First Semester Examination
Academic Session 2018/2019

December 2018/January 2019

# MAT363 - Statistical Inference
## *(Pentaabiran Statistik)*

Duration : 3 hours
*[Masa : 3 jam]*

Please check that this examination paper consists of SEVEN (7) pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH (7) muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

**Instructions** : Answer **FOUR (4)** questions.

*[**Arahan :** Jawab **EMPAT (4)** soalan.]*

In the event of any discrepancies, the English version shall be used.

*[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].*

**Question 1**

(a) Consider three events *A*, *B* and *E* such that: $P(A)=0.3$, $P(AB)=3P(ABE)$, $P(A\cup E)=0.5$, $P(AB^c)=0.03$, $P(BE)=0.1$ and $P(A^cB^cE^c)=0.48$.

(i) Obtain $P(ABE)$

(ii) Are events *AB* and *AE* independent?

(b) If the joint probability density of *X* and *Y* is given by

$$f(x)=\begin{cases}1+x & for \quad -1<x\le 0\\ 1-x & for \quad 0<x<1\\ 0 & elsewhere\end{cases}$$

and $U = X$ and $V = X^2$, show that

(i) $\operatorname{cov}(U,V)=0$

(ii) *U* and *V* are dependent

(c) The joint pdf of *X* and *Y* is given by $f(x,y)=\dfrac{1}{\sqrt{2\pi}}e^{-y}e^{-(x-y)^2/2}$, $0<y<\infty$, $-\infty<x<\infty$. Compute the joint moment generating function of *X* and *Y*.

[ 25 marks ]

***Soalan 1***

*(a)* *Pertimbangkan tiga peristiwa A, B dan E supaya:* $P(A)=0.3$, $P(AB)=3P(ABE)$, $P(A\cup E)=0.5$, $P(AB^c)=0.03$, $P(BE)=0.1$ *dan* $P(A^cB^cE^c)=0.48.$

*(i)* *Dapatkan* $P(ABE)$

*(ii)* *Adakah peristiwa AB dan AE tak bersandar?*

(b) *Jika ketumpatan kebarangkalian tercantum X dan Y diberi oleh*

$$f(x)=\begin{cases}1+x & \textit{untuk} \quad -1<x\le 0\\ 1-x & \textit{untuk} \quad 0<x<1\\ 0 & \textit{selainnya}\end{cases}$$

*dan* $U = X$ *dan* $V = X^2$ *, tunjukkan bahawa*

(i) $\operatorname{cov}(U,V)=0$

(ii) *U dan V adalah bersandar*

(c) *Fungsi ketumpatan kebarangkalian tercantum X dan Y adalah diberi oleh* $f(x,y)=\frac{1}{\sqrt{2\pi}}e^{-y}e^{-(x-y)^2/2}, \quad 0<y<\infty, -\infty<x<\infty$. *Terbitkan fungsi penjana momen tercantum X dan Y.*

*[ 25 markah ]*

**Question 2**

(a) Let $X_A$ be the indicator random variable for event *A* with probability *P*(*A*)=0.8. Let $\hat{P}(A)$ denote the relative frequency of event *A* in *n* independent trials.

(i) Find $E[X_A]$ and $\operatorname{var}[X_A]$

(ii) Use the Chebyshev's inequality to find the confidence coefficient $1-\alpha$ such that $P\left\{\left|\hat{P}_{100}(A)-P(A)\right|\le 0.1\right\}\ge 1-\alpha$

(b) Consider two random variables *X* and *Y* with the joint probability density

$$f(x,y)=\begin{cases}12xy(1-y) & 0<x<1, 0<y<1\\ 0 & \textit{elsewhere}\end{cases}$$

(i) Find the probability density of $Z = XY^2$

(ii) Find the joint probability density of $Z = XY^2$ and *U*=*Y*

[ 25 marks ]

***Soalan 2***

*(a)* *Biarkan* $X_A$ *sebagai pembolehubah rawak bagi peristiwa A dengan kebarangkalian P(A)=0.8. Biarkan* $\hat{P}(A)$ *mewakili kekerapan relatif untuk peristiwa A dalam n percubaan tak bersandar.*

*(i)* *Cari* $E[X_A]$ *dan* $\text{var}[X_A]$

*(ii)* *Guna ketaksamaan Chebyshev untuk mencari pekali keyakinan* $1-\alpha$ *supaya* $P\left\{\left|\hat{P}_{100}(A)-P(A)\right|\le 0.1\right\}\ge 1-\alpha$

*(b)* *Pertimbangkan dua pembolehubah rawak petunjuk X dan Y dengan ketumpatan kebarangkalian tercantum* $f(x,y)=\begin{cases}12xy(1-y) & 0<x<1, 0<y<1\\ 0 & \textit{selainnya}\end{cases}$

*(i)* *Cari ketumpatan kebarangkalian bersama bagi* $Z = XY^2$

*(ii)* *Cari ketumpatan kebarangkalian bersama bagi* $Z = XY^2$ *dan U=Y*

*[ 25 markah ]*

**Question 3**

(a) Let $X_1,...,X_n$ be a random sample from $\exp(\lambda)$.

