Second Semester Examination
2017/2018 Academic Session

May / June 2018

# MAT263 - PROBABILITY THEORY
## *(TEORI KEBARANGKALIAN)*

Duration : 3 hours
*[Masa : 3 jam]*

Please check that this examination paper consists of **TWELVE (12)** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **DUA BELAS (12)** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

**Instructions** : Answer **all ten (10)** questions.

*[**Arahan :** Jawab **semua sepuluh (10)** soalan.]*

In the event of any discrepancies, the English version shall be used.

*[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].*

**Question 1**

(a) A sport game has 15 participants, of which 7 are from Red group, 4 are from Blue group, 3 are from Green group and 1 is from Orange group. If the game results list just the groups of the participants in the order in which they are placed, how many outcomes are possible?

[5 marks]

(b) Suppose that an insurance company classifies people into one of three classes of risks (good, average or bad). The company's records indicate that the probabilities that a particular person will be involved in an accident over a 1-year span for these three classes are 0.05, 0.15 and 0.30, respectively. Twenty percent of the population is a good risk, 50 percent an average risk, and 30 percent a bad risk.

(i) What proportion of people have accidents in a fixed year?

(ii) If Abu had no accidents in 2017, what is the probability that he is good or average risk?

(iii) What is the conditional probability that Abu will have an accident in 2018, given that he has had an accident in 2017?

[25 marks]

***Soalan 1***

*(a) Suatu permainan sukan mempunyai 15 peserta, yang mana 7 adalah dari kumpulan Merah, 4 dari kumpulam Biru, 3 dari kumpulan Hijau and 1 dari kumpulan Oren. Jika keputusan permainan hanya menyenaraikan kumpulan peserta dalam turutan kedudukan, berapa banyakkah kemungkinan kesudahan?*

*[5 markah]*

*(b) Katakan sebuah syarikat insuran mengklasifikasikan penduduk ke dalam salah satu kumpulan risiko (baik, sederhana atau teruk). Rekod syarikat menunjukkan bahawa kebarangkalian seseorang akan terlibat dalam kemalangan bagi tempoh 1-tahun untuk ketiga-tiga kelas ini adalah 0.05, 0.15 and 0.30, masing-masing. Dua puluh peratus dari penduduk adalah dalam klasifikasi baik, 50 peratus dalam klasifikasi sederhana, dan 30 peratus dalam klasifikasi teruk.*

*(i) Apakah kadaran penduduk yang akan mengalami kemalangan dalam suatu tahun tertentu?*

*(ii) Jika Abu tidak terlibat dalam kemalangan pada tahun 2017, apakah kebarangkalian bahawa dia adalah seorang yang berisiko baik atau sederhana?*

*(iii) Apakah kebarangkalian bersyarat bahawa Abu akan mengalami kemalangan dalam tahun 2018, diketahui dia telah terlibat dalam kemalangan dalam tahun 2017?*

*[25 markah]*

**Question 2**

Let the distribution function of a random variable $X$ be

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{5} & 0 \le x < 1 \\ \frac{3}{5} & 1 \le x < 3 \\ \frac{4}{5} & 3 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

(a) Find the probability mass function of $X$ .

(b) Obtain the moment generating function of $X$ . Hence, compute the mean.

(c) Find the second factorial moment of $X$ . Hence, compute the variance.

[30 marks]

***Soalan 2***

*Katakan suatu fungsi taburan bagi pemboleh ubah rawak $X$ adalah*

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{5} & 0 \le x < 1 \\ \frac{3}{5} & 1 \le x < 3 \\ \frac{4}{5} & 3 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

*(a) Cari fungsi jisim kebarangkalian bagi $X$ .*

*(b) Dapatkan fungsi penjana momem bagi $X$ . Seterusnya, kira min.*

*(c) Cari momen faktoran kedua bagi $X$ . Seterusnya, kira varians.*

*[30 markah]*

**Question 3**

The number of misprint on a page of a book follows a Poisson process with $\lambda = \frac{1}{4}$.

(a) What is the probability that a page will have no misprint?

(b) What is the probability that a first page with misprint is the fifth page?

(c) If 20 pages is sampled with replacement, how many pages is expected with no misprint?

[30 marks]

***Soalan 3***

*Bilangan salah cetak pada suatu muka surat pada sebuah buku mengikut proses Poison dengan* $\lambda = \frac{1}{4}$.

