First Semester Examination
Academic Session 2018/2019

December 2018/January 2019

MAT514 - Mathematical Modelling
(Pemodelan Matematik)

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of TEN (10) pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEPULUH (10) muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **FOUR (4)** questions.

*[**Arahan**: Jawab **EMPAT (4)** soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai.]

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Question 1

(a) Explain the principles that are used in constructing the basic equations of convective heat and mass transfer.

(b) Define

(i) fluid,
(ii) incompressible flow,
(iii) no-slip condition.

[15 marks]

Soalan 1

(a) Terangkan prinsip-prinsip yang digunakan dalam pembinaan persamaan-persamaan asas pemindahan jisim dan haba secara olakan.

(b) Takrifkan

(i) bendalir,
(ii) aliran tak termampat,
(iii) syarat tak gelincir.

[15 markah]

Question 2

Consider a steady flow along a two-dimensional surface with a free stream velocity u_∞ and a free stream temperature T_∞ where x is measured along the surface and y normal to the surface. An infinitesimal stationary control volume of $\delta x \times \delta y$ within the boundary layer with various rates of energy transfer across the control surface of the fluid mixture is shown in Figure 1.

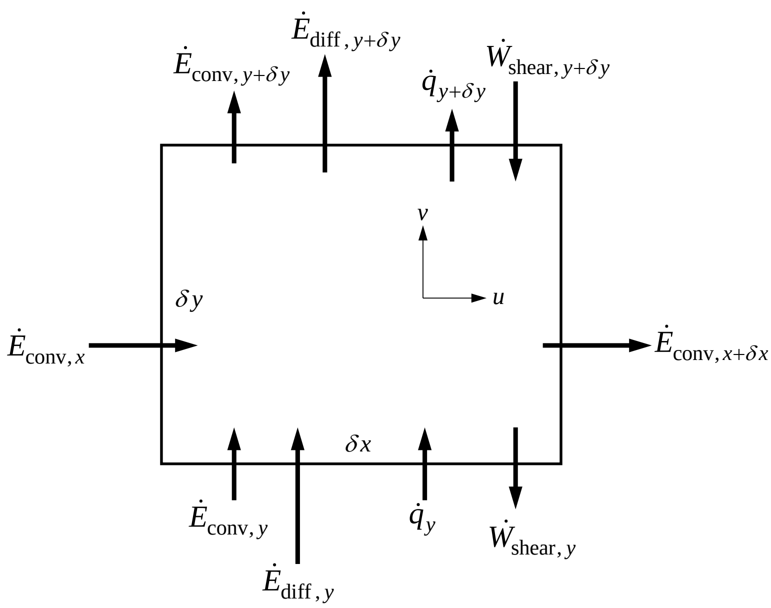


Figure 1: Control volume and energy transfer terms for development of the steady-flow energy differential equation of the boundary layer

Table 1: Basic rates of energy transfer

$\dot{E}_{\text{conv},y} = \rho v \delta x \left(i + \frac{1}{2}u^2\right)$ (convection rate: assuming $u^2 \gg v^2$),
$\dot{E}_{\text{diff},y} = -\left(\sum_j \gamma_j \frac{\partial m_j}{\partial y} i_j\right) \delta x$ (diffusion rate without Soret effect),
$\dot{q}_y = -k\left(\frac{\partial T}{\partial y}\right)\delta x$ (conduction heat transfer without Dufour effect),
$\dot{W}_{\text{shear},y} = (\tau_{yx} u)\,\delta x$ (shear force $\times$ velocity)

(a) The two-dimensional boundary layer continuity and momentum equations for this case are given, respectively, as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dP}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right). \tag{2}$$

Based on Figure 1, Table 1 and applying the related principle, the two dimensional boundary layer approximations, equations (1) and (2), show that the energy equation of the boundary layer is

$$\rho u\frac{\partial i}{\partial x} + \rho v\frac{\partial i}{\partial y} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\sum_j \gamma_j \frac{\partial m_j}{\partial y} i_j\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{dP}{dx}. \tag{3}$$

