Second Semester Examination
2017/2018 Academic Session

May/June 2018

MAT111 - Linear Algebra
[Aljabar Linear]

Duration : 3 hours
(Masa : 3 jam)

Please check that this examination paper consists of **NINE** (**9**) pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **SEMBILAN** (**9**) muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Instructions: Answer **all four** (**4**) questions.

*[**Arahan:** Jawab **semua empat** (**4**) soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

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Question 1

(a) Consider the following system of equations:

$$\begin{aligned} 3x_1 - x_2 + x_3 + 7x_4 &= 0, \\ -2x_1 + x_2 - x_3 - 3x_4 &= 0, \\ -2x_1 + x_2 - 7x_4 &= 0. \end{aligned}$$

(i) Write the coefficient matrix A of the system.

(ii) Solve the system using Gauss-Jordan elimination.

(iii) State a basis for the column space of A.

(iv) State a basis for the row space of A.

(v) State a basis for the null space of A.

(vi) State the rank and nullity of A.

(vii) Find a basis for the orthogonal complement to the subspace W of $\mathbb{R}^4$ spanned by the vectors $(3,-1,1,7),(-2,1,-1,-3),(-2,1,0,-7)$.

(b) Find all 2×2 diagonal matrices B that satisfy the equation

$$B^2 - 3B + 2I = \mathbf{0}.$$

(c) Let C be an $n \times n$ invertible matrix.

(i) Show that $\det(C^{-1}) = \frac{1}{\det(C)}$.

(ii) Find the determinant of C if $C^2 = 2C$.

[100 marks]

Soalan 1

(a) *Pertimbangkan sistem persamaan linear berikut:*

$$\begin{aligned} 3x_1 - x_2 + x_3 + 7x_4 &= 0, \\ -2x_1 + x_2 - x_3 - 3x_4 &= 0, \\ -2x_1 + x_2 - 7x_4 &= 0. \end{aligned}$$

(i) *Tuliskan matriks pekali A untuk sistem ini.*

(ii) *Selesaikan sistem ini menggunakan kaedah Gauss-Jordan.*

(iii) *Nyatakan asas bagi ruang lajur A.*

(iv) *Nyatakan asas bagi ruang baris A.*

(v) *Nyatakan asas bagi ruang nol A.*

(vi) *Nyatakan pangkat dan kenolan A.*

(vii) *Dapatkan asas bagi pelengkap berortogon subruang W dalam* $\mathbb{R}^4$ *yang direntang oleh vektor-vektor* $(3,-1,1,7),(-2,1,-1,-3),(-2,1,0,-7)$.

(b) *Dapatkan semua matriks pepenjuru* 2×2*, B yang memenuhi persamaan*

$$B^2 - 3B + 2I = \mathbf{0}.$$

(c) *Andaikan C suatu matriks* $n \times n$ *tersongsangkan.*

(i) *Tunjukkan bahawa* $\det(C^{-1}) = \frac{1}{\det(C)}$.

(ii) *Dapatkan penentu bagi C jika* $C^2 = 2C$.

[100 markah]

Question 2

(a) Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns, and let B be an invertible $n \times n$ matrix. Show that $A\mathbf{x} = \mathbf{0}$ has only trivial solution if and only if $(BA)\mathbf{x} = \mathbf{0}$ has only trivial solution.

(b) Let V be the set of 2×2 matrices of the form $\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$, where a and b are real numbers.

The addition and scalar multiplication operations on V are defined as:

$$\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} + \begin{bmatrix} 1 & c \\ d & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+c \\ b+d & 1 \end{bmatrix}$$

and

$$k \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1 & ka \\ kb & 1 \end{bmatrix}.$$

Assume that V is a vector space.

(i) Determine the zero vector Z in V.

(ii) Let A be $\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$. Find $-A$ such that $A + (-A) = Z$.

(iii) Find the basis of V.

(c) For each of the following, determine whether W is a subspace of $\mathbb{R}^3$. Prove your claim.

(i) $W = \{(2t, 0, -t+1) : t \in \mathbb{R}\}$.

(ii) $W = \{(x, y, z) : x - 2y + z = 0; x, y, z \in \mathbb{R}\}$.

(d) Consider $\mathbb{R}^2$ with the following inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 - u_2 v_1 - u_1 v_2 + 3u_2 v_2,$$

for all vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

(i) Find the length of the vector $(3, -4)$ using the inner product defined above.

(ii) State the Cauchy-Schwarz inequality for $\mathbb{R}^2$ with the defined inner product.

[100 marks]

Soalan 2

(a) *Andaikan* $A\mathbf{x} = \mathbf{0}$ *suatu sistem homogen dengan* n *persamaan linear dan* n *anu, dan andaikan* B *matriks* $n \times n$ *tersongsangkan. Tunjukkan bahawa* $A\mathbf{x} = \mathbf{0}$ *hanya mempunyai penyelesaian remeh jika dan hanya jika* $(BA)\mathbf{x} = \mathbf{0}$ *hanya mempunyai penyelesaian remeh.*

(b) *Andaikan* V *set yang mengandungi matriks* 2×2 *berbentuk* $\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$ *untuk* a *dan* b *sebarang nombor nyata. Operasi penambahan dan pendaraban skalar diberi seperti berikut:*

$$\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} + \begin{bmatrix} 1 & c \\ d & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+c \\ b+d & 1 \end{bmatrix}$$

dan

$$k\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1 & ka \\ kb & 1 \end{bmatrix}.$$

Andaikan bahawa V *ialah ruang vektor.*

(i) *Tentukan vektor sifar,* Z *dalam* V.

(ii) *Andaikan* A *ialah* $\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$. *Dapatkan* $-A$ *sedemikian* $A + (-A) = Z$.

(iii) *Dapatkan asas untuk* V.