*(a) Apakah kebarangkalian bahawa suatu muka surat tidak mengandungi sebarang salah cetak?*

*(b) Apakah kebarangkalian bahawa muka surat pertama dengan salah cetak adalah muka surat kelima?*

*(c) Jika 20 muka surat disampel dengan pengembalian, berapa banyak muka surat yang dijangka tidak mempunyai salah cetak?*

*[30 markah]*

**Question 4**

Customers arrive randomly at *Kedai Mahasiswa*. For each scenario below, state the probability density function of $X$, specify the mean and variance and find $P(X > 2)$.

(a) Given that one customer arrived during a particular 15-minute period, let $X$ be the time within the 15 minutes that the customer arrived.

(b) Suppose that the arrival of the customers follows a Poisson process with mean of 30 per hour.

 (i) Let $X$ denotes the waiting time until the first customer arrives after 8.00 am.

 (ii) Let $X$ denotes the waiting time until the 8th customer arrives.

[30 marks]

***Soalan 4***

*Pelanggan tiba secara rawak di Kedai Mahasiswa. Bagi setiap senario berikut, nyatakan fungsi ketumpatan kebarangkalian bagi* $X$ *, tentukan min and varians and cari* $P(X > 2)$ .

*(a)* *Diberi bahawa seorang pelanggan tiba dalam suatu tempoh 15-minit tertentu, katakan* $X$ *adalah masa dalam 15 minit pelanggan tiba.*

*(b)* *Katakan ketibaan pelanggan mengikut suatu proses Poisson dengan min 30 sejam.*

*(i)* *Katakan* $X$ *adalah masa menunggu sehingga pelanggan pertama tiba selepas 8.00 pagi.*

*(ii)* *Katakan* $X$ *masa menunggu sehingga pelanggan ke-8 tiba.*

*[30 markah]*

**Question 5**

(a) If $X$ is uniformly distributed on (0,1), find the probability density function of $Y = e^X$ .

[10 marks]

(b) The joint density of random variables $X$ and $Y$ is given by

$$f(x,y) = \begin{cases} \dfrac{2e^{-2x}}{x} & ; 0 \le x < \infty, \quad 0 \le y \le x \\ 0 & ; \text{otherwise} \end{cases}$$

Compute $Cov(X,Y)$ .

[20 marks]

***Soalan 5***

*(a)* *Jika* $X$ *adalah tertabur Unifom pada (0,1), cari fungsi ketumpatan kebarangkalian bagi* $Y = e^X$ .

*[10 markah]*

*(b)* *Taburan tercantum bagi pemboleh ubah rawak* $X$ *dan* $Y$ *diberi oleh*

$$f(x,y) = \begin{cases} \dfrac{2e^{-2x}}{x} & ; 0 \le x < \infty, \quad 0 \le y \le x \\ 0 & ; \text{di tempat lain} \end{cases}$$

*Hitung* $Cov(X,Y)$ .

*[20 markah]*

## **Question 6**

Suppose

$$f(x \mid y) = \begin{cases} a\dfrac{x}{y^2} & ; 0 < x < y, \quad 0 \le y \le 1 \\ 0 & ; \text{otherwise} \end{cases}$$

and

$$f_2(y) = \begin{cases} b\,y^4 & ; 0 < y < 1 \\ 0 & ; \text{otherwise} \end{cases}$$

(a) Obtain the constants $a$ and $b$.

(b) Find the joint probability density function of $X$ and $Y$.

(c) Compute $P(\frac{1}{2} < X < \frac{3}{4} \mid y = \frac{7}{8})$ and $P(\frac{1}{2} < X < \frac{3}{4})$.

[30 marks]

## ***Soalan 6***

*Andaikan*

$$f(x \mid y) = \begin{cases} a\dfrac{x}{y^2} & ; 0 < x < y, \quad 0 \le y \le 1 \\ 0 & ; \text{di tempat lain} \end{cases}$$

*dan*

$$f_2(y) = \begin{cases} b\,y^4 & ; 0 < y < 1 \\ 0 & ; \text{di tempat lain} \end{cases}$$

*(a) Dapatkan nilai pemalar $a$ dan $b$.*

*(b) Cari fungsi ketumpatan kebarangkalian tercantum bagi $X$ dan $Y$.*

*(c) Hitung $P(\frac{1}{2} < X < \frac{3}{4} \mid y = \frac{7}{8})$ dan $P(\frac{1}{2} < X < \frac{3}{4})$.*

*[30 markah]*

**Question 7**

Let $X$ and $Y$ be two random variables having the joint probability density function

$$f(x,y)=\begin{cases}15x^2y & ; \quad 0<x<y<1\\ 0 & ; \quad \text{elsewhere}\end{cases}$$

(a) Find $E(X \mid y)$.

(b) Compute $Var(X)$ and $Var(Y)$ .

(c) From (a) and (b), obtain the correlation of $X$ and $Y$ .

