1 fv, \quad \forall v \in H_0^1(0,1). \qquad \textit{(Masalah A)}$$

Jika u *menyelesaikan Masalah A, tunjukkan bahawa*

$$F(u) \le F(v), \quad \forall v \in H_0^1(0,1),$$

yang mana F *ialah fungsian kuadratik*

$$F(v) = \frac{1}{2}\int_0^1 (v')^2 - \int_0^1 fv \ .$$

[PETUNJUK:Biar $v = w + u$ *untuk sesuatu* $w \in H_0^1(0,1)$ *]*

[20 markah]

Question 7

Consider the differential equation

$$-u''(x) = x, \quad 0 < x < 1,$$
$$u(0) = 0; \quad u(1) = 0.$$

The weak formulation for the problem is given by

$$\int_0^1 u'v' = \int_0^1 vx, \quad \forall u, v \in H_0^1(0,1).$$

The domain of the problem is discretized into N line elements of equal length, h, with nodes at $a = x_0, x_1, x_2, \ldots, x_N = b$. Consider a finite dimensional approximation of u of the form

$$U_N(x) = \sum_{j=0}^{N} c_j \phi_j(x) \ ,$$

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where

$$\phi_j(x) = \begin{cases} \frac{1}{h}\left(x - x_{j-1}\right), & \text{if } x_{j-1} \le x < x_j \\ \frac{1}{h}\left(x_{j+1} - x\right), & \text{if } x_j \le x < x_{j+1} \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $c_0 = c_N = 0$, and, $c_j = U_N\left(x_j\right)$, for all $j = 1, 2, \ldots, N-1$,

(b) Derive the Galerkin finite element model for a general value of N.

(c) Let $N = 4$. Write down the Galerkin finite element model for this special case.

[40 marks]

Soalan 7

Pertimbangkan persamaan pembezaan

$$-u''(x) = x, \quad 0 < x < 1,$$
$$u(0) = 0; \quad u(1) = 0.$$

Formulasi lemah masalah ini diberikan oleh

$$\int_0^1 u'v' = \int_0^1 vx, \quad \forall u, v \in H_0^1(0,1).$$

Domain masalah ini didiskretkan kepada N *unsur linear yang mempunyai panjang yang sama, iaitu* h *dengan nod-nod pada* $a = x_0, x_1, x_2, \ldots, x_N = b$. *Pertimbangkan penghampiran berdimensi terhingga kepada* u *dalam bentuk*

$$U_N(x) = \sum_{j=1}^{N} c_j \phi_j(x) \ ,$$

dengan

$$\phi_j(x) = \begin{cases} \frac{1}{h}\left(x - x_{j-1}\right), & \text{if } x_{j-1} \le x < x_j \\ \frac{1}{h}\left(x_{j+1} - x\right), & \text{if } x_j \le x < x_{j+1} \\ 0, & \text{lain-lain.} \end{cases}$$

(a) *Tunjukkan bahawa* $c_0 = c_N = 0$, *dan,* $c_j = U_N(x_j)$, *untuk semua* $j = 1, 2, \ldots, N$.

(b) *Terbitkan model unsur terhingga Galerkin untuk suatu nilai umum* N .

(c) *Biar* $N = 4$. *Tuliskan model unsur terhingga Galerkin bagi kes spesifik ini.*

[40 markah]

Question 8

Consider the differential equation

$$-u''(x) - u = 1, \quad 0 < x < 1,$$
$$u(0) = 0; \quad u(1) = 0.$$

The weak formulation for the problem is given by

$$\int_0^1 u'v' - uv = \int_0^1 v, \quad \forall u, v \in H_0^1(0,1).$$

Consider using linear Lagrange interpolation to develop the Galerkin finite element model of the problem over a typical element $\Omega_i = [x_{i-1}, x_i]$.

(a) Represent the approximate solution over the element Ω_i as $U^i = U^i(x_{i-1})\varphi_0^i + U^i(x_i)\varphi_1^i$. Write down the Lagrange basis functions φ_0^i and φ_1^i.