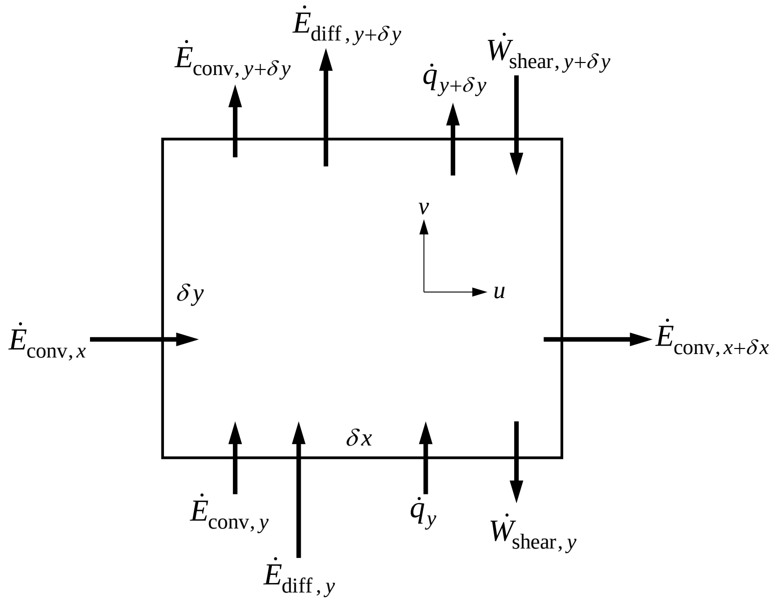
(b) State the assumption that can be made so that the mass diffusion term in equation (3) can be neglected.

[Note: (u,v) = velocity components along (x,y) axes; ρ = the density of the fluid; P = the pressure; i = the mixture enthalpy; i_j = the partial enthalpy of component j; γ_j = mass diffusion coefficient for component j in the mixture; m_j = mass concentration of component j in the mixture; k = the thermal conductivity; T = temperature of the fluid mixture; τ = shear stress]

[25 marks]

Soalan 2

Pertimbangkan suatu aliran mantap di sepanjang permukaan dua dimensi dengan halaju strim bebas u_∞ dan suhu strim bebas T_∞ dengan x diukur di sepanjang permukaan dan y serenjang terhadap permukaan. Satu unsur isipadu kawalan pegun $\delta x \times \delta y$ dalam lapisan *sempadan dengan beberapa kadar pemindahan tenaga merentasi permukaan kawalan bendalir campuran ditunjukkan dalam Rajah 1.*



Rajah 1: Isipadu kawalan dan sebutan-sebutan pemindahan haba untuk menerbitkan persamaan pembezaan tenaga bagi aliran lapisan sempadan yang mantap

Jadual 1: Kadar-kadar asas pemindahan tenaga

$$\dot{E}_{conv,y} = \rho v \delta x \left(i + \tfrac{1}{2}u^2\right) \text{ (kadar olakan: andaian } u^2 \gg v^2\text{)},$$

$$\dot{E}_{diff,y} = -\left(\sum_j \gamma_j \frac{\partial m_j}{\partial y} i_j\right) \delta x \text{ (kadar resapan tanpa kesan Soret)},$$

$$\dot{q}_y = -k\left(\frac{\partial T}{\partial y}\right) \delta x \text{ (pemindahan haba konduksi tanpa kesan Dufour)},$$

$$\dot{W}_{shear,y} = (\tau_{yx} u)\, \delta x \text{ (daya ricih} \times \text{halaju)}$$

(a) *Persamaan-persamaan keselanjaran dan momentum bagi lapisan sempadan dua dimensi untuk kes ini masing-masing diberikan seperti berikut:*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (1)$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dP}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right). \qquad (2)$$

Berdasarkan Rajah 1, Jadual 1 dan menggunakan prinsip yang berkaitan, penghampiran lapisan sempadan dua dimensi, persamaan-persamaan (1) dan (2), tunjukkan bahawa persamaan tenaga lapisan sempadan ialah

$$\rho u \frac{\partial i}{\partial x} + \rho v \frac{\partial i}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\sum_j \gamma_j \frac{\partial m_j}{\partial y} i_j\right) + \mu \left(\frac{\partial u}{\partial y}\right)^2 + u \frac{dP}{dx}. \quad (3)$$

(b) *Nyatakan andaian yang boleh dibuat supaya sebutan resapan jisim dalam persamaan (3) boleh diabaikan.*