(i) Provide the maximum likelihood estimate for $\lambda$

(ii) Provide the method of moments estimate for $\lambda$

(b) $Y_1,...,Y_n$ be a random sample of size $n$ from the pdf $f_Y(y;\theta)=\frac{1}{(r-1)!\theta^r}y^{r-1}e^{-y/\theta}$; $y$>0. It is given that $E(Y)=r\theta$ and $\text{var}(Y)=r\theta^2$.

(i) Is $\hat{\theta}=\frac{1}{r}\overline{Y}$ an unbiased estimator for $\theta$?

(ii) Show that $\hat{\theta}=\frac{1}{r}\overline{Y}$ is a minimum variance estimator for $\theta$.

[ 25marks ]

***Soalan 3***

*(a)* *Biarkan* $X_1,...,X_n$ *sebagai sampel rawak dari* $\mathrm{e}ks(\lambda)$.

*(i)* *Berikan anggaran kebolehjadian maksimum bagi* $\lambda$

*(ii)* *Berikan anggaran kaedah momen bagi* $\lambda$

*(b)* $Y_1,...,Y_n$ *merupakan sampel rawak bersaiz n dari fungsi ketumpatan kebarangkalian* $f_Y(y;\theta)=\frac{1}{(r-1)!\theta^r}y^{r-1}e^{-y/\theta}$ *; y>0. Adalah diberi bahawa* $E(Y)=r\theta$ *dan* $\mathrm{var}(Y)=r\theta^2$.

*(i)* *Adakah* $\hat{\theta}=\frac{1}{r}\overline{Y}$ *penganggar saksama bagi* $\theta$ *?*

*(ii)* *Tunjukkan bahawa* $\hat{\theta}=\frac{1}{r}\overline{Y}$ *merupakan penganggar varians minimum bagi* $\theta$.

*[ 25 markah ]*

**Question 4**

(a) Let $X$ be a single observation from a density function $f(x,\theta)=\theta x^{\theta-1}$, 0<$x$<1 and $\theta>0$. If $-\theta\log X$ is a pivotal quantity, use it to construct a confidence interval for $\theta$ with the level of confidence $1-\alpha$.

(b) Let $X_1,...,X_n$ be a random sample from the normal distribution $N(\mu,196)$.

(i) To test $H_0:\mu=56$ against $H_1:\mu\neq 56$, what is the critical region of size $\alpha=0.05$ specified by the likelihood ratio test criterion? $z_{0.025}=1.96$

(ii) If a sample of size $n$=100 yielded $\overline{x}=56.2$, is $H_0$ accepted?

(iii) Find the sample size if we are 95% confident that the maximum error of estimate of $\mu$ is 1.5.

(c) Let $X_1$ and $X_2$ be a random sample of size 2 from the Poisson distribution $f(x,\lambda)=\frac{\lambda^x e^{-\lambda}}{x!}$. Show that $T=X_1+X_2$ is a sufficient statistic.

[ 25 marks ]

***Soalan 4***

(a) *Biarkan X sebagai suatu cerapan tunggal dari fungsi ketumpatan* $f(x,\theta)=\theta x^{\theta-1}$, $0<x<1$ *dan* $\theta>0$. *Jika* $-\theta\log X$ *ialah kuantiti pangsian, gunakannya untuk membina satu selang keyakinan bagi* $\theta$ *dengan aras keyakinan* $1-\alpha$ .

(b) *Biarkan* $X_1,\ldots,X_n$ *sebagai satu sampel rawak dari taburan normal* $N(\mu,196)$.

