First Semester Examination
Academic Session 2018/2019

December 2018/January 2019

MSG489 - *Numerical Methods For Differential Equations*
(Kaedah Berangka Untuk Persamaan Pembezaan)

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of EIGHT (8) pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LAPAN (8) muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **FOUR (4)** questions.

*[**Arahan** : Jawab **EMPAT (4)** soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].

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Question 1

(a) Solve the following differential equation and determine the interval of validity for the solution,

(i) $\frac{dy}{dx} = 6y^2 x, \quad y(1) = \frac{1}{25}$

(ii) $y' + \frac{2y}{t} = \frac{\cos t}{t^2}, \quad y(\pi) = 0$

[30 marks]

(b) A 1500-gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and the water entering the tank has salt concentration of $\frac{1}{5}(1+\cos t)$ lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is on the tank when it overflows at $t = 300$?

[40 marks]

(c) The Runge-Kutta method of order four can be written in the form

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \frac{h}{6} f(t_i, w_i) + \frac{h}{3} f(t_i + \alpha_1 h, w_i + \delta_1 hf(t_i, w_i))$$

$$= + \frac{h}{3} f(t_i + \alpha_2 h, w_i + \delta_2 hf(t_i + \gamma_2 h, w_i + \gamma_3 hf(t_i, w_i)))$$

$$= + \frac{h}{6} f(t_i + \alpha_3 h, w_i + \delta_3 hf(t_i + \gamma_4 h, w_i + \gamma_5 hf(t_i + \gamma_6 h, w_i + \gamma_7 hf(t_i, w_i))))$$

Find the values of the constants $\alpha_1, \alpha_2, \alpha_3, \delta_1, \delta_2, \delta_3, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$ *and* γ_7

[30 marks]

Soalan 1

(a) *Selesaikan persamaan pembezaan berikut dan tentukan selang kesahan penyelesaiannya,*

(i) $\dfrac{dy}{dx} = 6y^2 x, \quad y(1) = \dfrac{1}{25}$

(ii) $y' + \dfrac{2y}{t} = \dfrac{\cos t}{t^2}, \quad y(\pi) = 0$

[30 markah]

(b) *Tangki 1500-galon pada mulanya mengandungi 600 galon air dengan 5 lbs garam dicampur ke dalamnya. Air memasuki tangki tersebut pada kadar 9 gal / jam dengan kepekatan garam* $\frac{1}{5}(1+\cos t)$ *lbs / gal. Jika larutan dalam tangki telah dicampur dengan sekata dan melimpah keluar dari tangki pada kadar 6 gal / jam, berapakah kandungan garam di dalam tangki apabila air melimpah semasa* $t = 300$ *?*

[40 markah]

(c) *Kaedah Runge-Kutta tertib keempat boleh ditulis dalam bentuk berikut*

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \frac{h}{6} f(t_i, w_i) + \frac{h}{3} f\left(t_i + \alpha_1 h, w_i + \delta_1 hf(t_i, w_i)\right)$$

$$= + \frac{h}{3} f\left(t_i + \alpha_2 h, w_i + \delta_2 hf\left(t_i + \gamma_2 h, w_i + \gamma_3 hf(t_i, w_i)\right)\right)$$

$$= + \frac{h}{6} f\left(t_i + \alpha_3 h, w_i + \delta_3 hf\left(t_i + \gamma_4 h, w_i + \gamma_5 hf\left(t_i + \gamma_6 h, w_i + \gamma_7 hf(t_i, w_i)\right)\right)\right)$$

Cari nilai pemalar-pemalar $\alpha_1, \alpha_2, \alpha_3, \delta_1, \delta_2, \delta_3, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$ *dan* γ_7.

[30 markah]

Question 2

(a) Consider the following initial value problem:

$$y' = te^{3t} - 2y,\ 0 \le t \le 1,\ \ y(0) = 0,\ h = 0.5$$

Approximate the solutions of $y(1.0)$ by using the following method

(i) Euler method

(ii) Midpoint method

(iii) Heun's method

(iv) Runge-Kutta method of order four

[40 marks]

(b) Consider the following initial value problem:

$$y' = 1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2,\ \ y(1) = 0$$

Actual solution is given by $y = t\,tan(ln\,t)$. Use starting values obtained from actual solution and compare the solutions to the actual values. Perform Adams-Bashforth 3rd order methods with $h = 0.1$ to approximate $y(1.4)$.

[30 marks]

(c) Taylor series have been widely used to study the behaviour of numerical approximation to differential equations. By using Taylor series expansion, prove the following approximation

$$\frac{d^2y}{dx^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \quad and \quad \frac{dy}{dx} \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

where h is a step size of the approximation

[10 marks]

(d) Use the finite difference method to approximate the solution to the following boundary value problem.

$$y'' = 4y' - y + xe^{-x} - x$$
$$0 \le x \le 3,\ y(0) = 0,\ y(3) = -4$$

Use $h = 1$

[20 marks]

Soalan 2

(a) *Pertimbangkan masalah nilai awalan yang berikut:*

$$y' = te^{3t} - 2y,\ 0 \le t \le 1,\ \ y(0) = 0,\ h = 0.5$$

Hitung penyelesaian kepada $y(1.0)$ *dengan menggunakan kaedah berikut:*

(i) *Kaedah Euler*

(ii) *Kaedah titik tengah*

(iii) *Kaedah Heun*

(iv) *Kaedah Runge-Kutta tertib keempat*

[40 markah]

(b) *Pertimbangkan masalah nilai awalan yang berikut:*

$$y' = 1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2,\ \ y(1) = 0$$

Penyelesaian sebenar soalan di atas ialah $y = t\tan(\ln t)$. *Gunakan nilai permulaan yang diperolehi daripada penyelesaian sebenar dan bandingkan penyelesaian dengan nilai sebenar. Laksanakan kaedah Adams-Bashforth tertib ketiga dengan* $h = 0.1$ *untuk menganggar nilai* $y(1.4)$.