[Nota: (u,v) = komponen-komponen halaju sepanjang paksi (x,y); ρ = ketumpatan bendalir; P = tekanan; i = entalpi campuran; i_j = entalpi separa komponen j; γ_j = pekali resapan jisim komponen j dalam campuran; m_j = kepekatan jisim komponen j dalam campuran; k = kekonduksian terma; T = suhu bendalir campuran; τ = tegasan ricih]

[25 markah]

Question 3

Assume that the momentum and energy equations with constant surface heat rate for axisymmetric flow in a circular tube of radius a in x-direction is given by the following equations, respectively:

$$r\frac{dP}{dx} - \frac{\partial}{\partial r}\left(r\,\mu \frac{\partial u}{\partial r}\right) = 0,$$

$$r\frac{dT_m}{dx} - \frac{1}{u}\frac{\partial}{\partial r}\left(r\,\alpha \frac{\partial T}{\partial r}\right) = 0,$$

where x and r are the axial and radial coordinates of circular tube, respectively, and u and T are the fully developed velocity and temperature profiles, respectively. The pressure P and the mass-averaged fluid temperature T_m are independent of r, and the dynamic viscosity μ and the thermal diffusivity α of the fluid are constants.

(a) Derive a fully developed temperature profile T by using the momentum and energy equations subject to the following boundary conditions:

$$u = 0, \quad T = T_s \text{ at the surface of tube, and}$$

$$\frac{\partial u}{\partial r} = \frac{\partial T}{\partial r} = \mu x r \text{ at the centerline of tube,}$$

where T_s is the fluid temperature at the surface of circular tube.

(b) What is the total flow rate Q if

$$Q = \int_0^a 2\pi r u \, dr?$$

(c) What is the temperature gradients near the surface of the tube?

[27 marks]

Soalan 3

Andaikan bahawa persamaan-persamaan momentum dan tenaga dengan kadar haba permukaan malar bagi aliran simetri sepaksi dalam tiub bulat mendatar berjejari a pada arah x masing-masing diberikan oleh persamaan-persamaan berikut:

$$r\frac{dP}{dx} - \frac{\partial}{\partial r}\left(r\,\mu\frac{\partial u}{\partial r}\right) = 0,$$

$$r\frac{dT_m}{dx} - \frac{1}{u}\frac{\partial}{\partial r}\left(r\,\alpha\frac{\partial T}{\partial r}\right) = 0,$$

dengan x dan r masing-masing ialah koordinat paksian dan jejarian tiub bulat, dan u dan T masing-masing ialah profil-profil halaju dan suhu terbangun penuh. Tekanan P dan suhu jisim-purata bendalir T_m tidak bergantung terhadap r, dan kelikatan dinamik μ dan resapan terma α adalah malar.

(a) Terbitkan profil suhu terbangun penuh T dengan menggunakan persamaan-persamaan momentum dan tenaga tertakluk kepada syarat-syarat sempadan berikut:

$$u = 0, \quad T = T_s \text{ pada permukaan tiub, dan}$$

$$\frac{\partial u}{\partial r} = \frac{\partial T}{\partial r} = \mu x r \text{ pada garis tengah tiub,}$$

dengan T_s ialah suhu bendalir pada permukaan tiub bulat.

(b) Apakah jumlah kadar aliran Q jika

$$Q = \int_0^a 2\pi r u\, dr?$$

(c) Apakah kecerunan suhu berhampiran permukaan tiub?

[27 markah]

Question 4

Consider a steady boundary layer flow and heat transfer past a sphere of radius b in a forced convection flow of a viscous and incompressible fluid of free stream velocity u_∞ and ambient temperature T_∞. It is assumed that the sphere is kept at the uniform heat flux q_w and the boundary layer approximations are valid. Under these assumptions, the dimensional basic equations are

$$\frac{\partial}{\partial \bar{x}}(\bar{r}\bar{u}) + \frac{\partial}{\partial \bar{y}}(\bar{r}\bar{v}) = 0, \tag{4}$$

$$\rho\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \rho\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \rho\bar{u}_e\frac{d\bar{u}_e}{d\bar{x}} + \mu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \tag{5}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \alpha\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}, \tag{6}$$

subject to the boundary conditions

$$\bar{u} = 0, \quad \frac{\partial \bar{T}}{\partial \bar{y}} = -\frac{q_w}{k} \quad \text{at} \quad \bar{y} = 0, \tag{7}$$
$$\bar{u} \to \bar{u}_e, \quad \bar{T} \to T_\infty \quad \text{as} \quad \bar{y} \to \infty,$$