(i) *Untuk menguji* $H_0:\mu=56$ *lawan* $H_1:\mu\neq 56$, *apakah rantau genting bersaiz* $\alpha=0.05$ *yang dispesifikasikan oleh kriteria ujian nisbah kebolehjadian? (* $z_{0.025}=1.96$ *).*

(ii) *Jika sampel bersaiz n=100 memberi* $\bar{x}=56.2$ , *adakah* $H_0$ *diterima?*

(iii) *Cari saiz sampel jika kita adalah 95% yakin bahawa ralat maksimum bagi anggaran* $\mu$ *ialah 1.5.*

(c) *Biarkan* $X_1$ and $X_2$ *merupakan pembolehubah rawak bersaiz 2 dari taburan Poisson distribution* $f(x,\lambda)=\dfrac{\lambda^x e^{-\lambda}}{x!}$ . *Tunjukkan bahawa* $T=X_1+X_2$ *adalah statistik mencukupi.*

*[ 25 markah ]*

Appendix

## Discrete Distributions

| | |
|---|---|
| **Bernoulli**<br>$0 < p < 1$ | $f(x) = p^x(1-p)^{1-x}, \quad x = 0, 1$<br>$M(t) = 1 - p + pe^t$<br>$\mu = p, \quad \sigma^2 = p(1-p)$ |
| **Binomial**<br>$b(n, p)$<br>$0 < p < 1$ | $f(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n$<br>$M(t) = (1 - p + pe^t)^n$<br>$\mu = np, \quad \sigma^2 = np(1-p)$ |
| **Geometric**<br>$0 < p < 1$ | $f(x) = (1-p)^{x-1}p, \quad x = 1, 2, 3, \ldots$<br>$M(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\ln(1-p)$<br>$\mu = \frac{1}{p}, \quad \sigma^2 = \frac{1-p}{p^2}$ |
| **Hypergeometric**<br>$N_1 > 0, \; N_2 > 0$<br>$N = N_1 + N_2$ | $f(x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}, \quad x \le n, x \le N_1, n - x \le N_2$<br>$\mu = n\left(\frac{N_1}{N}\right), \quad \sigma^2 = n\left(\frac{N_1}{N}\right)\left(\frac{N_2}{N}\right)\left(\frac{N-n}{N-1}\right)$ |
| **Negative Binomial**<br>$0 < p < 1$<br>$r = 1, 2, 3, \ldots$ | $f(x) = \binom{x-1}{r-1}p^r(1-p)^{x-r}, \quad x = r, r+1, r+2, \ldots$<br>$M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}, \quad t < -\ln(1-p)$<br>$\mu = r\left(\frac{1}{p}\right), \quad \sigma^2 = \frac{r(1-p)}{p^2}$ |
| **Poisson**<br>$0 < \lambda$ | $f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$<br>$M(t) = e^{\lambda(e^t - 1)}$<br>$\mu = \lambda, \quad \sigma^2 = \lambda$ |
| **Uniform**<br>$m > 0$ | $f(x) = \frac{1}{m}, \quad x = 1, 2, \ldots, m$<br>$\mu = \frac{m+1}{2}, \quad \sigma^2 = \frac{m^2-1}{12}$ |

## Continuous Distributions

| | |
|---|---|
| **Beta**<br>$0 < \alpha$<br>$0 < \beta$ | $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1$<br>$\mu = \frac{\alpha}{\alpha+\beta}, \quad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$ |
| **Chi-square**<br>$\chi^2(r)$<br>$r = 1, 2, \ldots$ | $f(x) = \frac{1}{\Gamma(r/2)2^{r/2}}x^{r/2-1}e^{-x/2}, \quad 0 \le x < \infty$<br>$M(t) = \frac{1}{(1-2t)^{r/2}}, \quad t < \frac{1}{2}$<br>$\mu = r, \quad \sigma^2 = 2r$ |
| **Exponential**<br>$0 < \theta$ | $f(x) = \frac{1}{\theta}e^{-x/\theta}, \quad 0 \le x < \infty$<br>$M(t) = \frac{1}{1-\theta t}, \quad t < \frac{1}{\theta}$<br>$\mu = \theta, \quad \sigma^2 = \theta^2$ |
| **Gamma**<br>$0 < \alpha$<br>$0 < \theta$ | $f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha}x^{\alpha-1}e^{-x/\theta}, \quad 0 \le x < \infty$<br>$M(t) = \frac{1}{(1-\theta t)^\alpha}, \quad t < \frac{1}{\theta}$<br>$\mu = \alpha\theta, \quad \sigma^2 = \alpha\theta^2$ |
| **Normal**<br>$N(\mu, \sigma^2)$<br>$-\infty < \mu < \infty$<br>$0 < \sigma$ | $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$<br>$M(t) = e^{\mu t + \sigma^2 t^2/2}$<br>$E(X) = \mu, \quad \mathrm{Var}(X) = \sigma^2$ |
| **Uniform**<br>$U(a, b)$<br>$-\infty < a < b < \infty$ | $f(x) = \frac{1}{b-a}, \quad a \le x \le b$<br>$M(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}, \quad t \ne 0; \quad M(0) = 1$<br>$\mu = \frac{a+b}{2}, \quad \sigma^2 = \frac{(b-a)^2}{12}$ |

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