[30 marks]

***Soalan 7***

*Katakan $X$ dan $Y$ adalah dua pemboleh ubah rawak yang mempunyai fungsi ketumpatan kebarangkalian tercantum*

$$f(x,y)=\begin{cases}15x^2y & ; \quad 0<x<y<1\\ 0 & ; \quad \text{di tempat lain}\end{cases}$$

*(a) Cari $E(X \mid y)$.*

*(b) Hitung $Var(X)$ dan $Var(Y)$.*

*(c) Dari (a) dan (b), dapatkan korelasi bagi $X$ dan $Y$ .*

*[30 markah]*

**Question 8**

Suppose $X_1,\ldots,X_5$ and $Y_1,\ldots,Y_{10}$ are random samples from an independent $N(2,5)$ and $N(2,30)$, respectively. Let $\bar{X}$ and $\bar{Y}$ denote the samples means and $S^2$ denotes the sample variance.

(a) Compute $P(\bar{X} > \bar{Y} + 1)$.

(b) Find a constant $c$ such that $P\left(\frac{\bar{X}-2}{S} < c\right) = 0.90$ .

(c) Find a constant $d$ such that $P\left(S > c\right) = 0.99$.

[30 marks]

***Soalan 8***

*Andaikan* $X_1,\ldots,X_5$ *dan* $Y_1,\ldots,Y_{10}$ *adalah sampel rawak dari* $N(2,5)$ *and* $N(2,30)$ *tak bersandar, masing-masing. Katakan* $\overline{X}$ *dan* $\overline{Y}$ *merupakan min-min sampel dan* $S^2$ *merupakan varians sampel.*

*(a)* *Hitung* $P(\overline{X} > \overline{Y} + 1)$.

*(b)* *Cari pemalar c supaya* $P\left(\frac{\overline{X}-2}{S} < c\right) = 0.90$ .

*(c)* *Cari pemalar* $d$ *supaya* $P(S > c) = 0.99$.

*[30 markah]*

**Question 9**

Suppose $X_1$ and $X_2$ are two random observations from a normal distribution with mean $\mu$ and variance $\sigma^2$. Let $Y_1 = a_1X_1 + a_2X_2$ and $Y_2 = b_1X_1 + b_2X_2$ are two arbitrary linear functions for $X_1$ and $X_2$.

(a) Find the joint moment generating function of $Y_1$ and $Y_2$.

(b) What is the joint probability distribution function of $Y_1$ and $Y_2$?

(c) What is the necessary condition so that $Y_1$ and $Y_2$ are independent?

[30 marks]

***Soalan 9***

*Andaikan* $X_1$ *dan* $X_2$ *adalah dua cerapan tak bersandar dari taburan normal dengan min* $\mu$ *dan varians* $\sigma^2$. *Katakan* $Y_1 = a_1X_1 + a_2X_2$ *dan* $Y_2 = b_1X_1 + b_2X_2$ *adalah sebarang dua fungsi linear bagi* $X_1$ *dan* $X_2$.

*(a)* *Cari fungsi penjana momen tercantum* $Y_1$ *dan* $Y_2$.

*(b)* *Apakah fungsi taburan kebarangkalian tercantum bagi* $Y_1$ *dan* $Y_2$ *?*

*(c)* *Apakah syarat yang perlu supaya* $Y_1$ *dan* $Y_2$ *adalah tak bersandar?*

*[30 markah]*

**Question 10**

Suppose $X$ and $Y$ are two independent random variables having

$$f(x)=\begin{cases} x^2 e^{\frac{-x}{2}} & ; \quad 0<x<\infty \\ 0 & ; \quad \text{elsewhere} \end{cases} \quad \text{and} \quad f(y)=\begin{cases} y e^{\frac{-y}{2}} & ; \quad 0<y<\infty \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Are $U$ and $V$ independent if $U = X+Y$ and $V=\dfrac{X}{X+Y}$? Justify your answer.

[30 marks]

**Soalan 10**

*Katakan $X$ dan $Y$ adalah dua pemboleh ubah rawak dengan*

$$f(x)=\begin{cases} x^2 e^{\frac{-x}{2}} & ; \quad 0<x<\infty \\ 0 & ; \quad \text{di tempat lain} \end{cases} \quad \textit{dan} \quad f(y)=\begin{cases} y e^{\frac{-y}{2}} & ; \quad 0<y<\infty \\ 0 & ; \quad \text{di tempat lain} \end{cases}$$

*Adakah $U$ dan $V$ tak bersandar jika $U = X+Y$ dan $V=\dfrac{X}{X+Y}$? Tentusahkan jawapan anda.*