where $\bar{x}$ and $\bar{y}$ are the Cartesian coordinates measured along the sphere and normal to it, respectively, $\bar{u}$ and $\bar{v}$ are velocity components along $\bar{x}$ and $\bar{y}$ axes, respectively, $\bar{T}$ is the temperature of the fluid, ρ is the constant density of the fluid, μ is the dynamic viscosity of the fluid, $\alpha = \mu/(\rho \mathrm{Pr})$ is the thermal diffusivity of the fluid, Pr is the Prandtl number and k is the themal conductivity. Here $\bar{r}$ is the radial distance from the symmetrical axis to the surface of the sphere and $\bar{u}_e$ is the local free stream velocity which are given by $b\sin(\bar{x}/b)$ and $(3/2)u_\infty \sin(\bar{x}/b)$, respectively. The non-dimensional variables are

$$x = \bar{x}/b, \quad y = \mathrm{Re}^{1/2}(\bar{y}/b), \quad u = \bar{u}/u_\infty, \quad v = \mathrm{Re}^{1/2}(\bar{v}/u_\infty),$$
$$r = \bar{r}/b, \quad \theta = \mathrm{Re}^{1/2}(\bar{T} - T_\infty)/(bq_w/k), \quad u_e = \bar{u}_e/u_\infty,$$

where $\mathrm{Re} = u_\infty b\rho/\mu$ is the Reynolds number.

Note: ψ is a stream function which is defined as $u = \frac{1}{r}\frac{\partial \psi}{\partial y}$ and $v = -\frac{1}{r}\frac{\partial \psi}{\partial x}$.

(a) By using non similarity variables of the following form

$$\psi = xrf(x,y), \quad \theta = g(x,y),$$

show that the boundary layer equations (4) – (6) can be reduced to the following system of differential equations

$$\frac{\partial^3 f}{\partial y^3} + \left(1 + \frac{x}{\sin x}\cos x\right) f\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial f}{\partial y}\right)^2 + \frac{9}{4}\frac{\sin x \cos x}{x}$$
$$= x\left(\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}\right), \tag{8}$$

$$\frac{1}{\mathrm{Pr}}\frac{\partial^2 g}{\partial y^2} + \left(1 + \frac{x}{\sin x}\cos x\right) f\frac{\partial g}{\partial y} = x\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right), \tag{9}$$

with the boundary conditions (7) becoming

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial g}{\partial y} = -1 \quad \text{at} \quad y = 0, \tag{10}$$
$$\frac{\partial f}{\partial y} \to \frac{3}{2}\frac{\sin x}{x}, \quad g \to 0 \quad \text{as} \quad y \to \infty.$$

(b) Table 1 shows the numerical results of shear stress $\left(\frac{\partial^2 f}{\partial y^2}\right)_{y=0}$ and surface temperature $(g)_{y=0}$ for the system of equations (8) and (9) subject to boundary conditions (10) when $x = 0$ with various values of Pr. Interpret the obtained results in this table.

Table 1: Numerical results of shear stress and surface temperature with various values of Pr

Pr	$\left(\frac{\partial^2 f}{\partial y^2}\right)_{y=0}$	$(g)_{y=0}$
0.7	2.4151	1.2270
1.0	2.4151	1.0711
7.0	2.4151	0.5280

[33 marks]

Soalan 4

Pertimbangkan suatu aliran lapisan sempadan dan pemindahan haba yang mantap terhadap sfera berjejari b dalam aliran olakan paksa bagi bendalir likat dan tak termampat dengan halaju strim bebas u_∞ dan suhu persekitaran T_∞. Andaikan bahawa sfera itu mempunyai fluks haba yang seragam q_w dan penghampiran lapisan sempadan adalah sah. Berdasarkan andaian tersebut, persamaan-persamaan asas berdimensi adalah

$$\frac{\partial}{\partial \bar{x}}(\bar{r}\bar{u}) + \frac{\partial}{\partial \bar{y}}(\bar{r}\bar{v}) = 0, \tag{4}$$

$$\rho\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \rho\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \rho\bar{u}_e\frac{d\bar{u}_e}{d\bar{x}} + \mu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \tag{5}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \alpha\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}, \tag{6}$$