(b) Show that the element stiffness matrix is given by

$$\mathbf{K}^i = \begin{pmatrix} \int_{x_{i-1}}^{x_i} \left\{ \left(\varphi_0^{i\prime}\right)^2 - \left(\varphi_0^i\right)^2 \right\} dx & \int_{x_{i-1}}^{x_i} \left\{ \left(\varphi_0^{i\prime}\varphi_1^{i\prime}\right) - \left(\varphi_0^i\varphi_1^i\right) \right\} dx \\ \int_{x_{i-1}}^{x_i} \left\{ \left(\varphi_1^{i\prime}\varphi_0^{i\prime}\right) - \left(\varphi_1^i\varphi_0^i\right) \right\} dx & \int_{x_{i-1}}^{x_i} \left\{ \left(\varphi_1^{i\prime}\right)^2 - \left(\varphi_1^i\right)^2 \right\} dx \end{pmatrix}$$

and the element load vector is given by

$$\mathbf{F}^i = \begin{pmatrix} \int_{x_{i-1}}^{x_i} \varphi_0^i dx \\ \int_{x_{i-1}}^{x_i} \varphi_1^i dx \end{pmatrix} .$$

(c) Suppose the domain of the problem is discretized into 4 line elements of length 0.25 so that the element nodes are given by $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1.0$. Show that the global finite element model of the problem is given by a linear system

$$\mathbf{Kc} = \mathbf{F} \ ,$$

where

$$\mathbf{K} = 4\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} - \frac{1}{24}\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}, \quad \mathbf{F} = \frac{1}{8}\begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

What is $\mathbf{c}$?

(d) Discuss how the boundary conditions of the problem can be used to reduce the global finite element model in part (c).

[40 marks]

Soalan 8

Pertimbangkan persamaan pembezaan

$$-u''(x) - u = 1, \quad 0 < x < 1,$$
$$u(0) = 0; \quad u(1) = 0.$$

Formulasi lemah bagi masalah ini diberikan oleh

$$\int_0^1 u'v' - uv = \int_0^1 v, \quad \forall u, v \in H_0^1(0,1).$$

Pertimbang menggunakan interpolasi linear Lagrange untuk membangunkan model unsur terhingga Galerkin bagi masalah ini atas unsur tipikal $\Omega_i = [x_{i-1}, x_i]$.

(a) Lambangkan penyelesaian hampiran atas unsur Ω_i *sebagai* $U^i = U^i(x_{i-1})\varphi_0^i + U^i(x_i)\varphi_1^i$. *Tuliskan fungsi asas Lagrange* φ_0^i *dan* φ_1^i.

(b) *Tunjukkan bahawa matriks kekukuhan unsur diberikan oleh*

$$\mathbf{K}^i = \begin{pmatrix} \int_{x_{i-1}}^{x_i} \left\{ \left(\varphi_0^{i\prime}\right)^2 - \left(\varphi_0^i\right)^2 \right\} dx & \int_{x_{i-1}}^{x_i} \left\{ \left(\varphi_0^{i\prime}\varphi_1^{i\prime}\right) - \left(\varphi_0^i\varphi_1^i\right) \right\} dx \\ \int_{x_{i-1}}^{x_i} \left\{ \left(\varphi_1^{i\prime}\varphi_0^{i\prime}\right) - \left(\varphi_1^i\varphi_0^i\right) \right\} dx & \int_{x_{i-1}}^{x_i} \left\{ \left(\varphi_1^{i\prime}\right)^2 - \left(\varphi_1^i\right)^2 \right\} dx \end{pmatrix}$$

dan vektor beban unsur diberikan oleh

$$\mathbf{F}^i = \begin{pmatrix} \int_{x_{i-1}}^{x_i} \varphi_0^i dx \\ \int_{x_{i-1}}^{x_i} \varphi_1^i dx \end{pmatrix}.$$

(c) *Andaikan domain masalah didiskretkan kepada 4 unsur linear yang panjangnya 0.25 supaya nod-nod unsur diberikan oleh* $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1.0$. *Tunjukkan bahawa model unsur terhingga global masalah ini diberikan oleh sistem linear*

$$\mathbf{Kc} = \mathbf{F} ,$$

dengan

$$\mathbf{K} = 4\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} - \frac{1}{24}\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}, \quad \mathbf{F} = \frac{1}{8}\begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

Apakah $\mathbf{c}$ *?*

(d) *Bincangkan bagaimana syarat-syarat sempadan masalah ini boleh digunakan untuk mengurangkan model unsur terhingga global di bahagian (c).*

[40 markah]

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