*[30 markah]*

**LAMPIRAN**

| ***DISCRETE DISTRIBUTIONS*** | |
|---|---|
| Bernoulli | $f(x) = p^x (1-p)^{1-x}, \quad x = 0,1$<br>$M(t) = 1 - p + pe^t$<br>$\mu = p, \quad \sigma^2 = p(1-p)$ |
| Binomial | $f(x) = \dfrac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, \quad x = 0,1,2,\ldots,n$<br>$M(t) = (1 - p + pe^t)^n$<br>$\mu = np, \quad \sigma^2 = np(1-p)$ |
| Geometric | $f(x) = (1-p)^x p, \quad x = 0,1,2,\ldots$<br>$M(t) = \dfrac{p}{1-(1-p)e^t}, \quad t < -\ln(1-p)$<br>$\mu = \dfrac{1-p}{p}, \quad \sigma^2 = \mu = \dfrac{(1-p)}{p^2}$ |
| Negative Binomial | $f(x) = \dfrac{(x+r-1)!}{x!(r-1)!} p^r (1-p)^x, \quad x = 0,1,2,\ldots$<br>$M(t) = \dfrac{p^r}{\left[1-(1-p)e^t\right]^r}, \quad t < \ln(1-p)$<br>$\mu = \dfrac{r(1-p)}{p}, \quad \sigma^2 = \dfrac{r(1-p)}{p^2}$ |
| Poisson | $f(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0,1,2,\ldots$<br>$M(t) = e^{\lambda(e^t - 1)}$<br>$\mu = \lambda, \quad \sigma^2 = \lambda$ |
| Hypergeometric | $f(x) = \dfrac{\binom{n_1}{x}\binom{n_2}{r-x}}{\binom{n}{t}}, \quad x \le r, \quad x \le n_1, \quad r - x \le n_2,$<br>$\mu = \dfrac{rn_1}{n}, \quad \sigma^2 = \dfrac{rn_1 n_2 (n-r)}{n^2(n-1)}$ |

| ***CONTINUOUS DISTRIBUTION*** | |
|---|---|
| Uniform | $f(x)=\frac{1}{b-a},\quad a\le x\le b$<br>$M(t)=\frac{e^{tb}-e^{ta}}{t(b-a)},\quad t\ne 0,\quad M(0)=1$<br>$\mu=\frac{a+b}{2},\quad \sigma^2=\frac{(b-a)^2}{12}$ |
| Exponential | $f(x)=\frac{1}{\theta}e^{-x/\theta},\ 0\le x\le\infty$<br>$M(t)=\frac{1}{1-\theta t},\quad t<1/\theta$<br>$\mu=\theta,\quad \sigma^2=\theta^2$ |
| Gamma | $f(x)=\frac{1}{\Gamma(\alpha)\theta^{\alpha}}x^{\alpha-1}e^{-x/\theta},\quad 0\le x\le\infty$<br>$M(t)=\frac{1}{(1-\theta t)^{\alpha}},\quad t<1/\theta$<br>$\mu=\alpha\theta,\quad \sigma^2=\alpha\theta^2$ |
| Chi Square | $f(x)=\frac{1}{\Gamma(r/2)2^{r/2}}r^{r/2-1}e^{-x/2},\quad 0\le x\le\infty$<br>$M(t)=\frac{1}{(1-2t)^{r/2}},\quad t<\frac{1}{2}$<br>$\mu=r,\quad \sigma^2=2r$ |
| Normal | $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\left[(x-\mu)^2/2\sigma^2\right]},\quad -\infty<x<\infty$<br>$M(t)=e^{\mu t+\sigma^2t^2/2}$<br>$E(X)=\mu,\quad Var(X)=\sigma^2$ |
| Beta | $f(x)=\frac{1}{B(\alpha,\beta)}\quad x^{\alpha-1}(1-x)^{\beta-1},\quad 0<x<1$<br>$\mu=\frac{\alpha}{\alpha+\beta},\quad \sigma^2=\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$ |

| | ***FORMULA*** |
|---|---|
| 1. | $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ |
| 2. | $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad \lvert r\rvert < 1$ |
| 3. | $\sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x} = (a+b)^n$ |
| 4. | $\sum_{x=0}^{n} \binom{n}{x}\binom{r-n}{r-x} = \binom{n}{r}$ |
| 5. | $\sum_{x=0}^{n} \binom{n+k-1}{k} w^k = (1-w)^{-n}, \quad \lvert w\rvert < 1$ |
| 6. | $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx, \quad \Gamma(\alpha) = (\alpha-1)!$ |
| 7. | $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx$ |
| 8. | $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ |
| 9. | Polar coordinates: $y = r\cos\theta$ <br> $z = r\sin\theta$ |

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