tertakluk kepada syarat-syarat sempadan

$$\bar{u} = 0, \quad \frac{\partial \bar{T}}{\partial \bar{y}} = -\frac{q_w}{k} \quad \textit{pada} \quad \bar{y} = 0, \tag{7}$$
$$\bar{u} \to \bar{u}_e, \quad \bar{T} \to T_\infty \quad \textit{apabila} \quad \bar{y} \to \infty,$$

dengan $\bar{x}$ dan $\bar{y}$ masing-masing adalah koordinat-koordinat Cartesan yang diukur di sepanjang sfera dan berserenjang dengannya, $\bar{u}$ dan $\bar{v}$ masing-masing ialah komponen-komponen halaju pada paksi $\bar{x}$ dan $\bar{y}$, $\bar{T}$ ialah suhu bendalir, ρ ialah ketumpatan bendalir malar, μ ialah kelikatan dinamik bendalir, $\alpha = \mu/(\rho Pr)$ ialah resapan terma bendalir, Pr ialah nombor Prandtl dan k ialah kekonduksian terma. Di sini $\bar{r}$ ialah jarak jejari dari paksi simetri dengan permukaan sfera dan $\bar{u}_e$ ialah halaju strim bebas setempat yang masing-masing diberikan oleh $b\sin(\bar{x}/b)$ dan $(3/2)u_\infty \sin(\bar{x}/b)$. Pemboleh-pemboleh ubah tak berdimensi adalah

$$x = \bar{x}/b, \quad y = Re^{1/2}(\bar{y}/b), \quad u = \bar{u}/u_\infty, \quad v = Re^{1/2}(\bar{v}/u_\infty),$$
$$r = \bar{r}/b, \quad \theta = Re^{1/2}(\bar{T} - T_\infty)/(bq_w/k), \quad u_e = \bar{u}_e/u_\infty,$$

dengan $Re = u_\infty b\rho/\mu$ ialah nombor Reynolds.

Nota: ψ adalah fungsi strim yang ditakrifkan sebagai $u = \dfrac{1}{r}\dfrac{\partial \psi}{\partial y}$ dan $v = -\dfrac{1}{r}\dfrac{\partial \psi}{\partial x}$.

(a) *Dengan menggunakan pemboleh-pemboleh ubah tak serupa berbentuk berikut*

$$\psi = xrf(x,y), \quad \theta = g(x,y),$$

tunjukkan bahawa persamaan-persamaan lapisan sempadan (4) – (6) boleh diturunkan kepada sistem persamaan pembezaan berikut

$$\frac{\partial^3 f}{\partial y^3} + \left(1 + \frac{x}{\sin x}\cos x\right) f \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial f}{\partial y}\right)^2 + \frac{9}{4}\frac{\sin x \cos x}{x} = x\left(\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2}\right), \qquad (8)$$

$$\frac{1}{Pr}\frac{\partial^2 g}{\partial y^2} + \left(1 + \frac{x}{\sin x}\cos x\right) f \frac{\partial g}{\partial y} = x\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right), \qquad (9)$$

dengan syarat-syarat sempadan (7) menjadi

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial g}{\partial y} = -1 \quad \text{pada} \quad y = 0, \qquad (10)$$
$$\frac{\partial f}{\partial y} \to \frac{3}{2}\frac{\sin x}{x}, \quad g \to 0 \quad \text{apabila} \quad y \to \infty.$$

(b) *Jadual 1 menunjukkan keputusan berangka untuk tegasan ricih $\left(\dfrac{\partial^2 f}{\partial y^2}\right)_{y=0}$ dan suhu permukaan $(g)_{y=0}$ bagi sistem persamaan (8) dan (9) tertakluk kepada syarat-syarat sempadan (10) bila $x = 0$ dengan beberapa nilai Pr. Tafsirkan keputusan yang diperoleh dalam jadual tersebut.*

Jadual 1: Keputusan berangka untuk tegasan ricih dan suhu permukaan haba permukaan dengan beberapa nilai Pr

Pr	$\left(\frac{\partial^2 f}{\partial y^2}\right)_{y=0}$	$(g)_{y=0}$
0.7	2.4151	1.2270
1.0	2.4151	1.0711
7.0	2.4151	0.5280

[33 markah]

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