First Semester Examination
Academic Session 2018/2019

December 2018/January 2019

MSS401 - Complex Analysis
(Analisis Kompleks)

Duration : 3 hours
(Masa : 3 jam)

Please check that this examination paper consists of EIGHT (8) pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN (8) muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer all **TEN (10)** questions.

*[**Arahan:** Jawab semua **SEPULUH (10)** soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

Question 1

(a) Define a complex number as well as the operations of addition and multiplication between two complex numbers.

[6 marks]

(b) Show that every complex number z can be expressed in its Cartesian form $z = a + ib$, where i is the unit imaginary number.

[6 marks]

(c) Show that no order exists on the field of complex numbers that will make it an ordered field.

[8 marks]

Soalan 1

(a) Takrifkan nombor kompleks serta operasi penambahan dan pendaraban dua nombor kompleks.

[6 markah]

(b) Tunjukkan bahawa setiap nombor kompleks z dapat diungkapkan dalam bentuk Cartesan $z = a + ib$, dengan i sebagai nombor khayalan unit.

[6 markah]

(c) Tunjukkan bahawa tidak wujud tertib pada medan nombor kompleks yang dapat menjadikan ia sebagai medan bertertib.

[8 markah]

Question 2

Solve each equation and express its solution in Cartesian form:

(a) $z^2 = 3 - 2(5^i)^{\frac{1}{2}}$,

[10 marks]

(b) $\sin z = 5i$.

[10 marks]

Soalan 2

Selesaikan setiap persamaan dan ungkapkan jawapan dalam bentuk Cartesan:

(a) $z^2 = 3 - 2(5^i)^{\frac{1}{2}}$,

[10 markah]

(b) $\sin z = 5i$.

[10 markah]

Question 3

(a) Define the logarithmic function $w = \log z$ on $\mathbb{C} \setminus \{0\}$. Explain why the principal logarithm $\operatorname{Log} z$ is discontinuous at all negative real numbers. Find the derivative of $\operatorname{Log} z$ and the largest domain where the function is analytic.

[15 marks]

(b) Show that $f(z) = e^z$ is periodic with period $2\pi i$. Hence, show that f is one-to-one in any open disk of radius π.

[10 marks]

(c) Show that $w = \cos z$ is an unbounded entire function.

[10 marks]

Soalan 3

(a) *Takrifkan fungsi logaritma* $w = \log z$ *pada* $\mathbb{C} \setminus \{0\}$*. Jelaskan sebab logaritma prinsipal* $\operatorname{Log} z$ *tak selanjar pada setiap nombor nyata negatif. Dapatkan terbitan bagi* $\operatorname{Log} z$ *dan domain terbesar supaya fungsi tersebut adalah analisis.*

[15 markah]

(b) *Tunjukkan bahawa* $f(z) = e^z$ *adalah berkala dengan kala* $2\pi i$*. Justeru, tunjukkan bahawa* f *adalah satu dengan satu pada sebarang cakera terbuka dengan jejari* π*.*

[10 markah]

(c) *Tunjukkan bahawa* $w = \cos z$ *merupakan fungsi seluruh yang tak terbatas.*

[10 markah]

Question 4

Let $\sin^{-1} z$ be the inverse function of $\sin z$. Prove that

$$\sin^{-1} z = \frac{1}{i} \log\left(iz + (1 - z^2)^{1/2}\right)$$

if the branch of $\sin^{-1} z$ is chosen with $\sin^{-1} 0 = 0$. Also show that

$$(\sin^{-1})'(z) = \frac{d}{dz} \sin^{-1} z = \frac{1}{(1 - z^2)^{1/2}}.$$

Classify each singularity of $(\sin^{-1})'(z)$.

[20 marks]

Soalan 4

Biarkan $\sin^{-1} z$ *ialah fungsi songsang bagi* $\sin z$. *Buktikan bahawa*

$$\sin^{-1} z = \frac{1}{i} \log\left(iz + (1-z^2)^{1/2}\right)$$

jika cabang bagi $\sin^{-1} z$ *dipilih dengan* $\sin^{-1} 0 = 0$. *Tunjukkan juga bahawa*

$$(\sin^{-1})'(z) = \frac{d}{dz} \sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}.$$

Kelaskan setiap titik singular $(\sin^{-1})'(z)$.

[20 markah]

Question 5

(a) Determine the region such that the function

$$f(z) = f(x+iy) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$$

is differentiable and analytic. Find the derivative of f if it exists.

[10 marks]

(b) Show that $u(x,y) = e^{x^2-y^2} \sin 2xy$ is harmonic in the complex plane $\mathbb{C}$. Find its harmonic conjugate v and find $f(z) = u(x,y) + iv(x,y)$.

[15 marks]

Soalan 5

(a) Tentukan rantau sedemikian hingga fungsi

$$f(z) = f(x+iy) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$$

adalah terbeza dan analisis. Dapatkan terbitan untuk f *jika wujud.*

[10 markah]

(b) Tunjukkan bahawa $u(x,y) = e^{x^2-y^2} \sin 2xy$ *adalah harmonik pada satah kompleks* $\mathbb{C}$. *Dapatkan konjugat harmonik* v *dan dapatkan* $f(z) = u(x,y) + iv(x,y)$.