(c) *Untuk setiap yang berikut, tentukan sama ada* W *adalah subruang untuk* $\mathbb{R}^3$. *Buktikan tuntutan anda.*

(i) $W = \{(2t, 0, -t+1) : t \in \mathbb{R}\}$.

(ii) $W = \{(x, y, z) : x - 2y + z = 0; x, y, z \in \mathbb{R}\}$.

(d) *Pertimbangkan* $\mathbb{R}^2$ *dengan hasil darab terkedalam berikut*

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 - u_2v_1 - u_1v_2 + 3u_2v_2,$$

untuk semua vektor $\mathbf{u} = (u_1, u_2)$ *dan* $\mathbf{v} = (v_1, v_2)$.

(i) *Dapatkan panjang vektor* $(3, -4)$ *menggunakan definisi hasil darab terkedalam tertakrif di atas.*

(ii) *Nyatakan ketaksamaan Cauchy-Schwarz untuk* $\mathbb{R}^2$ *dengan hasil darab terkedalam tersebut.*

[100 markah]

Question 3

(a) Let

$$\mathbf{u}_1 = (0,1,0), \quad \mathbf{u}_2 = \left(-\frac{4}{5},0,\frac{3}{5}\right), \quad \mathbf{u}_3 = \left(\frac{3}{5},0,\frac{4}{5}\right).$$

(i) Show that the set $S = \{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ is an orthonormal set.

(ii) Can S be a basis for $\mathbb{R}^3$? Explain.

(iii) Express the vector $\mathbf{u} = (1,1,1)$ as a linear combination of the vectors in S.

(iv) Find the coordinate vector $(\mathbf{u})_S$.

(b) Let $\mathbf{w_1}$ and $\mathbf{w_2}$ be distinct vectors in a vector space W. Show that the set $\{\mathbf{w_1},\mathbf{w_2}\}$ is linearly dependent if and only if $\mathbf{w_1}$ and $\mathbf{w_2}$ are multiple of each other.

(c) State the conditions for which a function $T : V \to W$ (from vector space V to vector space W) is a linear transformation.

(d) Let $S = \{\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = (1,1,0), \quad \mathbf{v}_3 = (1,0,0)\}$ be a basis for $\mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$T(\mathbf{v}_1) = (-1,2,4), \quad T(\mathbf{v}_2) = (0,3,2), \quad T(\mathbf{v}_3) = (1,5,-1).$$

Find a formula for $T(x_1,x_2,x_3)$, and use that formula to find $T(2,4,-1)$.

[100 marks]

Soalan 3

(a) Andaikan

$$\mathbf{u}_1 = (0,1,0), \quad \mathbf{u}_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right), \quad \mathbf{u}_3 = \left(\frac{3}{5}, 0, \frac{4}{5}\right).$$

(i) Tunjukkan bahawa set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *merupakan set ortonormal.*

(ii) Bolehkah S menjadi asas untuk $\mathbb{R}^3$*? Jelaskan.*

(iii) Tulis vektor $\mathbf{u} = (1,1,1)$ *sebagai gabungan linear vektor-vektor dalam S.*

(iv) Dapatkan vektor koordinat $(\mathbf{u})_S$.

(b) Andaikan $\mathbf{w_1}$ *dan* $\mathbf{w_2}$ *merupakan vektor yang berbeza dalam ruang vektor* W. *Tunjukkan bahawa set* $\{\mathbf{w_1}, \mathbf{w_2}\}$ *adalah bersandar linear jika dan hanya jika* $\mathbf{w_1}$ *dan* $\mathbf{w_2}$ *merupakan gandaan antara satu sama lain.*

(c) Nyatakan syarat untuk fungsi $T : V \to W$ *(dari ruang vektor* V *ke ruang vektor* W*) menjadi suatu transformasi linear.*

(d) Andaikan $S = \{\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = (1,1,0), \quad \mathbf{v}_3 = (1,0,0)\}$ *suatu asas bagi* $\mathbb{R}^3$ *dan* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *merupakan transformasi linear sedemikian*

$$T(\mathbf{v}_1) = (-1,2,4), \quad T(\mathbf{v}_2) = (0,3,2), \quad T(\mathbf{v}_3) = (1,5,-1).$$

Dapatkan formula untuk $T(x_1, x_2, x_3)$ *dan dengan menggunakan formula tersebut, dapatkan* $T(2,4,-1)$.

[100 markah]

Question 4

(a) Given $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}.$$

Find the least squares solution to the linear system $A\mathbf{x} = \mathbf{b}$.

(b) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(i) What is the characteristic equation of A?

(ii) Find the eigenvalues and also the bases for the corresponding eigenspaces of the matrix A.

(iii) Find the invertible matrix P and the diagonal matrix D such that $P^{-1}AP = D$.

(iv) Is P in part (iii) unique? Justify your answer.

(v) Find the eigenvalues and the associated eigenvectors of A^4.

[100 marks]

Soalan 4

(a) Diberi $A\mathbf{x} = \mathbf{b}$*, dengan*

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \quad \textit{dan} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}.$$

Dapatkan penyelesaian kuasa dua terkecil untuk sistem linear $A\mathbf{x} = \mathbf{b}$.

(b) Biarkan $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(i) Apakah persamaan cirian untuk A*?*

(ii) Dapatkan semua nilai eigen dan asas-asas yang sepadan dengan ruang eigen bagi matriks A*.*

(iii) Dapatkan matriks tersongsangkan P *dan matriks pepenjuru* D *sedemikian* $P^{-1}AP = D$*.*

(iv) Adakah P *dalam bahagian (iii) unik? Jelaskan jawapan anda.*

(v) Dapatkan semua nilai eigen dan asas bagi ruang eigen matriks A^4*.*

[100 markah]

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