[30 markah]

(c) *Siri Taylor telah digunakan secara meluas untuk mengkaji penyelesaian masalah pembezaan berangka. Dengan menggunakan pengembangan siri Taylor, buktikan anggaran berikut*

$$\frac{d^2y}{dx^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \quad dan \quad \frac{dy}{dx} \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

dengan h *ialah saiz langkah bagi anggaran*

[10 markah]

(d) *Gunakan kaedah beza terhingga untuk menganggarkan penyelesaian kepada masalah nilai sempadan berikut.*

$$y'' = 4y' - y + xe^{-x} - x$$
$$0 \le x \le 3,\ y(0) = 0,\ y(3) = -4$$

Guna $h = 1$

[20 markah]

Question 3

(a) Consider the following boundary value problem:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad 0<x<1, \quad 0<y<2$$
$$u(x,0)=x^2, \quad u(x,2)=(x-2)^2, \quad 0\le x\le 1$$
$$u(0,y)=y^2, \quad u(1,y)=(y-1)^2, \quad 0\le y\le 2$$

Solve the linear system by using finite difference method with central differencing approximation. Use $\Delta x=1/3$ and $\Delta y=0.5$. Perform two iterations using the Jacobi method to solve the system.

[50 marks]

(b) Find the solution of $\nabla^2 u=0$ in R subject to the boundary conditions $u(x,y)=x^2-y^2$ where R is the square, $0\le x\le 1, 0\le y\le 1$, using the five point formula. Assume the step length, $h=1/3$, is uniform along the axes.

[50 marks]

Soalan 3

(a) Pertimbangkan masalah nilai sempadan berikut:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad 0<x<1, \quad 0<y<2$$
$$u(x,0)=x^2, \quad u(x,2)=(x-2)^2, \quad 0\le x\le 1$$
$$u(0,y)=y^2, \quad u(1,y)=(y-1)^2, \quad 0\le y\le 2$$

Selesaikan sistem linear menggunakan kaedah beza terhingga dengan hampiran beza pusat. Gunakan $\Delta x=1/3$ dan $\Delta y=0.5$. Laksanakan dua lelaran menggunakan kaedah Jacobi bagi menyelesaikan sistem ini.

[50 markah]

(b) Dapatkan penyelesaian bagi $\nabla^2 u=0$ dalam R berdasarkan kepada syarat-syarat sempadan, $u(x,y)=x^2-y^2$ di mana R adalah segiempat sama, $0\le x\le 1, 0\le y\le 1$, menggunakan formula lima titik. Andaikan saiz langkah, $h=1/3$, adalah seragam sepanjang paksi-paksi.

[50 markah]

Question 4

(a) Use FTCS method with 4 sub intervals to solve:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0$$
$$u(x,0) = \sin(\pi x), \qquad 0 \le x \le 1$$

at $t = 0.1$ and $t = 0.2$. Use $\Delta t = 0.1$. Is the FTCS method stable for this problem? Justify your answer.

[50 marks]

(b) The upward velocity of a rocket is given at three different times in the following table:

Time, t (s)	Velocity, v (m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3, \ \ \textit{where} \ \ 5 \le t \le 12$$

Find the values of a_1, a_2 and a_3 using the Gauss Seidel method. For this purpose, assume an initial guess of the solution as $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ and conduct two iterations.

[50 marks]

Soalan 4

(a) Gunakan kaedah FTCS dengan 4 sub selang untuk menyelesaikan:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0$$
$$u(x,0) = \sin(\pi x), \qquad 0 \le x \le 1$$

pada $t = 0.1$ dan $t = 0.2$. Gunakan $\Delta t = 0.1$. Adakah kaedah FTCS stabil untuk masalah ini? Justifikasi jawapan anda.

[50 markah]

(b) *Kelajuan tujahan sebuah roket diberikan pada tiga masa yang berlainan dalam jadual berikut:*

Masa, t (s)	*Kelajuan, v (m/s)*
5	*106.8*
8	*177.2*
12	*279.2*

Data kelajuan dianggarkan dengan suatu polynomial sebagai:

$$v(t) = a_1 t^2 + a_2 t + a_3 \text{, } \textit{dengan } 5 \le t \le 12$$

Dapatkan nilai-nilai a_1, a_2 *dan* a_3 *menggunakan kaedah Gauss Seidel. Untuk tujuan ini, andaikan nilai awalan bagi penyelesaian sebagai* $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ *dan jalankan dua lelaran.*

[50 markah]

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