[15 markah]

Question 6

(a) Suppose $f = u + iv$ is analytic in a domain D. Explain why u and v have continuous partial derivatives up to the second order. Also show that $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$ in D.

[15 marks]

(b) Let $0 < r < R$, $z = re^{i\theta}$ and

$$P(re^{i\theta}) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}.$$

Show that $P(z)$ is harmonic in the disk $|z| < R$. By considering

$$\oint_{|z|=r} \frac{R+z}{Rz - z^2}\, dz,$$

show that

$$\frac{1}{2\pi}\int_0^{2\pi} P(re^{i\theta})d\theta = 1.$$

[15 marks]

Soalan 6

(a) Biarkan $f = u + iv$ analisis pada domain D. Jelaskan sebab terbitan separa u and v yang selanjar wujud sehingga peringkat kedua. Tunjukkan juga $u_{xy} = u_{yx}$ dan $v_{xy} = v_{yx}$ pada D.

[15 markah]

(b) Andaikan $0 < r < R$, $z = re^{i\theta}$ dan

$$P(re^{i\theta}) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}.$$

Tunjukkan bahawa $P(z)$ adalah harmonik pada cakera $|z| < R$. Dengan mempertimbangkan

$$\oint_{|z|=r} \frac{R+z}{Rz - z^2}\, dz,$$

tunjukkan bahawa

$$\frac{1}{2\pi}\int_0^{2\pi} P(re^{i\theta})d\theta = 1.$$

[15 markah]

Question 7

(a) Let γ be a smooth curve in the complex plane and f be a continuous function over γ. Define $\int_\gamma f(z)\,dz$.

[5 marks]

(b) Evaluate

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}.$$

[15 marks]

Soalan 7

(a) *Biarkan γ suatu lengkung licin pada satah kompleks dan f adalah fungsi selanjar pada γ. Takrifkan $\int_\gamma f(z)\,dz$.*

[5 markah]

(b) *Nilaikan*

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}.$$

[15 markah]

Question 8

Evaluate the following integrals over the given positively oriented simple closed contour:

(a) $$\oint_{|z|=\pi} \frac{e^{2z}}{(z^2+1)(z-3)}\,dz\ ,$$

[10 marks]

(b) $$\oint_{|z|=1} \frac{z^6+1}{z^3(2z-1)^2(z-2)}\,dz\ .$$

[10 marks]

Soalan 8

Nilaikan kamiran berikut pada kontur tertutup ringkas berarah positif yang diberikan:

(a) $$\oint_{|z|=\pi} \frac{e^{2z}}{(z^2+1)(z-3)}\,dz\ ,$$

[10 markah]

(b) $$\oint_{|z|=1} \frac{z^6+1}{z^3(2z-1)^2(z-2)}\,dz\ .$$

[10 markah]

Question 9

(a) Classify each singularity of

$$f(z) = \frac{ze^{iz}}{(z^2+9)(z^2+4)}$$

and find its residue. Hence, evaluate

$$\int_0^{\infty} \frac{x\sin x}{(x^2+9)(x^2+4)}\,dx.$$

[20 marks]

(b) Find two Laurent series expansions for

$$f(z) = \frac{z-a}{z^2}, \quad a \in \mathbb{C}\backslash\{0\}$$

in the power of $z-a$.

[15 marks]

Soalan 9

(a) *Kelaskan setiap titik singular*

$$f(z) = \frac{ze^{iz}}{(z^2+9)(z^2+4)}$$

dan dapatkan nilai reja. Justeru, nilaikan

$$\int_0^\infty \frac{x\sin x}{(x^2+9)(x^2+4)}\,dx.$$

[20 markah]

(b) *Dapatkan dua kembangan siri Laurent bagi*

$$f(z) = \frac{z-a}{z^2}, \quad a \in \mathbb{C}\backslash\{0\}$$

dalam kuasa $z-a$.

[15 markah]

Question 10

(a) Let f be an entire function satisfying $|f(z)| \leq Me^x$, $z = x+iy$ for each z. Show that there exists a constant a with $|a| \leq M$ such that $f(z) = ae^z$.

[8 marks]

(b) Find the maximum value of $|z^2+z-2|$ on the disk $|z| \leq 1$.

[10 marks]

(c) Prove that $f(z) = \displaystyle\int_0^1 \frac{e^{tz}}{e^{t^2}+1}\,dt$ is an entire function.

[7 marks]

Soalan 10

(a) *Biarkan f fungsi seluruh yang memenuhi $|f(z)| \leq Me^x$, $z = x + iy$ untuk setiap z. Tunjukkan wujud pemalar a dengan $|a| \leq M$ sedemikian hingga $f(z) = ae^z$.*

[8 markah]

(b) *Dapatkan nilai maksimum bagi $|z^2 + z - 2|$ pada cakera $|z| \leq 1$.*

[10 markah]

(c) *Buktikan bahawa $f(z) = \displaystyle\int_0^1 \frac{e^{tz}}{e^{t^2}+1}\,dt$ ialah fungsi seluruh.*

[7 markah]

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